AN ELEMENTARY TREATISE ON ALGEBRA, IN THEORY AND PRACTICE.

WITH ATTEMPTS TO SIMPLIFY
SOME OF THE MORE DIFFICULT PARTS OF THAT SCIENCE, PARTICULARLY
THE SOLUTION OF CUBIC EQUATIONS AND OF THE
HIGHER ORDERS.

WITH
NOTES AND ILLUSTRATIONS,
CONTAINING A VARIETY OF PARTICULARS RELATING TO THE DISCOVERIES
AND IMPROVEMENTS THAT HAVE BEEN MADE IN THIS BRANCH OF
ANALYSIS, AND, IT IS BELIEVED, MORE NEW AND EN-
TERTAINING QUESTIONS AND SOLUTIONS THAN
CAN BE FOUND IN ANY OTHER WORK
ON THE SAME SUBJECT.

TO WHICH IS ADDED
AN APPENDIX, ON THE APPLICATION OF ALGEBRA TO
GEOMETRY.

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"A Key to Hutton's Mathematics, containing the Questions and their Solutions;"
"Arithmetical and Algebraical Amusements," "Arithmetic and Key," &c.

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PREFACE.

ALGEBRA is allowed to be the grand pillar on which the whole science of the Mathematics depends. It is the foundation on which the glorious superstructure of the abstruse sciences has been reared, and is therefore of undoubted importance and utility. Indeed, volumes have been written on the elegance and importance of this study, but the subject is inexhausted; and the author deems no apology necessary for dwelling a few moments on a subject which has occupied much of his time, and which has proved a source of the purest gratification while engaged in its pursuits.

We have said that Algebra is the foundation of all the abstruse sciences; the assertion is not made at random, but admits of ample proofs and clear demonstrations. Unlike arithmetic in this respect, it does not merely consist in understanding the common routine of mechanical and mercantile pursuits, but its study expands the intellect, enlarges the reasoning faculties, and accustoms the juvenile mind to patient attention and accurate reasoning. By its operations, the laws which govern the planetary system have been calculated, their order, harmony, and regularity have been displayed, and their general characteristics have been developed; the trackless ocean has been traversed, its boundaries have been determined, and a communication has been opened with every corner of our globe. Hence the increasing taste for these studies is readily accounted for, and we cease to wonder at the numerous compilations which have appeared, in order to facilitate the labors of the student. Unfortunately, however, the increasing demand for works of this nature has called into existence an ephemeral race of authors, who, instead of lessening the labors of the student, have strown his path with new difficulties, and harassed and perplexed him with unmeaning and tautological rules and expressions, insomuch that many have given up its pursuit on the very threshold, and others, having proceeded a short distance, and found new and increasing difficulties at every step, have closed the volume, and with it all attempts at mathematical studies, in hopeless despair.

It may be here mentioned, that some of these works on algebra are mere copies of others, and in one instance a person of this city, though, to our credit be it spoken, not an American, having republished a European work, claimed the merit of it as his own, after having *copied verbatim* from another, and with so little judgment that the *very errors* of the *press*, in the London edition, appear glaringly, and without comment, in the work which this individual (who shall be nameless) claimed as exclusively his own.

As it respects the method which should be adopted in pursuing

the study of algebra, it should be remarked, that it is of primary importance that the student should in the first place make himself master of the common rules of arithmetic. Unless this is accomplished, it would be mere waste of time to attempt proceeding in the study of this science, which, although beautifully simple in its rules, is precise and accurate in its investigations. It is also the nature of all mathematical sciences, and of algebra in particular, to advance in continued progression, patiently but steadily; and hence the obvious necessity of the student's learning every thing thoroughly; and of perfectly understanding every rule and question as he advances, before he proceeds to another.

By these means the study, instead of being a toil, will become a pleasure, what at first appeared difficult will become easy, and the student will find new beauties allure him at every step, and cheer him through every difficulty. Let the diligent student also bear in mind that genius without application is useless, and that continued and untiring perseverance can accomplish almost every thing, however arduous it may appear at the outset. When, however, to a genius and a taste for mathematical pursuits is added a persevering industry, the juvenile mind ascends the ladder of science to the topmost round, and gazes eagle-eyed on that imperishable tablet where are recorded the names of a Newton, Bowditch, Galileo, Leibnitz, Lagrange, and Laplace, who, by their works, have not only shed a lustre upon the science, but have gained for themselves an immortality that shall endure till time shall be no longer. Our own country too, though it is comparatively young, and has made but little progress in the mathematical sciences, has yet produced a few men who would have done honor to any age or country; and when the green sod of the valley shall have covered their mortal remains, when the present generation shall have passed away and been forgotten, their names will acquire unfading lustre, and be hailed by generations yet unborn, as the luminaries of science, and as the benefactors of the human race. Posterity must pass their eulogia, to posterity they must look for their fame and their immortality. And here, could I find suitable language, I might pay a passing tribute to the memory of him who sleeps beneath the waves of the Atlantic, the young, the accomplished, the lamented Fisher.* But what avails it? He hath passed from his sphere of usefulness; his bright and glorious career is finished; but his memory yet lives, and will be cherished in the hearts of his countrymen as a legacy never to be forgotten.

It only remains now for the author to state the reasons which have induced him to this compilation, and the manner in which it has been treated.

* He was a passenger in the packet-ship Albion, lost on the coast of Ireland.

To make an excuse, or to offer an apology for a work of this nature, is, I believe, unnecessary, because, if it should prove unworthy of the public patronage, no excuse will palliate its defects, and no prefatorial apology will be received in extenuation for accidental or wilful error. Hence, apology is needless; but the author believes it to be a duty he owes to the public to state the reasons which have caused this compilation, and they are briefly these: In his own opinion, and in that of some of the best mathematicians of our city, the different algebras at present in use in this country are defective in many particulars. The rules, it is believed, are generally not the best that could be given, and in some cases are tediously abstruse and perplexing; and though there is generally an ample sufficiency of theory, yet there is not practice enough to engage without wearying the attention of the student, and to excite without overburdening his reasoning faculties. These objections it has been the aim of the author to remove, and to treat the subject on a clear and rational foundation, with as much simplicity as possible, and, instead of making a mystery of the science, and putting new and imaginary difficulties in the way of the student, to remove them by every possible method.

For these reasons the questions are worked out at full length, and every thing explained in the operation; and it is believed that the time is gone by when the best method of teaching algebra was thought to be by giving the student a difficult question, and leaving him to ponder and pore for weeks over a set of, to him, unmeaning symbols and figures.

The solutions of the greatest part of the difficult questions, being all given, will be of utility, also, in an economical point of view; as to all the algebras now in use, Keys, containing solutions, have been published, and by the necessity of purchasing them the student is subjected to additional expense.

The student will find a greater number of new equations than have ever been published in any treatise before. As they are problems which possess a degree of interest to the mind of the young student, it has been thought they would prove highly useful by blending amusement with study. The method of solving the irreducible case of cubic equations is new and of great importance, as it solves the most difficult questions with the greatest ease.

The author, in conclusion, would take this opportunity of returning his grateful thanks to the public for the liberal patronage bestowed on his former publications; and in offering this new work to their notice, he fearlessly depends on the candor and impartiality of an enlightened and liberal public.

JOHN D. WILLIAMS.

Dighton, March 1, 1839.

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☞ It is to be distinctly understood that I am indebted to the following works, viz. Simson's Algebra, Emerson, Bonnycastle, Euler, Saunderson, Maclaurin, Bland, Bridge, Wood, Nicholson, Newton's Universal Arithmetic, Ward, Doddridge, De Moivre's, Clairaut, Wolfius, Peacock, Muller, Hayes, Fermat, Pierse's, Diophantus, Bachet, Waring, and others, which my limits will not permit me to name.

1. A and B purchase 900 (a) acres of land at the rate of \$2 p
acre, which they paid equally between them; but on dividi
the same, A got that part of the farm which contained the hou
and agreed to pay \$$\frac{45}{100}$ or \$$\frac{9}{20}$ per acre more than B; $\therefore$ 45 ct
how many acres had A and B, and at what price? Let x, and a—
denote the number of acres A and B each had; then $\frac{a}{x}$; $\frac{a}{a-x}+\frac{9}{2}$
and $\frac{a}{x}=\frac{a}{a-x}+\frac{9}{20}$; will be the price A and B each paid per ac
and the equation. Reduced I have $20a^2-200x=20ax+9ax-9x$
$\therefore x^2-\frac{49}{9}ax=\frac{20}{9}a^2$, by Art. 70, page 141. $x=2490-2090=4$
$=$A's share, and $900-400=500=$B's share. And $\frac{900}{400}-\frac{900}{500}=$
$225-180=45$ cents. See Flint's Surveying.

Let x, and $900-x$ be the acres A and B each bought, and let
and $y+45$ represent the cents per acre given by B and A respec
ively, then $xy+45x=900y-xy$, (1) and $xy+45x+900y-xy=$
180000, (2) and by (1) and (2) I have $\frac{900y}{2y+45}=4000-20y$; $\therefore y^2-$
$155y=4500$, by Art. 70. $y=180$ cents the price of one acre o
B's, and $180+45=2.25$ cents, the price of one acre of A's lan
Otherwise let x, and y, denote the number of acres A and B eac
bought, then the price of one acre of each man's land will be ex
pressed by $\frac{90000}{x}$, and $\frac{90000}{y}$; $\therefore$ by the question $x+y=90$
and $\frac{90000}{x}=\frac{90000}{y}+45$, or $\frac{90000}{900-y}=\frac{90000}{y}+45$; or
$y^2+3100y=1800000$, and $y=500$, as before.

2. A man buys 80 pounds of pepper and 36 pounds of saffro
so that for \$8, he had 14 pounds of pepper more than he had o
saffron for \$26, and what he laid out amounted to \$188. Ho
many pounds of pepper had he for \$8, and how many of saffro
for \$26. Ans. 20 pounds of pepper and 6 of saffron

Let x denote number of pounds of pepper that he bought fo
\$8, and $y=$number of pounds of saffron for \$26, then by th
question if x pounds cost \$8 what will 80 pounds cost, and if y
pounds of saffron cost 26 dollars what will 36 pounds of saffro
cost, and consequently their sum must $=188$, that is $\frac{8.80}{x}+\frac{36.2}{y}$
$=188$ or $\frac{160}{x}+\frac{234}{y}=47$, (1) and $x-y=14$, (2) or $\frac{160}{14+y}+\frac{23}{y}$
$=47$, $160y+234y+3276=47y^2+658y$, or $47y^2+264y=3276$
part 70; $\therefore y=6$ and $x=14+6=20=$the number of pounds he
bought of pepper, and 6 of saffron. Ans.

Investigation of the Rules of Compound Interest.

Let x = annuity, rent, or pension, n = number of times that interest is to be paid for the annuity or sum lent, r = rate of interest of 1 dollar for 1 time, m = amount of the annuity, or sum lent, for n times, at r interest, p = principal, sum used, or present worth of a sum before it is due, (of compound interest.) First in amounts let $q = 1 + r$ = amount of one dollar for 1 time. Now a = last year's amount, and $1 : q :: a : aq$ = last year but 1 amt. $1 : q :: aq : aq^2$ = last but 2 year's amount. $1 : q :: aq^2 : aq^3$ = last but 3, and so on to aq^{n-1} = first year's amount; therefore $a + aq + aq^2 + aq^3$, &c., $+ aq^{n-1} = m$; but $a : aq :: m - aq^{n-1} : m - a$. then $m - a = mq - aq^n$, or $m - a = m + mr - aq^n$, $\therefore aq^n - a = mr$, then put $A = q^n$, $(A-1)a = mr$, $\therefore a = \frac{mr}{*A-1}$; $m = \frac{(A-1)a}{r}$; $r = \frac{(A-1)a}{m}$.

2d. In discounts, $q : 1 :: a : \frac{a}{q}$ = first year's present worth, and $q : 1 :: \frac{a}{q} : \frac{a}{q^2}$ = 2d year's present worth; $q : 1 :: \frac{a}{q^2} : \frac{a}{q^3}$ = 3d year, and so on $\frac{a}{q^n}$ = nth year's present worth, therefore I have $\frac{a}{q} + \frac{a}{q^2} + \frac{a}{q^3}$ + &c. to $\frac{a}{q^n} = p$; but $\frac{a}{q} : \frac{a}{q^2} :: p - \frac{a}{q^n} : p - \frac{a}{q}$; (Euc. 12, V.) $\therefore \frac{pq^n - a}{q^n} = pq - a$, or $\frac{pq^n - a}{q^n} = p + pr - a$; $\therefore aq^n - a = prq^n$, or $(A-1)a = prA$; $\therefore A = \frac{a}{a - rp}$; hence $(A-1)a = mr = prA$; $\therefore A = \frac{mr + a}{a} = \frac{a}{a - rp} = \frac{m}{p} = (1+r)^n$.

Case 1. Given a, m, and p, to find r.

Since $A = \frac{m}{p}$; $\therefore r = \frac{(A-1)a}{m}$. $\quad *A = \frac{mr + a}{a}$.

Case 2. Given p, m, and n, to find r.

Since $(r+1)^n = \frac{m}{p}$; $\therefore r = \left(\frac{m}{p}\right)^{\frac{1}{n}} - 1$.

Case 3. Given a, m, and n, to find r.

Since $(A-1)a = mr$; therefore $\frac{(A-1)}{nr} = \frac{m}{na} = \frac{(1+r)^n - 1}{nr}$; now $(1+r)^n = 1 + nr + n\left(\frac{n-1}{2}\right)r^2 + n\left(\frac{n-1}{2}\right) \times \left(\frac{n-2}{3}\right)r^3$, &c.; $\therefore \frac{m}{na} = 1 + \frac{n-1}{2}r + \frac{n-1}{2} \times \frac{n-2}{3}r^2$, &c., and $\left(\frac{m}{an}\right)^{\frac{2}{n-1}} = \left(1 + \frac{n-1}{2}r + \frac{n-1}{2} \times \frac{n-2}{3}r^2\right)^{\frac{2}{n-1}}$; which by the binomial theo-

Case 10. Given m, n, and r; required p.

Since $A=\frac{m}{p}$; $\therefore p=\frac{m}{A}$, ... $\left\{\begin{array}{l} p(1+r)^n=m \text{ in Logarithms} \\ P=M-n\times L(1+r). \end{array}\right.$

Case 11. Given a, m, and r, to find p.

Since $\frac{mr+a}{a}=\frac{m}{p}$; $\therefore p=\frac{ma}{mr+a}$.

Case 12. Given a, m, and n, to find p.

Find $D=\left(\frac{m}{na}\right)^{\frac{2}{n-1}}$; $E=\frac{2}{\frac{1}{2}(n+1)}$; $F=\sqrt{\{2(D-1)+E\}\times E}$.

Now $F-E=r$, by 3d, but $(1+r)^n=\frac{m}{p}$; $\therefore p=\frac{m}{(1+r)^n}$.

Case 13. Given p, n, and r, to find a.

Since $(A-1)a=prA$; $\therefore a=\frac{prA}{A-1}$.

Case 14. Given p, m, and n, to find a.

Since $A=(r+1)^n=\frac{m}{p}$; $\therefore r+1=\left(\frac{m}{p}\right)^{\frac{1}{n}}$. Hence $A-1$, and $B-1=r$, are known; $\therefore a=\frac{m\times(B-1)}{A-1}$.

Case 15. Given m, n, and r, to find a.

Since $(A-1)a=mr$, therefore $a=\frac{mr}{A-1}$.

Case 16. Given p, m, and r, to find a.

Since $A=\frac{m}{p}$, $A-1$ is given, $\therefore a=\frac{mr}{A-1}$.

Case 17. Given m, p, and r, to find n.

Put L for the logarithm, and L′ for the arith. comp. of a log.

Since $(1+r)^n=\frac{m}{p}$, therefore $n=\frac{L.m+L'.p}{L.(r+1)}=\frac{M-p}{L(1+r)}$.

Case 18. Given a, p, and r, to find the value of n.

Since $(1+r)^n=\frac{a}{a-pr}$, therefore $n=\frac{L.a-L.(a-pr)}{L.(r+1)}$.

Case 19. Given a, p, and m, to find n.

Since $A=(r+1)^n=\frac{m}{p}$; then $L.(r+1)^n=L.m-L.p=L.A$.

Hence A, and $A-1$, are known; Also $L.(A-1)+L.a+L'.m=L.(B-1)$ by Case 14th hence $r=B-1$ and $r+1$, are known therefore $n=\frac{L.A.}{L.(r+1)}$.

Case 20. Given a, m, and r, to find n.

Since $A=\frac{mr+a}{a}=(1+r)^n$; $\therefore n=\frac{L.(mr+a)-L.a}{L.(1+r)}$.

rem will become $=1+r+\frac{n+1}{12}r^2$ nearly.

Let $D=\left(\frac{m}{an}\right)^{\frac{2}{n-1}}=1+r+\frac{n+1}{12}r^2$; then $r^2+\frac{12r}{n+1}=(D-1)\times\frac{12}{n+1}$; let $2E=\frac{12}{n+1}$; $r+E=(\sqrt{\{(2\times[D-1]+E)\times E\}}=F$; therefore $r=F-E$. In this solution, 1st find $D=\left(\frac{m}{na}\right)^{\frac{2}{n-1}}$; 2d, find $E=\frac{6}{n+1}$; 3d, find $F=\sqrt{\{2(D-1)+E\}\cdot E}$; 4th, find $r=F-E$.

Case 4. Given a, p, and n, to find r.

Since $(A-1)a=Apr$, $\therefore \frac{p}{na}=\frac{A-1}{Anr}=\frac{1-A^{-1}}{nr}=\frac{1-(1+r)^{-n}}{nr}$; now $(1+r)^{-n}=1-nr+n\times\frac{n+1}{2}r^2-n\times\frac{n+1}{2}\times\frac{n+2}{3}r^3+$ &c.; therefore $\frac{p}{na}=1-\frac{n+1}{2}r+\frac{n+1}{2}\times\frac{n+2}{3}r^2$, nearly; then $\left(\frac{p}{na}\right)^{-\frac{2}{n+1}}=\left(1-\frac{n+1}{2}r+\frac{n+1}{2}\times\frac{n+2}{3}r^2\right)^{-\frac{2}{n+1}}$; which, by binomial theorem, will become $=1+r-\frac{n-1}{12}r^2$, nearly. Now, $\left(\frac{p}{na}\right)^{-\frac{2}{n+1}}=\left(\frac{na}{p}\right)^{\frac{2}{n+1}}$; let $G=\left(\frac{na}{p}\right)^{\frac{2}{n+1}}=1+r-\frac{n-1}{12}r^2$; therefore $r^2-\frac{12}{n-1}r=-(G-1)\times\frac{12}{n-1}$. Let $2H=\frac{12}{n-1}$, then $r-H=(\sqrt{\{H-2(G-1)\}\times H}=)K$; $\therefore r=H-K$.

Case 5. Given a, n and r, to find m.

Since $(A-1)a=mr$, therefore $m=\frac{(A-1)a}{r}$.

Case 6. Given p, n, and r, to find m.

Since $A=\frac{m}{p}$, $\therefore\ m=pA$.

Case 7. Given a, p, and r; find m.

Since $\frac{m}{p}=\frac{a}{a-rp}$; $\therefore m=\frac{ap}{a-rp}$.

Case 8. Given a, p, and n, to find m.

Find $G=\left(\frac{na}{p}\right)^{\frac{2}{n+1}}$; $H=\frac{3}{\frac{1}{2}(n-1)}$; $K=\sqrt{\{H-(2G-1)\}\times H}$.
Now $H-K=r$, by the 4th, but $(r+1)^n=\frac{n}{p}$; $\therefore m=(r+1)^n\times p$.

Case 9. Given a, n, and r, to find p.

Since $(A-1)a=prA$; therefore $p=\frac{(A-1)\times a}{Ar}$

$2 \times 4 \times 5$, or $2.4.5$, is the continued product of 2, 4 and 5. Likewise, $7 \times a \times b$, or $7.a.b.$, or $7ab$, is the continued product of 7, a, and b.

9. $\div$ divided by. This character is the sign of division, and signifies that the former of the quantities between which it is placed is to be divided by the latter; thus, $a \div b$ means that the quantity a is to be divided by b. The division of one quantity by another is frequently represented by placing the dividend over the divisor, with a line between them, in which case the expression is called a fraction; thus, $\frac{\text{Dividend, } a}{\text{Divisor, } b}$, signifies a divided by b; then a is the numerator and b the denominator of the fraction.

10. A quantity in the denominator of a fraction is also expressed by placing it in the numerator, and prefixing the negative sign to its index; thus, $a^{-1}, a^{-2}, a^{-3}, a^{-n}$, signify $\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^n}$, respectively; these are called the negative powers of a.

11. Points are generally made use of to denote proportion; thus, $a : b :: c : d$, signifies that a bears the same proportion to b that c bears to d.

$=$ equal to. This sign means that the quantities between which it is placed are equal to each other; $ax - by = cd + ad$, signifies that the quantity $ax - by$ is equal to the quantity $cd + ad$.

12. The sign $\backsim$ between two quantities means or signifies their difference; thus, $a \backsim x$, is $a \backsim x$ or $x \backsim a$, according as a or x is the greater; $a \overset{+}{\backsim} x$ signifies the sum or difference of a and x.

13. A vinculum ——, is a line drawn over several quantities, and signifies that the terms under it are to be taken as one whole, and to be affected with the same operation. The modern method of expressing the same thing is by the parenthesis ( ) or bracket, [ ]. Thus $(a+b) \times x$, $\overline{a+b} \times x$, or $[a+b] \times x$, means that the quantity represented by $a+b$ is to be multiplied by the quantity represented by x. Let $a = 3$, $b = 4$, then $(a+b) \times x = 7x$, and if $x = 1$, then $7x = 7$.

14. The powers of algebraic quantities are denoted by placing a small figure, called the index or (indices) exponent of the power, to the right hand of the letter, as a^2, a^3, a^4, &c.; so when the index is a fraction the numerator shows the power to which the quantity is first to be raised, and the denominator expresses the root to be extracted; thus $a^{\frac{2}{3}}$ denotes the cube root of a square; $a^{\frac{3}{2}}$ is the square root of a^3; and $a^{\frac{1}{2}}$ denotes the square root of a, the same as $\sqrt{a}$; $a^{\frac{1}{3}}$, the cube root of a, the same as $\sqrt[3]{a}$; $a^{\frac{3}{5}}$ the fifth root of

the cube of a, or $\sqrt[5]{a^3}$; and $a^{\frac{5}{9}}$ is the ninth root of a fifth power of a, or $\sqrt[9]{a^5}$; and $(a+b)^{\frac{1}{n}}$ denotes the nth root of the sum of a and b.

15. A surd, or irrational quantity, is that of which the value cannot be accurately expressed in numbers; thus 2, 3, 5, and 7, the root of quantities, are denoted by the radical $\sqrt{}$, with the proper index in it. The signs $\sqrt{}$, or $\sqrt[2]{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, &c., are used to express the square, cube, biquadrate, &c., roots of the quantities before which they are placed; thus, $\sqrt{2}$, $\sqrt[2]{a}$, $\sqrt[3]{a^3}=a$, $\sqrt[4]{a^4}=a$.

16. The reciprocal of any quantity is that quantity inverted, or unity divided by it; so the reciprocal of a, or $\frac{a}{1}$, is $\frac{1}{a}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

17. A rational quantity is that which has no radical sign or index annexed to it, as a, $\frac{2}{5}a$, or $8a$.

18. The words *therefore*, *consequently*, or *that is*, are usually expressed by the symbol $\therefore$; thus the sentence therefore $a+b$ is equal to $c+d$, is expressed by $\therefore a+b=c+d$.

19. Positive or affirmative quantities are those which are to be added, as a or $+a$, or $3ax$, or $+3ax$. See ex. 1, p. 5. For when a quantity is found without a sign it is understood to be positive, or to have the sign $+$ prefixed, that is, always when it is a leading quantity; and a quantity without any coefficient is supposed to have one or unity before it; thus, $a=$ once a, ($1a$.)

20. Negative quantities are those which are to be subtracted, as Ex. 11, viz: $-2by^2, -6by^2, -by^2, -8by^2, -by^2$.

21. When two quantities are multiplied together, each, considered separately, is termed the coefficient of the other; but when one of them is a known quantity, it alone is termed the coefficient. Thus in the quantity ab, a is the coefficient of b, and b of a; but in $4x$, the numeral 4 is the coefficient of x, and x is never termed the coefficient of 4. Sometimes the coefficient is a compound quantity, as $(a+b)x$, or $ax+bx$; here the coefficient is $a+b$.

22. Like signs are such as are all positive, ($+$), plus, or all negative, ($-$), minus.

23. Unlike signs are when some are positive, plus, ($+$), and others minus, ($-$).

24. Like quantities are such as do not differ except in their coefficients; as $2ax^2$, and $-3ax^2$. Unlike quantities are those which differ in letters or power, as 2, a and b, or $6a^{\frac{1}{2}}x^{\frac{1}{2}}$, and $4a^2y^2$.

25. A monomial, or simple quantity, is a quantity consisting of one term only, as a; $3bx$; $-2xy$; $+3xy$.

26. A binomial quantity consists of two terms, as $a+x$, or $a-b$, the latter of which, being connected by the sign $-$, is sometimes called a residual.

27. A trinomial consists of three, as $a + bx - 3y$.

28. A quadrinomial is a quantity consisting of four terms, as $ax^2 + 3xy - 3y + bxy^2$, or $a + b - x + y$.

29. A multinomial, or polynomial, is a quantity consisting of many terms, or of an indefinite number of terms, as $a + b - c + d - x + y$, &c.

30. When two equal quantities are compounded with the sign $=$ between them, the comparison is called an equation; thus $3 + 4 = 7$, or $2x + 4 = 8$, is an equation; the quantity on the left hand side of the sign $=$ is called the first member of an equation, and the quantity on the right hand of it the second. In the equation $2x + b = a$; $2x + b$ is the first member, and a is the second. When the quantities which compose a number are separated by the signs $+$ or $-$, each quantity, so separated, is called a term; thus, the first member of the equation $2x + b = a$, contains two terms, viz. $2x$ and b; also a and b are the terms of ab or $\frac{b}{a}$, and a, b, x, y are the terms of the proportion $a : b :: x : y$.

31. A quantity is said to be a multiple of another when it contains it a certain number of times exactly; thus $16a$ is a multiple of $4a$, as it contains it exactly four times.

32. An equation of the third, fourth, &c., degree, is one in which the highest power of the unknown quantity is the third, fourth, &c., powers; and in general, an equation, in which the unknown quantity is called an equation of the mth degree, and each of the two members of an equation is called a side.

33. If equal quantities be either increased or diminished by the same quantity, the results will be equal, or, in other words, if each side of an equation be either increased or diminished by the same quantity, the result will be an equation.

34. If each side of an equation be either multiplied or divided by the same quantity, the result will be an equation.

35. If each side of an equation be either involved to the same power, or evolved to the same root, the result will be an equation.

36. Hence, generally, whatever operations be performed on one side of an equation, if the same operations be performed on the other side, the result will be an equation.

37. Notation in Algebra is the method of representing any proposed quantity by means of certain symbols.

1. What is the numeral value of $a + b - c$, supposing $a = 7$, $b = 5$, and $c = 8$?

Here $a + b - c = 7 + 5 - 8 = 12 - 8 = 4$, Ans.

2. What is the value of $\frac{ax + by}{b + x}$, where $a = 4$; $b = 6$; $x = 3$, and $y = 7$?

Here $ax + by$, or $4 \times 3 + 6 \times 7 = 12 + 42 = 54$.

And $b + x$, or $6 + 3 = 9$. Hence $\frac{ax+by}{b + x} = \frac{54}{9} = 6$, Ans.

3. What is the numeral value of $\frac{ab}{c} + \frac{ax - d^3}{b} - \sqrt{ab^2}$, supposing $a = 9$; $b = 4$; $c = 3$; $d = 2$; and $x = 8$?

$$\frac{ab}{c} + \frac{ax - d^3}{b} - \sqrt{ab^2} = \frac{(9 \times 4)}{3} + \frac{(9 \times 8) - 2^3}{4} - \sqrt{(9 \times 4 \times 4)} = 16.$$

Here, ab, or $9 \times 4 = 36$, which, divided by c or 3, gives 12, the value of the first term. Then ax, or $9 \times 8 = 72$, from which subtracting d^3 or 8, there remains 64; which, divided by b, or 4, gives 16, the value of the second term. Therefore the sum of the first and second terms is 28. Then ab^2, or $9 \times 4 \times 4 = 144$, the square root of which is 12, the value of the third term; and this subtracted from the sum of the former terms, because connected by the sign $-$, gives 16, the value of the whole expression.

ADDITION.

38. Addition in Algebra is the method of finding the sum of several algebraic quantities, and connecting them together by their proper signs. This rule is generally divided into three cases.

Case I. *To add like quantities with like signs.*

Rule. Add the coefficients of the several quantities together, and to their sum prefix the common signs and annex the common letter or letters.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.	Ex. 5.	Ex. 6.
a	$7a$	$-4ax$	$5ax$	$6x + 3y$	$4x + 7y$
$3a$	$2a$	$-2ax$	$9ax$	$2x + 5y$	$x + y$
$4a$	$9a$	$-6ax$	$14ax$	$8x + 8y$	$5x + 8y$

Ex. 7.	Ex. 8.	Ex. 9.	Ex. 10.	Ex. 11.
$3a$	$-3ax$	$2ay$	$2b + 3y$	$-2by^2$
$5a$	$-6ax$	$5ay$	$5b + 7y$	$-6by^2$
a	$-ax$	$4ay$	$b + 2y$	$-by^2$
$7a$	$-2ax$	$7ay$	$8b + y$	$-8by^2$
$12a$	$-7ax$	$16ay$	$4b + 4y$	$-by^2$
$28a$	$-19ax$	$34ay$	$20b + 17y$	$18by^2$

Ex. 12.
$$\begin{array}{r} 3ax^2 \\ 2ax^2 \\ 12ax^2 \\ 9ax^2 \\ 12ax^2 \\ \hline 38ax^2 \end{array}$$

Ex. 13.
$$\begin{array}{r} a - 2x^2 \\ a - 6x^2 \\ 4a - x^2 \\ 3a - 5x^2 \\ 7a - x^2 \\ \hline 16a - 15x^2 \end{array}$$

Ex. 14.
$$\begin{array}{r} 7x - 4y \\ x - 8y \\ 3x - y \\ x - 3y \\ 4x - y \\ \hline 16x - 17y \end{array}$$

Ex. 15.
$$\begin{array}{r} 2a + x^2 \\ 3a + x^2 \\ a + 2x^2 \\ 9a + 3x^2 \\ 4a + x^2 \\ \hline 19a + 8x^2 \end{array}$$

Ex. 16.
$$\begin{array}{r} 3a \\ 9a \\ 5a \\ 12a \\ a \\ 2a \\ \hline 32a \end{array}$$

Ex. 17.
$$\begin{array}{r} -3bx \\ -5bx \\ -4bx \\ -2bx \\ -7bx \\ -bx \\ \hline -22bx \end{array}$$

Ex. 18.
$$\begin{array}{r} bxy \\ 2bxy \\ 5bxy \\ bxy \\ 3bxy \\ 6bxy \\ \hline 18bxy \end{array}$$

Ex. 19.
$$\begin{array}{r} 3z \\ 2z \\ 4z \\ z \\ 5z \\ \hline 15z \end{array}$$

Ex. 20.
$$\begin{array}{r} 3x + 5xy \\ x + xy \\ 2x + 4xy \\ 5x + 2xy \\ 4x + 3xy \\ \hline 15x + 15xy \end{array}$$

Ex. 21.
$$\begin{array}{r} 2ax - 4y \\ 4ax - y \\ ax - 3y \\ 5ax - 5y \\ 7ax - 2y \\ \hline 19ax - 15y \end{array}$$

Ex. 2
$$\begin{array}{l} 5 \\ 14 \\ 22 \\ 17 \\ 1\frac{1}{2} \\ \frac{1}{2} \\ \hline 59 \end{array}$$

Ex. 23.
$$\begin{array}{r} 2r + 3a - 4b \\ 3r + 2a - 5b \\ 4r + 8a - 7b \\ 9r + 4a - 6b \\ 5r + 7a - 9b \\ \hline 23r + 24a - 31b \end{array}$$

Ex. 24.
$$\begin{array}{r} 7r^2 + 3ry - 5bc \\ 9r^2 + 2ry - 7bc \\ 11r^2 + 5ry - 4bc \\ r^2 + 4ry - bc \\ r^2 + 9ry - 2bc \\ \hline 29r^2 + 23ry - 19bc \end{array}$$

Ex. 25.
$$\begin{array}{r} 4a^3 - 3a^2 + 1 \\ 2a^3 - a^2 + 17 \\ 5a^3 - 2a^2 + 4 \\ 3a^3 - 7a^2 + 3 \\ a^3 - a^2 + 10 \\ \hline 15a^3 - 14a^2 + 35 \end{array}$$

Ex. 26.
$$\begin{array}{r} -12y \\ -7y \\ -2y \\ -4y \\ -y \\ -3y \\ \hline -29y \end{array}$$

Ex. 27.
$$\begin{array}{r} 4a - 4b \\ 5a - 5b \\ 6a - b \\ 3a - 2b \\ 2a - 7b \\ 8a - b \\ \hline 28a - 20b \end{array}$$

Ex. 28.
$$\begin{array}{r} 30 - 13x^{\frac{1}{2}} - 3xy \\ 23 - 10x^{\frac{1}{2}} - 4xy \\ 14 - 14x^{\frac{1}{2}} - 7xy \\ 10 - 16x^{\frac{1}{2}} - 5xy \\ 16 - 20x^{\frac{1}{2}} - xy \\ \hline 93 - 73\sqrt{x} - 20xy \end{array}$$

Ex. 29.
$$\begin{array}{r} 5xy - 3x + 4a \\ 8xy - 4x + 3a \\ 3xy - 5x + 5a \\ xy - 2x + a \\ 4xy - x + 7a \\ \hline 21xy - 15x + 20a \end{array}$$

CASE II. *To add like quantities with unlike or different signs.*

RULE. Add all the positive or plus quantities into one sum, and all the negative or minus into another sum; subtract the less of these sums from the greater; to their difference prefix the sign of the greater sum, whether + or —, and annex the common letter or letters.

Ex. 1.
$$\begin{array}{l} +5a \\ -3a \\ \hline +2a \text{ sum.} \end{array}$$

Ex. 2.
$$\begin{array}{l} -5x \\ +3x \\ \hline -2x \text{ sum.} \end{array}$$

Ex. 3.
$$\begin{array}{l} +3a \\ -2a \\ \hline +1a \text{ sum.} \end{array}$$

Ex. 4.
$$\begin{array}{l} +9y \\ -6y \\ \hline +3y \text{ sum.} \end{array}$$

Ex. 5.
$$\begin{array}{l} -6y\sqrt{x} \\ +3y\sqrt{x} \\ \hline -3y\sqrt{x} \text{ sum} \end{array}$$

Ex. 6.

$$\begin{array}{r} -3a \\ +7a \\ +8a \\ -a \\ \hline +11a \end{array}$$

Ex. 7.

$$\begin{array}{rr} 2a & -3r^2 \\ -7a & +5r^2 \\ -3a & +r^2 \\ +a & -3r^2 \\ \hline -7a & * \end{array}$$

Ex. 8.

$$\begin{array}{rr} 6r & +5ay \\ -3r & +2ay \\ r & -6ay \\ 2r & +ay \\ \hline 6r & +2ay \end{array}$$

Ex. 9.

$$\begin{array}{r} -2a^2 \\ -3a^2 \\ -8a^2 \\ +10a^2 \\ +13a^2 \\ \hline +10a^2 \end{array}$$

Ex. 10.

$$\begin{array}{rr} 3ay & -7 \\ -ay & +8 \\ +2ay & -9 \\ -3ay & -11 \\ +12ay & +13 \\ \hline +13ay & -6 \end{array}$$

NOTE. In the 6th example, the sum of the positive or plus quantities exceeds the sum of the negative by $11a$; consequently the sign is $+$, according to the rule. In the 7th example, the sum of the positive or $+$ (plus) quantities is less by $7a$ than the sum of the negative or $-$ (minus) quantities; consequently, the sign is $-$, according to the rule. In the 9th example, the sum of the positive terms is $23a^2$, and the sum of the negative ones is $-13a^2$; their difference, therefore, is $+10a^2$, which is the sum required. The other examples are wrought in a similar manner. If the positive and negative quantities be equal, the sum is nothing, and they are said to destroy each other. See example 7, right hand column.

Ex. 11.

$$\begin{array}{r} +9x \\ -6x \\ +7x \\ -x \\ \hline +9x \end{array}$$

Ex. 12.

$$\begin{array}{r} -4a^2 \\ +a^2 \\ -9a^2 \\ +7a^2 \\ \hline -5a^2 \end{array}$$

Ex. 13.

$$\begin{array}{r} -6a\sqrt{x} \\ +4a\sqrt{x} \\ +a\sqrt{x} \\ -5a\sqrt{x} \\ \hline -6a\sqrt{x} \end{array}$$

Ex. 14.

$$\begin{array}{r} +6ax \\ -2ax \\ +7ax \\ -ax \\ \hline +10ax \end{array}$$

Ex. 15.

$$\begin{array}{rr} +6a & +4x \\ +4a & +8x \\ -5a & -2x \\ -7a & -3x \\ \hline -2a & +7x \end{array}$$

Ex. 16.

$$\begin{array}{rr} -3ab & +7x \\ +3ab & -10x \\ +3ab & -6x \\ -ab & -2x \\ +2ab & +7x \\ \hline +4ab & -4x \end{array}$$

Ex. 17.

$$\begin{array}{r} -2a\sqrt{x} \\ +a\sqrt{x} \\ -3a\sqrt{x} \\ +7a\sqrt{x} \\ -4a\sqrt{x} \\ \hline -a\sqrt{x} \end{array}$$

Ex. 18.

$$\begin{array}{rr} -6a^2 & +2b \\ +2a^2 & -3b \\ -5a^2 & -8b \\ +4a^2 & -2b \\ -3a^2 & +9b \\ \hline -8a^2 & -2b \end{array}$$

Ex. 19.

$$\begin{array}{rr} 6ax^2 & +5x^{\frac{1}{2}} \\ -2ax^2 & -6x^{\frac{1}{2}} \\ +3ax^2 & -10x^{\frac{1}{2}} \\ -7ax^2 & +3x^{\frac{1}{2}} \\ +ax^2 & +11x^{\frac{1}{2}} \\ \hline +ax^2 & +3x^{\frac{1}{2}}=3\sqrt{x} \end{array}$$

Ex. 20.

$$\begin{array}{rrr} -6x & +4x^2 & -8 \\ +8x & +x^2 & +1 \\ +5x & -3x^2 & +9 \\ +7x & -5x^2 & -7 \\ \hline 14x & -3x^2 & -5 \end{array}$$

Ex. 21.

$$\begin{array}{rr} 7\sqrt{y} & -4(a+b) \\ 6\sqrt{y} & +2(a+b) \\ 2\sqrt{y} & +(a+b) \\ \sqrt{y} & -3(a+b) \\ \hline 16\sqrt{y} & -4(a+b) \end{array}$$

Ex. 22.

$$\begin{array}{rr} a(a+b) & -3\sqrt{(a-x)} \\ -4a(a+b) & +7\sqrt{(a-x)} \\ -2a(a+b) & -8\sqrt{(a-x)} \\ 5a(a+b) & +14\sqrt{(a-x)} \\ \hline * & +10\sqrt{(a-x)} \end{array}$$

CASE III. *To add quantities when some are like and others unlike; or when all the quantities are unlike.*

RULE. Add the like quantities together, according to cases 1

and 2; connect the unlike quantities *in any order*, with proper signs and coefficients prefixed.

Ex. 1.

$$\begin{array}{rr} +4ab & +\ 4 \\ -4ab & +\ 12 \\ +7ab & -\ 14 \\ +ab & +\ 3 \\ -5ab & -\ 10 \\ \hline +3ab & -\ 5 \end{array}$$

Ex. 2.

$$\begin{array}{r} -3ax^{\frac{1}{2}} \\ +ax^{\frac{1}{2}} \\ +5ax^{\frac{1}{2}} \\ -6ax^{\frac{1}{2}} \\ \hline -3\sqrt{a^2x}=3ax^{\frac{1}{2}} \end{array}$$

Ex. 3.

$$\begin{array}{r} +10\sqrt{ax} \\ -3\sqrt{ax} \\ +4\sqrt{ax} \\ -12\sqrt{ax} \\ \hline -\sqrt{ax} \end{array}$$

Ex. 4

$$\begin{array}{rl} +3y & + \\ -y & - \\ +4y & + \\ -2y & + \\ \hline 4y & + \end{array}$$

Ex. 5.

$$\begin{array}{r} -5a \\ +4a \\ +6a \\ -3a \\ +a \\ \hline +3a \end{array}$$

Ex. 6.

$$\begin{array}{r} +3ax^2 \\ +4ax^2 \\ -8ax^2 \\ -6ax^2 \\ +5ax^2 \\ \hline -2ax^2 \end{array}$$

Ex. 7.

$$\begin{array}{rr} +8x^3 & +3y \\ -5x^3 & +4y \\ -16x^3 & +5y \\ +3x^3 & -7y \\ +2x^3 & -2y \\ \hline -8x^3 & +3y \end{array}$$

Ex. 8.

$$\begin{array}{r} -3a^2 \\ -5a^2 \\ -10a^2 \\ +10a^2 \\ +14a^2 \\ \hline 6a^2 \end{array}$$

Ex. 9

$$\begin{array}{r} +3b \\ +9b \\ -10b \\ -19b \\ -2b \\ \hline -19b \end{array}$$

Ex. 10.

$$\begin{array}{l} 2a^3 - 3ab + 2b^3 - 3a^2 \\ -2a^3 + a^3 + 3b^3 - 5c^3 \\ 100 + 5ab - 2b^3 + 4c^3 \\ 16a^2 + 20ab - bc - 80 \\ \hline 13a^2 + 22ab + 3b^3 + \\ a^3 - c^3 + 20 - bc. \end{array}$$

Ex. 11.

$$\begin{array}{l} +3a^2x^2-\sqrt{ax}-xy^{\frac{1}{2}} \\ -xy+5\sqrt{ab}-\sqrt{ab} \\ -\sqrt{xy}-12b^3+a^2x^2 \\ +4(ax)^{\frac{1}{2}}-2b^3-cd \\ \hline 4a^2x^2-xy-\sqrt{xy}+3\sqrt{a} \\ 4\sqrt{ab}-14b^3-xy^{\frac{1}{2}}- \end{array}$$

Ex. 12.

$$\begin{array}{ll} \text{Add} & \frac{5a}{b}-\frac{3c^2}{a}+\frac{7\sqrt{bc}}{x}-9\left(\frac{ab+x}{d}\right) \\ \text{And} & \frac{8a}{b}+\frac{7c^2}{a}-\frac{12\sqrt{bc}}{x}+6\left(\frac{ab+x}{d}\right) \\ \hline \text{Sum} & \frac{13a}{b}+\frac{4c^2}{a}-\frac{5\sqrt{bc}}{x}-3\left(\frac{ab+x}{d}\right) \end{array}$$

Ex. 13.

$$\begin{array}{l} 3a^2+4bc-e^2+ \\ -5a^2+6bc+2e^2- \\ -4a^2-9bc-10e^2+ \\ \hline -6a^2+bc-9e^2+ \end{array}$$

Ex. 14.

$$\begin{array}{l} +3a^2y \\ -2xy^2 \\ -3y^2x \\ -8x^2y \\ +2xy^2 \\ \hline 3a^2y-3y^2x-8x^2y \end{array}$$

Ex. 15.

$$\begin{array}{ll} 2\sqrt{x} & -17y \\ 3\sqrt{xy} & +10x \\ 2x^2y & +25y \\ 12xy & +18x^{\frac{1}{2}} \\ -8y & -\sqrt{xy} \\ \hline \multicolumn{2}{l}{20\sqrt{x}+12xy+} \\ \multicolumn{2}{l}{2x^2y+2\sqrt{xy}+10x} \end{array}$$

Ex. 16.

$$\begin{array}{ll} 2a^2 & -3a\sqrt{x} \\ x^2 & -2a^{\frac{1}{2}}x^{\frac{1}{2}} \\ 3a^2 & -13xy \\ xy & +32x^2 \\ 20 & -65x^2 \\ \hline \multicolumn{2}{l}{37a^2-3a\sqrt{x}-12xy-} \\ \multicolumn{2}{l}{64x^2-2a^{\frac{1}{2}}x^{\frac{1}{2}}+} \end{array}$$

Ex. 17.	Ex. 18.	Ex. 19.	Ex. 20.
$5xy$	$2ax\ -30$	$2xy-2x^2$	$6xy-12x^2$
$4ax$	$3x^2\ -\ 2ax$	$3x^2+\ xy$	$3xy-\ 4x^2$
$-\ xy$	$5x^2\ -\ 3x^{\frac{1}{2}}$	$x^2+\ xy$	$2xy+\ 4x^2$
$-4ax$	$3\sqrt{x}+10$	$4x^2-3xy$	$3xy+\ 4x^2$
$\overline{4xy}$	$\overline{8x^2\ -20}$	$\overline{6x^2+\ xy}$	$\overline{14xy-\ 8x^2}$

Ex. 21.	Ex. 22.
$8a^2x^2-3ax$	$+\ ax^{\frac{1}{2}}+10b^2-\ 3a^2x$
$7ax\ -5xy$	$-\ ax^2\ \ -b^2+\ 2a^2x^2$
$9xy\ -5ax$	$+3ax^2+50\ +\ 2a^2x$
$2a^2x^2+\ xy$	$-\ ax^{\frac{1}{2}}+a^2x^2+\ 120$
$\overline{10a^2x^2+5xy-ax}$	$\overline{+2ax^2+9b^2+\ 3a^2x^2-a^2x+170}$

Ex. 23.	Ex. 24.	Ex. 25.	Ex. 26.
$+3x$	$4x$	$+6xy-12x^2$	$4ax-130x+3\sqrt{x}$
$-4x$	$5x$	$-4x^2+3xy$	$5x^2+\ 3ax+\ 9x^2$
$-2x$	$3x$	$+4x^2-2xy$	$7xy-\ 4x^{\frac{1}{2}}+90$
$+6y$	$-\ 2y$	$-3xy+4x^2$	$\sqrt{x}+40\ -\ 6x^2$
$\overline{6y-3x}$	$\overline{12x-2y}$	$\overline{4xy-8x^2}$	$\overline{7ax+8x^2+7xy}$

NOTE. When quantities with literal coefficients are to be added together, it may be done by placing the coefficients, with their proper signs, under a vinculum, or between brackets, and then subjoining the common quantity to the sum or difference thus arising, as in the margin.

$$\begin{array}{l} ax+b \\ cx+d \\ \hline (a+c)x+b+d \end{array} \qquad \begin{array}{l} ax^2+bx \\ cx^2-dx \\ \hline (a+c)x^2+(b-d)x \end{array}$$

Ex. 1.	Ex. 2.
$ax+by^2$	x^2+adz
$cdx+ady^2$	$\frac{1}{2}x^2-nz$
$bx-cy^2$	bx^2+cez
	dx^2-mz
$\overline{(a+cd+b)x+(b+ad-c)y^2}$	$\overline{(d+b+1\frac{1}{2})x^2+(ad-n+ce-m)z}$

Ex. 3.	Ex. 4.
$ax+dy^2$	$\sqrt{x}+by$
$by-dx$	$ax-z$
$-by^2+my$	$amy+c\sqrt{x}$
$\overline{(a-d)x+(d-b)y^2+(b+m)y}$	$dz+y$
	$\overline{(am+b+1)y+(c+1)\sqrt{x}+(d-1)z+ax}$

EXAMPLES FOR PRACTICE.

1. Required the sum of $\frac{a+b}{2}$ and $\frac{a-b}{2}$. Answer, a.

2. Add $5x-3a+b+7$, and $\frac{-4a-3x+2b-9}{2}$, together.

3. Add $3a+2b-5$, $a+b-c$, and $6a-2c+3$, together.
4. Add $2a+3b-4c-9$, and $5a-3b+2c-10$, together.
5. Add x^3+ax^2+bx+2, and x^3+cx^2+dx-1, together.

SUBTRACTION.

39. SUBTRACTION in Algebra is the method of finding ference between two algebraical quantities, and connectin quantities together with their proper signs.

RULE 1. Set those quantities from which the subtraction made in one line, and the quantities to be subtracted ir below them, observing to place like quantities under eac when they occur.

RULE 2. Subtraction in Algebra is performed by simply ing the signs (+ into —, and — into +) of the lower quantities to be subtracted, and then adding or connectin as in addition, and the result will be the difference require

Ex. 1.			Ex. 2.
From $8a$	or changing the	To $8a$	From $+6a+5$
Take $3a$	sign of $3a$	Add $-3a$	Take $-3a+2$
The answer or remainder,		$+5a$	$+9a+3$

Here the quantity to be subtracted is $-3a+2y-3$; w fore change its signs according to the 2d rule, and it t $+3a-2y+3$, and this added to the other quantity gi remainder $+9a+3y-4$.

Ex. 3.	Ex. 4.	Ex. 5.	Ex.
$5a^2-2b$	x^2-2y+3	$5xy+8x-2$	$+8y^2-$
$2a^2+5b$	$4x^2+9y-5$	$3xy-8x-7$	$-y^2+$
$3a^2-7b$	$-3x^2-11y+8$	$2xy+16x+5$	$+9y^2-$

Ex. 7.	Ex. 8.	Ex. 9.	Ex. 10.
$10-8x-3xy$	$5xy-18$	$-5x^2y-8b$	$4\sqrt{ax}-2x^2$
$-x+3-xy$	$-xy+12$	$+3x^2y-7b$	$3\sqrt{ax}-5xy$
$7-7x-2xy$	$6xy-30$	$-8x^2y-b$	$\sqrt{ax}-2x^2y$

Ex. 11.	Ex. 12.
$5x^2+\sqrt{x}-4y$	$20ax-5\sqrt{x}+3a$
$6x^2-8x-x^2$	$4ax+5\sqrt{x}-a$
$-x^2+8x+2\sqrt{x}-4y$	$16ax-10\sqrt{x}+4a$

Ex. 13.

$$\begin{array}{r}
5ab+2b^2-c+bc-b \\
-2ab+b^2\quad+bc \\
\hline
7ab+b^2-c-b \\
\hline
\end{array}$$

Ex. 14.

$$\begin{array}{r}
ax^3-bx^2+cx-d \\
+bx^2+ex-2d \\
\hline
ax^3-2bx^2+(c-e)\,x+d \\
\hline
\end{array}$$

Ex. 15.

$$\begin{array}{r}
-6a+13x-4b-12c \\
-9a+4x+4b-5c \\
\hline
3a+9x-8b-7c \\
\hline
\end{array}$$

Ex. 16.

$$\begin{array}{r}
6x^2y-3\sqrt{(xy)} \\
3x^2y+3\sqrt{(xy)} \\
\hline
3x^2y-6\sqrt{(xy)} \\
\hline
\end{array}$$

Ex. 17.

$$\begin{array}{r}
-4xy^3+12a\sqrt[n]{(ax+6)} \\
7xy^3+4a\sqrt[n]{(ax+6)} \\
\hline
-11xy^3+8a\sqrt[n]{(ax+6)} \\
\hline
\end{array}$$

Ex. 18. From the sum of $4ax-150+4x^{\frac{1}{2}}$, $5x^2+3ax+10x^{\frac{1}{2}}$, and $90-2ax-12\sqrt{(x)}$, take the sum of $2ax-80+7x^2$, $7x^{\frac{1}{2}}-8ax-70$, and $30-4\sqrt{(x)}-2x^2+4a^2x^2$.

$$\begin{array}{r}
4ax-150+4x^{\frac{1}{2}} \\
3ax+5x^2+10x^{\frac{1}{2}} \\
-2ax+90-12x^{\frac{1}{2}} \\
\hline
\end{array}
\qquad \text{and} \qquad
\begin{array}{r}
2ax-80+7x^2 \\
-8ax-70+7x^{\frac{1}{2}} \\
4a^2x^2+30-4x^{\frac{1}{2}}-2x^2 \\
\hline
\end{array}$$

$$\begin{array}{rr}
\text{Then,} & 5ax-60+5x^2+2x^{\frac{1}{2}} \\
 & -6ax-120+5x^2+3x^{\frac{1}{2}}+4a^2x^2 \\
\hline
\text{Ans.} & 11ax+60-\sqrt{x}-4a^2x^2. \\
\hline
\end{array}
\qquad
\begin{array}{r}
-6ax-120+3x^{\frac{1}{2}}+5x^2+4a^2x^2 \\
\hline
\end{array}$$

EXAMPLES WITH LITERAL COEFFICIENTS.

NOTE. When the quantities that are to be subtracted have literal coefficients, the operation may be performed by placing the coefficients, with their proper signs, between brackets, as in addition, and then subjoining the common quantity, the same as in the margin.

$$\begin{array}{r}
ax-b \\
cx-d \\
\hline
(a-c)x+d-b
\end{array}
\qquad
\begin{array}{r}
ax^2+bx \\
cx^2-cx \\
\hline
(a-c)x^2+(b+c)x
\end{array}$$

Ex. 19. From $pxy+qxz-rz^2+s$

Take $mxy-pqxz-nz^2+a$

Remainder, $(p-m)xy+(1+p)\,qxz+(n-r)z^2+s-a.$

Ex. 20. From $a(x-y)^{\frac{1}{2}}+bxy+c(a+x)^2$

Take $(x-y)^{\frac{1}{2}}-bxy+(a+c)\,(a+x)^2.$

Remainder, $(a-1)\,(x-y)^{\frac{1}{2}}+2bxy-a(a+x)^2.$

EXAMPLES FOR PRACTICE.

1. Required the difference of $\frac{a+b}{2}$, and $\frac{a-b}{2}$. Answer, b.
2. From $3x-2a-b+7$, take $8-3b+a+4x$.
3. From $3a+b+c-2d$, take $b-8c+2d-8$.
4. From $5ab+2b^2-c+bc-b$, take $b^2-2ab+bc$.
5. From $ax^3-bx^2+cx-5d$, take $bx^2+ex-12d$.

MULTIPLICATION.

40. Multiplication in Algebra is the method of finding the product of *two* or *more* indeterminate algebraic quantities, and is generally divided into three *cases*.

CASE I. *When both factors are simple quantities.*

RULE. Multiply the coefficients of the two quantities togethe and annex to the result (or product) all the letters in both factor which will give the whole product required. If the signs of th factors be like, that is, both $+$ or both $-$, the sign of the product $+$; but if they are unlike, or one of them $-$ and the other $+$ the sign of the product is $-$: and this is commonly expressed b saying, like signs give plus, and unlike signs minus.

	Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.	Ex. 5.
Multiply	$4a$	$4abc$	$9xy^2$	$-3abc$	$-6a^2bc$
by	$3b$	$3ac$	$-2x$	$5a^2b$	$-2b^2x^2$
	$12ab$	$12a^2bc^2$	$-18x^2y^2$	$-15a^3b^2c$	$+12a^2b^3cx^2$

Ex. 6.	Ex. .7	Ex. 8.	Ex. 9.	Ex. 10.	Ex. 11.
$12a$	$-2a$	$+5a$	$-9x^2$	$-6a^2x$	$-a^2xy$
$3b$	$+4b$	$-6x$	$-5bx$	$+5x$	$+2xy^2$
$36ab$	$-8ab$	$-30ax$	$+45bx^3$	$-30a^2x^2$	$-2a^2x^2$

Ex. 12.	Ex. 13.	Ex. 14.	Ex. 15.	Ex. 16.	Ex. 17.
$6\sqrt{ax}$	$12x^{\frac{1}{4}}y$	$-4cdx$	$-4x^3y^2$	$6axy^2$	$-7ax^2y$
$4b$	$-4a$	$-2c$	$-4x^2y^3$	$3a^2bx^2$	$-2ac^2x$
$24b\sqrt{ax}$	$-48ax^{\frac{1}{4}}y$	$+8c^2dx$	$+16x^5y^5$	$18a^3by^2x^3$	$+14a^2c^2x^3$

Ex. 18.	Ex. 19.	Ex. 20.	Ex. 21.	Ex. 22.	Ex. 23.
$7ab$	$-7axy$	$-2xy^2$	$+12a^2x$	$3a^2b$	$-6xyz$
$-5ac$	$+6ay$	$-\ xy$	$-2x^2y$	$2b^3a^2$	$+\ ay^2z$
$-35a^2bc$	$-42a^2xy^2$	$+2x^2y^3$	$-24a^2x^3y$	$6a^4b^4$	$-6axy^3z$

CASE II. *When one factor is a compound and the other simple.* RULE. Multiply each or every term of the compound factor separately by the simple factor, and to each product prefix its proper sign, and the result will be the whole product.

Ex. 24.	Ex. 25.	Ex. 26.	Ex. 27.
$3a-2b$	$6xy-8$	$12x-ab$	a^2-2x+1
$4a$	$3x$	$5a$	$4x$
$12a^2-8ab$	$18x^2y-24x$	$60ax-5a^2b$	$4a^2x-8x^2+4$

Ex. 28.	Ex. 29.	Ex. 30.
$35x-7a$	$12xy-ax+6$	$31xy^2-4\sqrt{x}+a$
$-2x$	$3xy$	$-2\sqrt{b}$
$-70x^2+14ax$	$36x^2y^2-3ax^2y+18xy$	$-62xy^2\sqrt{b}+8\sqrt{bx}-2a\sqrt{}$

Ex. 31.

Multiply $12x^3y+2xy^2+xy$
by $3ax$

$36ax^4y+6ax^2y^2+3ax^2y$

Ex. 32.

factor $4abx+3cy-abc$ compound.
simple $3xy^2$ or single factor.

$12abx^2y^2+9xcy^3-3abcxy^2$

Ex. 33.

$$\begin{array}{l} 3x^2-xy+2y^2 \\ 5x^2 \\ \hline 15x^4-5x^3y+10x^2y^2 \end{array}$$

Ex. 34.

$$\begin{array}{l} 3y^2+y-2 \\ xy \\ \hline 3xy^3+xy^2-2xy \end{array}$$

Ex. 35.

$$\begin{array}{l} 13x^2-a^2b \\ -2a \\ \hline -26ax^2+2a^3b \end{array}$$

36. Mult. $5ab+3a-2$
by $5xy$

$25abxy+15axy-10xy$

37. $12a^3-2a^2+4a-1$
$3x$

$36a^3x-6a^2x+12ax-3x$

Ex. 38.

$$\begin{array}{l} a+xb \\ 4x^2 \\ \hline 4ax^2+4bx^3 \end{array}$$

Ex. 39.

$$\begin{array}{l} 3x^3-2x^2+4 \\ -14ax \\ \hline -42ax^4+28ax^3-56ax \end{array}$$

Ex. 40.

$$\begin{array}{l} 25xy+3a^2 \\ 13x^2 \\ \hline 325x^3y+39a^2x^2 \end{array}$$

Ex. 41.

Multiply $9a^2x+3a-x+1$
by $-x^2$

$-9a^2x^3-3ax^2+x^3-x^2$

Ex. 42.

Mult. $4x^2y+3x-2y$
by $-3yx$

$-12x^3y^2-9x^2y+6xy^2$

CASE III. *When both factors are compound quantities.*

RULE. Multiply every part of the multiplicand by each part of the multiplier, placing the products one after the other, with their proper signs; then add the several products together, as in common multiplication.

Ex. 1. Multiply $a+b$
by $a+b$

$$\begin{array}{ll} \text{1st, by } a\cdot\cdot & a^2+ab \\ \text{2d, by } b\cdot\cdot & \quad\;\; ab+b^2 \\ \hline \text{Product} & a^2+2ab+b^2 \end{array}$$

Ex. 2.

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad\;\; -ab-b^2 \\ \hline a^2 \quad * \quad -b^2 \end{array}$$

Ex. 3.

$$\begin{array}{l} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad\;\; -a^2b-ab^2-b^3 \\ \hline a^3 \quad * \quad * \quad -b^3 \end{array}$$

When we have two or more quantities to multiply together, it is indifferent which two we begin with; for the products will always be the same, as will appear from the following example.

Let it be proposed to find the value or product of the four following factors, viz:

(I.)	(II.)	(III.)	(IV.)
$(a+b)$	(a^2+ab+b^2)	$(a-b)$	and (a^2-ab+b^2)

1st. Multiply the factors I. and II.

$$\begin{array}{l} a^2+ab+b^2 \\ a+b \\ \hline a^3+a^2b+ab^2 \\ \quad\;\; +a^2b+ab^2+b^3 \\ \hline a^3+2a^2b+2ab^2+b^3 \end{array}$$

Next the factors III. and IV.

$$\begin{array}{l} a^2-ab+b^2 \\ a-b \\ \hline a^3-a^2b+ab^2 \\ \quad\;\; -a^2b+ab^2-b^3 \\ \hline a^3-2a^2b+2ab^2-b^3 \end{array}$$

It remains now to multiply the first product I. II. by the second product III. IV.

$$\begin{array}{l}
a^3+2a^2b+2ab^2+b^3 \\
a^3-2a^2b+2ab^2-b^3 \\
\hline
a^6+2a^5b+2a^4b^2+a^3b^3 \\
\quad -2a^5b-4a^4b^2-4a^3b^3-2a^2b^4 \\
\qquad +2a^4b^2+4a^3b^3+4a^2b^4+2ab^5 \\
\qquad\quad -a^3b^3-2a^2b^4-2ab^5-b^6 \\
\hline
a^6 \quad * \quad * \quad * \quad * \quad * \quad -b^6
\end{array}$$

2d. Change the order of the question; that is, multiply the factors I. and III., then II. and IV. together.

Then multiply the products I. III., and the II. IV.

$$\begin{array}{l}
a+b \\
a-b \\
\hline
a^2+ab \\
\quad -ab-b^2 \\
\hline
a^2 \quad * \quad -b^2
\end{array}
\qquad
\text{Then }
\begin{array}{l}
a^2+ab+b^2 \\
a^2-ab+b^2 \\
\hline
a^4+a^3b+a^2b^2 \\
\quad -a^3b-a^2b^2-ab^3 \\
\qquad +a^2b^2+ab^3+b^4 \\
\hline
a^4 \quad * \quad +a^2b^2 \quad * \quad +b^4
\end{array}
\qquad
\begin{array}{l}
a^4+a^2b^2+b^4 \\
a^2-b^2 \\
\hline
a^6+a^4b^2+a^2b^4 \\
\quad -a^4b^2-a^2b^4-b^6 \\
\hline
a^6 \quad * \quad * \quad -b^6
\end{array}$$

which is the product required.

3d. Again multiply the I. factor by the IV., and next the II. by the III.

It remains to multiply the product I. IV. and II. III.

$$\begin{array}{l}
a^2-ab+b^2 \\
a+b \\
\hline
a^3-a^2b+ab^2 \\
\quad +a^2b-ab^2+b^3 \\
\hline
a^3 \quad * \quad * \quad +b^3
\end{array}
\qquad
\text{Next }
\begin{array}{l}
a^2+ab+b^2 \\
a-b \\
\hline
a^3+a^2b+ab^2 \\
\quad -a^2b-ab^2-b^3 \\
\hline
a^3 \quad * \quad * \quad -b^3
\end{array}
\qquad
\begin{array}{l}
a^3+b^3 \\
a^3-b^3 \\
\hline
a^6+a^3b^3 \\
\quad -a^3b^3-b^6 \\
\hline
a^6 \quad * \quad -b^6
\end{array}$$

as in the two foregoing cases.

It will be proper to illustrate this example by a numerical application. Suppose $a=3$, and $b=2$, we shall have $a+b=5$, and $a-b=1$; further, $a^2=9$, $ab=6$, and $b^2=4$, therefore $a^2+ab+b^2=19$, and $a^2-ab+b^2=7$: so that the product required is that of $5\times19\times1\times7=665$. Now $a^6=729$, and $b^6=64$; consequently the product is $a^6-b^6=665$, as we have already seen.

Ex. 4.

$$\begin{array}{l}
\frac{1}{5}x^2-15 \\
\frac{1}{5}x^2-15 \\
\hline
\frac{1}{25}x^4-3x^2 \\
\quad -3x^2+225 \\
\hline
\frac{1}{25}x^4-6x^2+225
\end{array}$$

Ex. 5.

$$\begin{array}{l}
-135+6x^2-\frac{1}{25}x^4 \\
-135+6x^2-\frac{1}{25}x^4 \\
\hline
+18225-810x^2+\frac{27}{5}x^4 \\
\qquad -810x^2+36x^4-\frac{6}{25}x^6 \\
\qquad\qquad \frac{27}{5}x^4-\frac{6}{25}x^5+\frac{6}{625}x^8 \\
\hline
\text{Ans. } 18225-1620x^2+\frac{234}{5}x^4-\frac{12}{25}x^6+\frac{1}{625}x^8
\end{array}$$

Ex. 6.

$$\begin{array}{l}
-5s^2+as-b\\
-5s^2+as-b\\
\hline
25s^4-5as^3+5bs^2\\
\quad -5as^3+a^2s^2-abs\\
\qquad\qquad 5bs^2-abs+b^2\\
\hline
25s^4-10as^3-(10b+a^2)s^2-2abs+b^2
\end{array}$$

Ex. 7.

$$\begin{array}{l}
2a^2-3ax+4x^2\\
5a^2-6ax-2x^2\\
\hline
10a^4-15a^3x+20a^2x^2\\
\quad -12a^3x+18a^2x^2-24ax^3\\
\qquad\qquad -4a^2x^2+6ax^3-8x^4\\
\hline
10a^4-27a^3x+34a^2x^2-18ax^3-8x^4
\end{array}$$

Ex. 8.

$$\begin{array}{l}
5x^3+4ax^2+3a^2x+a^3\\
2x^2-3ax+a^2\\
\hline
10x^5-8ax^4+6a^2x^3+2a^3x^2\\
-15ax^4-12a^2x^3+9a^3x^2-3a^4x\\
\qquad +5a^2x^3+4a^3x^2+3a^4x+a^5\\
\hline
10x^5-7ax^4-a^2x^3-3a^3x^2 \ * \ +a^5
\end{array}$$

Ex. 9.

$$\begin{array}{l}
2a^2-3ax+4x^2\\
5a^2-6ax-2x^2\\
\hline
10a^4-15a^3x+20a^2x^2\\
-12a^3x+18a^2x^2-24ax^3\\
-4a^2x^2+6ax^3-8x^4\\
\hline
10a^4-27a^3r+34a^2x^2-18ax^3-8x^4
\end{array}$$

Ex. 10.

$$\begin{array}{l}
5x^3+4ax^2+3a^2x+a^3\\
2x^2-3ax+a^2\\
\hline
10x^5+8ax^4+6a^2x^3+2a^3x^2\\
-15ax^4-12a^2x^3-9a^3x^2-3a^4x\\
\qquad -5a^2x^3+4a^3x^2+3a^4x+a^5\\
\hline
10x^5-7ax^4-a^2x^3-3a^3x^2 \ * \ +a^5
\end{array}$$

Ex. 11.

$$\begin{array}{l}
3x^2-2x+5\\
6x-7\\
\hline
18x^3-12x^2+30x\\
\quad -21x^2+14x-35\\
\hline
18x^3-33x^2+44x-35
\end{array}$$

Ex. 12.

$$\begin{array}{l}
x^2+y\\
x^2+y\\
\hline
x^4+x^2y\\
\quad +x^2y+y^2\\
\hline
x^4+2x^2y+y^2
\end{array}$$

Ex. 13.

$$\begin{array}{l}
x^2+xy+y^2\\
y^2-xy+y^2\\
\hline
x^4+x^3y+x^2y^2\\
\quad -x^3y-x^2y^2-xy^3\\
\qquad\quad +x^2y^2+xy^3+y^4\\
\hline
x^4 \ * \ +x^2y^2 \ * \ +y^4
\end{array}$$

Ex. 14.

$$\begin{array}{l}
x^3-ax^2+bx-c\\
x^2-dx+e\\
\hline
x^5-ax^4+bx^3-cx^2\\
\quad -dx^4+adx^3-bdx^2+cdx\\
\qquad\quad +ex^3-aex^2+bex-ce\\
\hline
x^5-(a+d)x^4+(b+ad+e)x^3-(c+bd+ae)x^2+(cd+be)x-ce
\end{array}$$

Ex. 15.

$$\begin{array}{l}
x^2+xy+y^2\\
x-y\\
\hline
x^3+x^2y+xy^2\\
-x^2y-xy^2-y^3\\
\hline
x^3 \ * \ * \ -y^3
\end{array}$$

Ex. 16.

$114-\sqrt{\{196x^2-(x^2+24)^2\}}$ or, $114-\sqrt{(196x^2-x^4-48x^2-576)}$

or mult. $114-\sqrt{(148x^2-x^4-576)}$

by $114-\sqrt{(148x^2-x^4-576)}$

$$\begin{array}{l} 114^2-114\sqrt{(148x^2-x^4-576)} \\ \quad -114\sqrt{(148x^2-x^4-576)}+(148x^2-x^4-576) \\ \hline \end{array}$$

Prod. $12996-228\sqrt{(148x^2-x^4-576)}+(148x^2-x^4-576)$

Ex. 17.

$$\begin{array}{l} 3x^3+2x^2y^2+3y^3 \\ 2x^3-3x^2y^2+5y^3 \\ \hline 6x^6+4x^5y^2+6x^3y^3 \\ \quad -9x^5y^2-6x^4y^4-9x^2y^5 \\ \quad\quad +15x^3y^3+10x^2y^5+15y^6 \\ \hline 6x^6-5x^5y^2-6x^4y^4+21x^3y^3+x^2y^5+15y^6 \end{array}$$

Ex. 18.

$$\begin{array}{l} a^4+a^2c^2+c^4 \\ a^2-c^2 \\ \hline a^6+a^4c^2+a^2c^4 \\ \quad -a^4c^2-a^2c^4-c^6 \\ \hline a^6 \quad * \quad * \quad -c^6 \end{array}$$

NOTE. When the quantities that are to be multiplied together have literal co-efficients, proceed as before, putting the sum, or difference of the co-efficients of the resulting terms, between brackets, as in the former rule.

$$\begin{array}{l} ax+b \\ cx-d \\ \hline acx^2+bcx \\ \quad -adx-bd \\ \hline acx^2+(bc-ad)x-bd \end{array} \qquad \begin{array}{l} a-bx \\ c-dx \\ \hline ac-bcx \\ \quad -adx+bdx^2 \\ \hline ac-(bc+ad)x+bdx^2 \end{array}$$

And if several compound quantities are to be multiplied together, multiply the first by the second, and then that product by the third, and so on to the last factor, as below.

$$\begin{array}{l} a+2x \\ a-3x \\ \hline a^2+2ax \\ \quad -3ax-6x^2 \\ \hline a^2-ax-6x^2 \\ a+4x \\ \hline a^3-a^2x-6ax^2 \\ \quad +4a^2x-4ax^2-24x^3 \\ \hline a^3+3a^2x-10ax^2-24x^3 \end{array} \qquad \begin{array}{l} 3a-x \\ 2a+4x \\ \hline 6a^2-2ax \\ \quad +12ax-4x^2 \\ \hline 6a^2+10ax-4x^2 \\ 4a-2x \\ \hline 24a^3+40a^2x-16ax^2 \\ \quad -12a^2x-20ax^2+8x^3 \\ \hline 24a^3+28a^2x-36ax^2-8x^3 \end{array}$$

To this we may add, that it is usual, in some cases, to write down the quantities that are to be multiplied together, between brackets, or under a vinculum, without performing the whole operation; as $3ab(a+b)\times a\sqrt{(a^2-b^2)}$

Ex. 19.

$28-\sqrt{(20x^2-10x)}$ multiply
$28-\sqrt{(20x^2-10x)}$ by

$784-28\sqrt{(20x^2-10x)}$
$\quad -28\sqrt{(20x^2-10x)}+20x^2-10x$

$784-56\sqrt{(20x^2-10x)}+20x^2-10x$ product. Multiply
$28-\sqrt{(20x^2-10x)}$ again by this.

$21952-1568\sqrt{(20x^2-10x)}+560x^2-280x$
$\quad -784\sqrt{(20x^2-10x)}+56(20x^2-10x)-(20x^2+10x)\sqrt{(20x^2-10x)}$

$21952-2352\sqrt{(20x^2-10x)}+1680x^2-840x+(10x-20x^2)\sqrt{(20x^2-10x)}$

add $-2352\sqrt{(20x^2-10x)}$
and $+(10x-20x^2)\sqrt{(20x^2-10x)}$

sum $+(10x-20x^2-2352)\sqrt{(20x^2-10x)}$

Ex. 20.

$2352+20x^2-10x$ multiply
$2352+20x^2-10x$ by

$5531904+47040x^2-23520x$
$\quad +47040x^2+400x^4-200x^3$
$\quad\quad -23520x-200x^3+100x^2$

$5531904-47040x+94180x^2-400x^3+400x^4$ sum.
Multiply $20x^2-10x$ again.

$110638080x^2-940800x^3+1883600x^4-8000x^5+8000x^6$
$\quad -55319040x+470400x^2-941800x^3+4000x^4-4000x^5$

$111108480x^2-55319040x-1882600x^3+1887600x^4-12000x^5+8000x^6$

Ex. 21.

$a^2+b^2+c^2-ab-ac-bc$
$a+b+c$

$a^3+ab^2+ac^2-a^2b-a^2c-abc$
$\quad +a^2b+b^3+bc^2-ab^2-abc-b^2c$
$\quad\quad a^2c+b^2c+c^3-abc-ac^2-bc^2$

a^3 * $+b^3$ * $-3abc+c^3$

Ex. 22.

$a^3+3a^2x+3ax^2+x^3$
$a^3-3a^2x+3ax^2-x^3$

$a^6+3a^5x+3a^4x^2+a^3x^3$
$\quad -3a^5x-9a^4x^2-9a^3x^3-3a^2x^4$
$\quad +3a^4x^2+9a^3x^3+9a^2x^4+3ax^5$
$\quad\quad -a^3x^3-3a^2x^4-3ax^5-x^6$

a^6 * $-3a^2x^4$ * $+3a^2x^4$ * $-x^6$

Ex. 23.

$$
\begin{array}{l}
7x^3-1676x^2+840x-21952 \text{ multiply} \\
7x^3-1676x^2+840x-21952 \text{ by} \\
\hline
49x^6-11732x^5+5880x^4-153664x^3 \\
\qquad -11732x^5+2808976x^4-1407840x^3+36791552x^2 \\
\qquad\qquad +5880x^4-1407840x^3+705600x^2-18439680x \\
\qquad\qquad\qquad -153664x^3+36791552x^2-18439680x+481890304 \\
\hline
49x^6-23464x^5+2820736x^4-3123008x^3+74288704x^2-36879360x+481890304
\end{array}
$$

NOTE. The products of the *powers* of the same quantity are found by adding their indices. Thus: $a^1 \times a^1 = a^2$, or $a \times a = a^{1+1} = a^2$; $a^1 \times a^2 = a^{1+2} = a^3$; $a^2 \times a^3 = a^5$; $a^m \times a^n = a^{m+n}$; $a^{\frac{1}{3}} \times a^{\frac{1}{4}} = a^{\frac{1}{3}+\frac{1}{4}} = a^{\frac{7}{12}}$; $a^{\frac{1}{m}} \times a^{\frac{1}{n}} = a^{\frac{m+n}{mn}}$; $x^n \times x = x^{n+1}$.

Obs. 1. In multiplication, as well as in addition and subtraction, the order of the letters is of no consequence. Thus, if *abc* be multiplied by *d*, the product is *abcd*, *bacd*, *cbda*, &c., each of which is of the same value; but it is usual to arrange them according to the order of the alphabet.

Obs. 2. In algebra it is customary to begin the multiplication on the left; but because the steps are merely indicated, it is of no consequence where the operation commences.

Obs. 3. It may be useful to observe, that, according to Euclid, Lib. II. Prop. V., the product of the sum and difference of any two quantities is equal to the difference of their squares; thus, (1.) $(a+b)(a-b)=a^2-b^2$

(2.) $(a^2+b^2)(a^2-b^2)=a^4-b^4=(a^2+b^2)(a+b)(a-b)$.

(3.) $(a^4+b^4)(a^4-b^4)=a^8-b^8=(a^4+b^4)(a^2+b^2)(a^2-b^2)=(a^4+b^4)(a^2+b^2)(a+b)(a-b)$

These compositions and decompositions of quantities are often found to be of great utility in the solution of equations.

Ex. 24.

$$
\begin{array}{l}
\textbf{Multiply } ex+mx^2+nx^3+rx^4 \\
\qquad \text{by } ax+bx^2+cx^3+dx^4 \\
\hline
aex^2+amx^3+anx^4+arx^5 \\
\qquad bex^3+bmx^4+bnx^5+brx^6 \\
\qquad\qquad +cex^4+cmx^5+cnx^6+crx^7 \\
\qquad\qquad\qquad dex^5+dmx^6+dnx^7+drx^8 \\
\hline
aex^2+(am+be)x^3+(an+bm+ce)x^4+(ar+bn+cm+de)x^5 \\
\qquad +(br+cn+dm)x^6+(cr+dn)x^7+drx^8
\end{array}
$$

DIVISION.

41. Division in Algebra is the method of finding the quotient arising from the division of one algebraic quantity by another. Division is generally divided into three cases, namely, when the divisor and dividend are both simple quantities; when the divisor is a simple quantity and the dividend a compound one; and when the divisor and dividend are both compound quantities.

Case I. *When the divisor and dividend are both simple terms.*

Rule. Place the divisor in the form of a denominator under the dividend; cancel those letters which are common to both, and divide the coefficients by any number that will divide them without a remainder, and the result will be the quotient required.

Rule II. Divide the coefficients as in common arithmetic, and to the quotient annex those letters in the dividend which are not found in the divisor.

A general rule for the signs in all the cases of division:

When the signs of the divisor and dividend are alike, (that is, both + or both —,) the sign of the quotient will be +. When they are unlike, (that is, the one + and the other —,) the sign of the quotient will be —.

The above rule briefly expressed in one view, is as follows:

	Div'r.	Div'd.	Quo't.		Div'r.	Div'd.	Quo't.
1.	+ Plus	+ Plus	+ Plus	3	—	+	— Both minus.
2.	— Minus	— Minus	+ Plus		+	—	—

AGAIN, THUS:

$$\frac{+ab}{+a} = +b; \quad \frac{-ab}{-a} = +b; \quad \frac{-ab}{+a} = -b; \quad \frac{+ab}{-a} = -b;$$

and these four are all the cases that can possibly happen with regard to the variation of the signs.

Powers and roots of the same quantity are divided by subtracting their idices, that is, subtract the index of the diviser from the index of the dividend.

Thus, $\frac{a^2}{a} = a^{2-1} = a^1$, or a; $\frac{a^5}{a^3} = \frac{a^2 \times a^3}{a^3} = a^2 = a^{5-3}$; $\frac{a^3}{a^5} = a^{3-5} = a^{-2}$, $\frac{a^{-5}}{a^{-2}} = a^{-5+2} = a^{-3}$; $\frac{a^{-3}}{a^{-6}} = a^{-3+6} = a^{+3}$; $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}} = a^{\frac{1}{2}-\frac{1}{2}} = a^0$; $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}} = a^{\frac{1}{2}-\frac{1}{3}} = a^{\frac{1}{6}}$; $\frac{a^{\frac{3}{4}}}{a^{\frac{2}{5}}} = a^{\frac{3}{4}-\frac{2}{5}} = a^{\frac{7}{20}}$; $\frac{a^m}{a^n} = a^{m-n}$;

$\frac{a^{-m}}{a^n} = a^{-m-n}$; $\frac{a^{-m}}{a^{-n}} = a^{-m+n}$.

1. Divide $16x^2$ by $8x$.

$$\frac{16x^2}{8x}=2x, \text{ Ans.}$$

2. Divide $12ax$ by $3a$.

$$\frac{12ax}{3a}=4x, \text{ Ans.}$$

3. Divide $-10x^2$ by $-2x$.

$$\frac{-10x^2}{-2x}=+5x, \text{ Ans.}$$

4. Divide $-18ax^2y$ by $-8ax$.

$$\frac{-18ax^2y}{-8ax}=+\frac{9xy}{4}, \text{ Ans.}$$

5. Divide $12a^2x^2$ by $-8a^2x$.

$$\frac{12a^2x^2}{-8a^2x}=-\frac{3x}{2}, \text{Ans.}$$

6. Divide $-15ay^2$ by $3ay$.

$$\frac{-15ay^2}{3ay}=-5y, \text{ Ans.}$$

7. Divide $6ax$ by $2\sqrt{ax}$.

$$\frac{6ax}{2\sqrt{(ax)}}=3\sqrt{(ax)}, \text{ Ans.}$$

8. Divide $15a^2x^2$ by $-3ax^2$.

$$\frac{15a^2x^2}{-3ax^2}=-5a, \text{ Ans.}$$

When the coefficients of the divisor and dividend are both fractions.

RULE. Invert the coefficients of the divisor, that is, put the denominator of the divisor in the numerator's place, and the numerator in the denominator's place; then proceed as in multiplication of fractions in arithmetic.

9. Divide $12a^2x^2$ by $3a^2x$.

$$\frac{12a^2x^2}{-3a^2x}=-4x, \text{ Ans.}$$

10. Divide $-15ab^{\frac{2}{3}}$ by $-3ab^{\frac{1}{2}}$.

$$\frac{-15ab^{\frac{2}{3}}}{-3ab^{\frac{1}{2}}}=+\frac{15ab^{\frac{2}{3}}}{3ab^{\frac{1}{2}}}=5b^{\frac{2}{3}-\frac{1}{2}}=5b^{\frac{1}{6}}.$$

11. Divide $18a^2b^2x^{\frac{1}{2}}$ by $-6ab^2x^{\frac{1}{4}}$

$$\frac{18a^2b^2x^{\frac{1}{2}}}{-6ab^2x^{\frac{1}{4}}}=-3ax^{\frac{1}{4}}.$$

Divide $-13a^{\frac{1}{2}}b^3c$ by $-7a^{\frac{1}{3}}b^2c^{\frac{1}{2}}$.

$$\frac{-13a^{\frac{1}{2}}b^3c}{-7a^{\frac{1}{3}}b^2c^{\frac{1}{2}}}=\frac{13a^{\frac{1}{2}-\frac{1}{3}}\,bc^{1-\frac{1}{2}}}{7}=\frac{13a^{\frac{1}{6}}bc^{\frac{1}{2}}}{7}$$

13. *Divide $ax^{\frac{1}{3}}$ by $-\frac{3}{5}a^{\frac{1}{2}}x^{\frac{1}{5}}$.

14. Divide $27a^{\frac{1}{3}}$ by $-\frac{3}{5}a^{\frac{1}{2}}a^{\frac{1}{4}}$.

$$\frac{27a^{\frac{1}{3}}}{-\frac{3}{5}a^{\frac{1}{2}}x^{\frac{1}{4}}}=\frac{27\times5}{1\times3}a^{\frac{1}{2}}x^{\frac{1}{12}}=-45a^{\frac{1}{2}}a^{\frac{1}{12}}, \text{ Ans.}$$

15. Divide $-\frac{3}{6}x^{\frac{1}{2}}$ by $\frac{1}{6}x^{\frac{1}{3}}$.

$$\frac{-\frac{3}{6}x^{\frac{1}{2}}}{\frac{1}{6}x^{\frac{1}{3}}}=-3x^{\frac{3}{6}-\frac{2}{6}}=-3x^{\frac{1}{6}}$$

16. Divide $-\frac{2}{3}a^{\frac{1}{2}}$ by $\frac{4}{5}a^{\frac{1}{2}}$.

$$\frac{-\frac{2}{3}a^{\frac{1}{2}}}{\frac{4}{5}a^{\frac{1}{2}}}=-\frac{2}{3}\times\frac{5}{4}=-\frac{5}{6}, \text{ Ans.}$$

$$*\ \frac{ax^{\frac{1}{3}}}{-\frac{3}{5}a^{\frac{1}{2}}x^{\frac{1}{5}}}=-\frac{5}{3}a^{\frac{1}{2}}x^{\frac{5}{15}-\frac{3}{15}}=-\frac{5}{3}a^{\frac{1}{2}}x^{\frac{2}{15}}$$

CASE II. *When the divisor is a simple quantity and the dividend a compound one.*

RULE. Divide each term of the dividend separately by the divisor, and put down such as will not divide in the form of a fraction, and the result will be the quotient required.

1. Divide $18a^2b-54a^2$ by $6a^2$.

$$\frac{18a^2b-54a^2}{6a^2}=3b-9.$$

2. Divide $16ax-40x^3$ by $8x$.

$$\frac{16ax-40x^3}{8x}=2a-5x^2.$$

3. Divide $12a^2x+4ax^2-16a$ by $4a$.

$$\frac{12a^2x+4ax^2-16a}{4a}=3ax+x^2-4, \text{ Ans.}$$

4. Divide $3abc+12abx-9a^2b$ by $3ab$.

$$\frac{3abc+12abx-9a^2b}{3ab}=c+4x-3a, \text{ Ans.}$$

5. Divide $40a^3b^3+60a^2b^2-17ab$ by $-ab$.

$$\frac{40a^3b^3+60a^2b^2-17ab}{-ab}=-40a^2b^2-60ab+17, \text{ Ans.}$$

6. Divide $3x^3+6x^2+3ax-15x$, by $3x$. Ans., $x^2+2x+a-5$.

7. Divide $14a^2-7ab+21ax-28a$, by $7a$.

$$\frac{14a^2-7ab+21ax-28a}{7a}=2a-b+3x-4, \text{ Ans.}$$

8. Divide $-10ab+30ab^3-6a^2b^2$, by $-2ab$.

Here $\dfrac{-10ab+30ab^3-6a^2b^2}{-2ab}=5-15b^2+3ab$, Ans.

9. Divide $15a^2bc-12acx^2+5ad^2$, by $-5ac$.

$$\frac{15a^2bc-12acx^2+5ad^2}{-5ac}=-3ab+\frac{12x^2}{5}-\frac{d^2}{c}, \text{ Ans.}$$

10. Divide $9x^3-3x-10+2y$ by $-3x$.

$$\frac{9x^3-3x-10+2y}{-3x}=-3x^2+1+\frac{10}{3x}-\frac{2y}{3x}, \text{ Ans.}$$

11. Divide $-8x^3+5x-12+2b$ by $-2x$.

$$\frac{-8x^3+5x-12+2b}{-2x}=+4x^2-\frac{5}{2}+\frac{6}{x}-\frac{b}{x}, \text{ Ans.}$$

12. Divide $2a-15+7a+3b$ by $3a$.

$$\frac{2a-15+7a+3b}{3a}=3+\frac{b-5}{a}, \text{ Ans.}$$

13. Divide $a^{n+1}x-a^{n+2}x-a^{n+3}x-a^{n+4}x-$, &c., by a^n.

$$\frac{a^{n+1}x-a^{n+2}x-a^{n+3}x-a^{n+4}x}{a^n}=ax-a^2x-a^3x-a^4x-, \text{ Ans.}$$

Case III. ***When the divisor and dividend are both compound quantities.***

Rule. Arrange both divisor and dividend according to the powers of the same letters, beginning with the highest; then find how often the first term of the divisor is contained in the first term of the dividend, and place the result in the quotient; multiply the whole divisor by this quantity, and place the product under the corresponding or like terms in the dividend, and subtract it from them; to the remainder bring down as many terms of dividend as are requisite for the next operation, and divide as before, and so on till the work is finished, as in common arithmetic.

Dividend. $2x+3$ Divisor.

$32x^5+243 \quad 16x^4-24x^3+36x^2-54x+81$ quotient.

$$\begin{array}{l}
32x^5+48x^4 \\
\hline
-48x^4+243 \\
-48x^4-72x^3 \\
\hline
72x^3+243 \\
72x^3+108x^2 \\
\hline
-108x^2+243 \\
-108x^2-162x \\
\hline
162x+243 \\
162x+243 \\
\hline
\end{array}$$

Divis. Divid. Quotient.

$$\begin{array}{l}
3x-6)6x^4-96(2x^3+4x^2+8x+16 \\
6x^4-12x^3 \\
\hline
12x^3-96 \\
12x^3-24x^2 \\
\hline
24x^2-96 \\
24x^2-48x \\
\hline
48x-96 \\
48x-96 \\
\hline
\end{array}$$

Dividend. $2x-3a$ Divisor.

$48x^3-76ax^2-64a^2x+105a^3 \quad 24x^2-2ax-35a^2$ quot.

$$\begin{array}{l}
48x^3-72ax^2 \\
\hline
-4ax^2-64a^2x \\
-4ax^2+6a^2x \\
\hline
-70a^2x+105a^3 \\
-70a^2x+105a^3 \\
\hline
\end{array}$$

Dividend. Divisor.

$4x^4-9x^2+6x-1 \quad 2x^2+3x-1$

$4x^4+6x^3-2x^2 \quad 2x^2-3x+1$ quot.

$$\begin{array}{l}
-6x^3-7x^2+6x \\
-6x^3-9x^2+3x \\
\hline
+2x^2+3x-1 \\
2x^2+3x-1 \\
\end{array}$$

Dividend.

$x^4+y^4 \quad x+y$ divisor.

$x^4+x^3y \quad x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y}$ quot.

$$\begin{array}{l}
-x^3y+y^4 \\
-x^3y-x^2y^2 \\
\hline
+x^2y^2+y^4 \\
x^2y^2+xy^3 \\
\hline
-xy^3+y^4 \\
-xy^3-y^4 \\
\end{array}$$

Remainder $2y^4$

Dividend. $x^3-y^3 \quad x+y$ Divisor.

$x^3+x^2y \quad x^2-xy+y^2-\dfrac{2y^3}{x+y}$ quot.

$$\begin{array}{l}
\hline
-x^2y-y^3 \\
-x^2y-xy^2 \\
\hline
xy^2-y^3 \\
xy^2+y^3 \\
\hline
\end{array}$$

Remainder $-2y^3$

Dividend. Quotient.

$$\begin{array}{l}
a+2b)a^4+4a^2b+8b^4(a^3-2a^2b+4ab+4ab^2-(8b^2+8b^3) \\
\quad a^4+2a^3b \\
\quad -2a^3b+4a^2b \\
\quad -2a^3b-4a^2b^2 \\
\qquad +4a^2b+4a^2b^2 \\
\qquad +4a^2b+8a^2b^2 \\
\qquad\quad +4a^2b^2-8ab^2 \\
\qquad\quad +4a^2b^2+8ab^3 \\
\qquad\qquad -8ab^2-8ab^3 \\
\qquad\qquad -8ab^2-16b^3 \\
\qquad\qquad\quad -8ab^2+16b^3+8b^4 \\
\qquad\qquad\quad -8ab^2 \quad -16b^4 \\
\qquad\qquad\qquad +16b^3+24b^4 \text{ rem.}
\end{array}$$

Divisor. Divd.

$$\begin{array}{l}
x+1)x^5+1(x^4-x^3+x^2-x+1 \\
\quad x^5+x^4 \\
\quad -x^4+1 \\
\quad -x^4-x^3 \\
\qquad x^3+1 \\
\qquad x^3+x^2 \\
\qquad\quad -x^2+1 \\
\qquad\quad -x^2-x \\
\qquad\qquad x+1 \\
\qquad\qquad x+1
\end{array}$$

Quotient.

Div. Divd. Quotient.

$$\begin{array}{l}
x-1)x^6-1(x^5+x^4+x^3+x^2+x+1 \\
\quad x^6-x^5 \\
\quad\ x^5-1 \\
\quad\ x^5-x^4 \\
\qquad x^4-1 \\
\qquad x^4-x^3 \\
\qquad\ x^3-1 \\
\qquad\ x^3-x^2 \\
\qquad\quad x^2-1 \\
\qquad\quad x^2-x \\
\qquad\qquad x-1 \\
\qquad\qquad x-1
\end{array}$$

Divd. Div. Quo.

$$\begin{array}{l}
a+x)a^5+x^5(a^4-a^3x+a^2x^2-ax^3+x^4 \\
\quad a^5+a^4x \\
\quad -a^4x+x^5 \\
\quad -a^4x-a^3x^2 \\
\qquad a^3x^2+x^5 \\
\qquad a^3x^2+a^2x^3 \\
\qquad\quad -a^2x^3+x^5 \\
\qquad\quad -a^2x^3-ax^4 \\
\qquad\qquad ax^4+x^5 \\
\qquad\qquad ax^4+x^5
\end{array}$$

Div.

$$\begin{array}{l}
a^2-2ab+2b^2)a^4+4b^4(a^2+2ab+2b^2 \\
\quad a^4-2a^3b+2a^2b^2 \\
\quad 2a^3b-2a^2b^2+4b^4 \\
\quad 2a^3b-4a^2b^2+4ab^3 \\
\qquad 2a^2b^2-4ab^3+4b^4 \\
\qquad 2a^2b^2-4ab^3+4b^4
\end{array}$$

Divisor. Divd.

$$\begin{array}{l}
a^2+ab\sqrt{2}+b^2)a^4+b^4(a^2-ab\sqrt{2}+b \\
\quad a^4+a^3b\sqrt{2}+a^2b^2 \\
\quad -a^3b\sqrt{2}-a^2b^2 \\
\quad -a^3b\sqrt{2}-2a^2b^2-ab^3\sqrt{2} \\
\qquad a^2b^2+ab^3\sqrt{2}+b^4 \\
\qquad a^2b^2+ab^3\sqrt{2}+b^4
\end{array}$$

$$\begin{array}{l}
2x^2-3ax+a^2)4x^4-9a^2x^2+6a^3x-a^4(2x^2+3ax-a^2 \\
\quad 4x^4-6ax^3+2a^2x^2 \\
\quad\ 6ax^3-11a^2x^2+6a^3x \\
\quad\ 6ax^3-9a^2x^2+3a^3x \\
\qquad -2a^2x^2+3a^3x-a^4 \\
\qquad -2a^2x^2+3a^3x-a^4
\end{array}$$

$x+y)x^2+2xy+y^2(x+y$

x^2+xy

$xy+y^2$

$xy+y^2$

dividend. $a+x$ divisor

$a^3+5a^2x+5ax^2+x^3(a^2+4ax+$

a^3+a^2x quotient.

$4a^2x+5ax^2$

$4a^2x+4ax^2$

ax^2+x^3

ax^2+x^3

Dividend.

$x-3)x^3-9x^2+27x-27(x^2-6x+9$ quotient.

x^3-3x^2

$-6x^2+27x$

$-6x^2+18x$

$9x-27$

$9x-27$

Divisor. Dividend.

$y-8)2y^3-19y^2+26y-16(2y^2-3y+$ quotient

$2y^3-16y^2$

$-3y^2+26y$

$-3y^2+24y$

$2y-16$

$2y-16$

$b-y)b^4-3y^4(b^3+b^2y+by^2+y^3-\frac{2y^4}{b-y}$ quotient.

b^4-b^3y

b^3y-3y^4

$b^3y-b^2y^2$

$b^2y^2-3y^4$

$b^2y^2-by^3$

by^3-3y^4

by^3-y^4

$-2y^4$ remainder.

$a+x)a^3-x^3(a^2-ax+x^2-\frac{2x^3}{a+x}$ quotient

a^3+a^2x

$-a^2x-x^3$

$-a^2x-ax^2$

ax^2-x^3

ax^2+x^3

$-2x^3$

Divis. Divid.

$a-x)a^3+5a^2x+5ax^2+x^3(a^2+6ax+11x^2+\frac{12x^3}{a-x}$ quotient.

a^3-a^2x

$6a^2x-6ax^2$

$6a^2x+5ax^2$

$11ax^2+x^3$

$11ax^2-11x^3$

Remainder $12x^3$

Divis. Divid.

$x-a)x^3-3ax^2+3a^2x-a^3(x^2-2ax+a^2$

x^3-ax^2

$-2ax^2+3a^2x$

$-2ax^2+2a^2x$

a^2x-a^3

a^2x-a^3

Divisor. Dividend. Quot

$a^2-2ax+x^2)a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5(a^3-3a$

$a^5-2a^4x+a^3x^2$ $+3ax^2-$

$-3a^4x+9a^3x^2-10a^2x^3$

$-3a^4x+6a^3x^2-3a^2x^3$

$3a^3x^2-7a^2x^3+5ax^4$

$3a^3x^2-6a^2x^3+3ax^4$

$-a^2x^3+2ax^4-x^5$

$-a^2x^3+2ax^4-x^5$

From the preceding rules are deduced the following useful theorems, viz:

1. By the rule for addition, if the sum of any two quantities, a and b, be added to their difference, the sum will be twice the greater.*

2. By the rule for subtraction, if the difference of any two quantities be taken from their sum, the remainder will be twice the less.†

3. By multiplication, example 2, article 21, if the sum of any two quantities be multiplied by their difference, the product will be the difference of their squares.

* For $\left\{\begin{array}{r} \ldots\ldots\ldots a+b \\ \text{added to } \underline{a-b} \\ \text{gives } 2a \end{array}\right.$ † and $\left\{\begin{array}{r} \ldots\ldots\ldots\ldots a+b \\ \text{diminished by } \underline{a-b} \\ \text{gives } \quad 2b \end{array}\right.$

$$a+x)a\ldots\ldots(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\frac{x^4}{a^4}-\frac{x^5}{a^5}+\frac{x^6}{a^6}-, \text{ \&c.}$$

$$\begin{array}{r} \underline{a+x} \\ -x \\ \underline{-x-\frac{x^2}{a}} \\ +\frac{x^2}{a} \end{array}$$

$$a-x)a\ldots\ldots(1+\frac{x}{a}+\frac{x^2}{a}+, \text{ \&c.}‡$$

$$\begin{array}{r} \underline{a-x} \\ +x \\ x-\frac{x^2}{a} \end{array}$$

‡ Now, it is easy to perceive that the next, or sixth term of the quotient will be $-\frac{x^5}{a^5}$, and the seventh term $\frac{x^6}{a^6}$, and so on, alternately plus and minus; this is called the law of continuation of the series. And the sum of all the terms, when infinitely continued, is said to be equal to the fraction $\frac{a}{a+x}$. Thus we say the vulgar fraction $\frac{2}{3}$, when reduced to a decimal, is $=.66666$, &c. infinitely continued. The terms in the quotient are found by dividing the remainders by a, the first term of the divisor; thus, the first remainder, $-x$, divided by a, gives $-\frac{x}{a}$, the second term in the quotient; and the second remainder, $+\frac{x^2}{a}$, divided by a, gives $+\frac{x^2}{a^2}$, the third term, &c.

EXAMPLES FOR PRACTICE.

1. Divide x^3-48x^2+200 by $x+2$. Ans. $x^2-50x+100$.
2. Divide $x^3-22x-24$ by $x+4$. Ans. x^2-4x-6.
3. Divide $x^3+9x^2+4x-80$ by $x+5$. Ans. $x^2-4x-16$.
4. Divide $x^3+39x^2+249x+289$ by $x+1$. Ans. $x^2-40x+289$.
5. Divide $x^4-38x^3+210x^2+538x+289$ by $x+1$.
   Answer $x^3-39x^2+249x+289$.

$$a\pm b)\ a\ \left(1\pm\frac{b}{a}+\frac{b^2}{a^2}\pm\frac{b^3}{a^3}+\frac{b^4}{a^4},\ \text{\&c. answer.}\right.$$

$$\begin{array}{l}
a\pm b \\ \hline
\mp b \\
\mp b-\frac{b^2}{a} \\ \hline
\pm\frac{b^2}{a} \\
\pm\frac{b^2}{a}\pm\frac{b^3}{a^2} \\ \hline
\mp\frac{b^3}{a^2} \\
\mp\frac{b^3}{a^2}-\frac{b^4}{a^3} \\ \hline
\text{Rem. } +\frac{b^4}{a^3}
\end{array}$$

$$a^2+2ab+b^2)a^2\dots\left(1-\frac{2b}{a}+\frac{3b^2}{a^2}-\frac{4b^3}{a^3}\right.$$

$$\begin{array}{l}
a^2+2ab+b^2 \\ \hline
-2ab-b^2 \\
-2ab-4b^2-\frac{2b^3}{a} \\ \hline
+3b^2-\frac{2b^3}{a} \\
+3b^2-\frac{6b^3}{a}+\frac{3b^4}{a^2} \\ \hline
-\frac{4b^3}{a}-\frac{3b^4}{a^2} \\
-\frac{4b^3}{a}-\frac{8b^4}{a^2}-\frac{4b^5}{a^3} \\ \hline
\text{Remainder } +\frac{5b^4}{a^2}+\frac{4b^5}{a^3}
\end{array}$$

$$a^2+x^2)a^2-x^2\left(1-\frac{2x^2}{a^2}+\frac{2x^4}{a^4}-\frac{2x^6}{a^6}+\frac{2x^8}{a^8}-\frac{2x^{10}}{a^{10}}+\ \text{\&c.}\right.$$

$$\begin{array}{l}
a^2+x^2 \\ \hline
-2x^2 \\
-2x^2-\frac{2x^4}{a^2} \\ \hline
+\frac{2x^4}{a^2} \\
+\frac{2x^4}{a^2}+\frac{2x^6}{a^4} \\ \hline
-\frac{2x^6}{a^4} \\
-\frac{2x^6}{a^4}-\frac{2x^8}{a^6}
\end{array}$$

$$a^m-x^n)a^m-x^m(a^{m-n}+a^{m-2n}x^n+a^{m-3n}x^{2n},+\text{\&c.}$$

$$\begin{array}{l}
a^m-a^{m-n}x^n \\ \hline
+a^{m-n}x^n \\
+a^{m-n}x^n-a^{m-2n}x^{2n} \\ \hline
+a^{m-2n}x^{2n} \\
+a^{m-2n}x^{2n}-a^{m-3n}x^{3n} \\ \hline
\text{Rem. } +a^{m-3n}x^{3n}
\end{array}$$

Divisor. Quotient.

$$a+x)b(\frac{b}{a}-\frac{bc}{a^2}+\frac{bc^2}{a^3}-\frac{bc^3}{a^4}+\frac{bx^4}{a^5}, \&c. = (1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\frac{x^4}{a^4})\times\frac{b}{x}$$

$$\begin{array}{l}
b+\frac{bx}{a} \\
\hline
\frac{bx}{a} \\
\frac{bx}{a}-\frac{bx^2}{a^2} \\
\hline
+\frac{bx^2}{a^2} \\
+\frac{bx^2}{a^2}+\frac{bx^3}{a^3} \\
\hline
-\frac{bx^3}{a^3} \\
-\frac{bx^3}{a^3}-\frac{bx^4}{a^4} \\
\hline
+\frac{bx^4}{a^4} \\
+\frac{bx^4}{a^4}+\frac{bx^5}{a^5} \\
\hline
\text{Remainder } -\frac{bx^5}{a^5}
\end{array}$$

$$x+b\,)\,a^2(\frac{a^2}{x}-\frac{a^2b}{x^2}+\frac{a^2b^2}{x^3}-\frac{a^2b^3}{x^4}, \&c. \text{ answer.}$$

$$\begin{array}{l}
a^2+\frac{ba^2}{x} \\
\hline
-\frac{ba^2}{x} \\
-\frac{ba^2}{x}-\frac{a^2b^2}{x^2} \\
\hline
+\frac{a^2b^2}{x^2} \\
+\frac{a^2b^2}{x^2}+\frac{a^2b^2}{x^3} \\
\hline
-\frac{a^2b^3}{x^3} \\
-\frac{a^2b^3}{x^3}+\frac{a^2b^4}{x^4} \\
\hline
\text{R. } -\frac{a^2b^4}{x^4}
\end{array}$$

$$a+b)2ab(2b-\frac{2b^2}{a}+\frac{2b^3}{a^2}-\frac{2b^4}{a^3}\&c.$$

Quotient.

$$\begin{array}{l}
2ab+2b^2 \\
\hline
-2b^2 \\
-2b^2-\frac{2b^3}{a} \\
\hline
+\frac{2b^3}{a} \\
+\frac{2b^3}{a}+\frac{2b^4}{a^2} \\
\hline
-\frac{2b^4}{a^2} \\
-\frac{2b^4}{a^2}-\frac{2b^5}{a^3} \\
\hline
\text{Rm. } +\frac{2b^5}{a^3}
\end{array}$$

6. Divide $x^5+6x^4-10x^3-112x^2-207x-110$ by $x+5$.
Answer $x^4+x^3-15x^2-37x-22$.

7. Divide $x^4+x^3-15x^2-37x-22$ by $x+2$.
Answer $x^3-x^2-13x-11$, and $x^2-2x-11$.

8. Divide $x^3-x^2-13x-11$ by $x+1$. Ans. $x^2-2x-11$.

9. Divide $x^5+6x^4-10x^3-112x^2-207x-110$ by $x+2$.
Ans. $x^4+4x^3-18x^2-76x-55$.

10. Divide $x^4+4x^3-18x^2-76x-55$ by $x+5$.
Ans. $x^3-x^2-13x-11$.

11. Divide $x^5+6x^4-10x^3-112x^2-207x-110$ by $x+1$.
Ans. $x^4+5x^3-15x^2-97x-110$.

12. Divide $x^4+5x^3-15x^2-97x-110$ by $x+2$ and by $x+5$.
Ans. $x^2+3x^2-21x-55$.

13. Divide $x^4-19x^3+123x^2-302x+200$ by $x-4$.
Ans. $x^3-15x^2+63x+50$.

14. Divide $x^4-27x^3+262x^2+356x-1200$ by $x+3$.
Answer $x^3-30x^2+252x-400$.

ON THE REDUCTION OF ALGEBRAIC FRACTIONS.

(42.) The rules managing algebraic fractions being of the same nature as vulgar fractions in common arithmetic, the operations are performed exactly in the same manner.

Case I. *To reduce a mixed quantity to an improper fraction.*

Rule. Multiply the integer, or whole part, by the denominator of the fraction, and to the product add the numerator; then under their sum place the original denominator.

1. Reduce $a+\frac{b}{a}$ and $a-\frac{x}{b}$ each to improper fractions.

$a+\frac{b}{a}=\frac{a\times a+b}{a}=\frac{a^2+b}{a}$ Ans. $a-\frac{x}{b}=\frac{ab-x}{b}$ Ans.

2. Reduce $4x-\frac{3x-5}{4}$ and $4ax-\frac{b^2}{2ax}$ to improper fractions.

Here $4x-\frac{3x-5}{4}=\frac{16x-3x+5}{4}=\frac{13x+5}{4}$ Ans.

and $4ax-\frac{b^2}{2ax}=\frac{4ax\times 2ax-b^2}{2ax}=\frac{8a^2x^2-b^2}{2ax}$ Ans.

3. Reduce $x-\frac{a^2-x^2-c}{x}$ and $y-\frac{b^2-y^2-d}{y}$ to improper fractions.

$x-\frac{a^2-x^2-c}{x}=\frac{x^2-a^2+x^2+c}{x}=\frac{2x^2-a^2+c}{x}$ Ans. $\frac{2y^2-b^2+d}{y}$

4. Reduce $5a-\frac{3x-b}{a}$ and $x-\frac{ax+x^2}{2a}$ to improper fractions.

Here $5a-\frac{3x-b}{a}=\frac{5a^2-(3x-b)}{a}=\frac{5a^2-3x+b}{a}$ Ans.

and $x-\frac{ax+x^2}{2a}=\frac{2ax-(ax+x^2)}{2a}=\frac{ax-x^2}{2a}$ Ans.

5. Reduce $5x-\frac{2x+5}{4}$ and $5x-\frac{2x-5}{4}$ to improper fractions.

$5x-\frac{2x+5}{4}=\frac{5x\times 4-(2x+5}{4}=\frac{20x-2x-5}{4}=\frac{18x-5}{4}$ Ans.

and $5x-\frac{2x-5}{4}=\frac{5x\times 4-(2x-5)}{4}=\frac{20x-2x+5}{4}=\frac{18x+5}{4}$ An.

6. Reduce $a-x+\frac{a^2-ax-b}{x}$ to an improper fraction.

$a-x+\frac{a^2-ax-b}{x}=\frac{ax-x^2+a^2-ax-b}{x}=\frac{a^2-x^2-b}{x}$ Ans.

7. Reduce $2+3x-\frac{x-6}{4x}$ and $x-y-\frac{3x-4y-2}{5}$ to improper fractions. Ans. $\frac{12x^2+7x+6}{4x}$ Ans. $\frac{2x-y+2}{5}$

8. Reduce $1-\frac{5x}{y}$, $x+\frac{x^2}{b}$, and $3x-\frac{2x-5}{5}$ to improper fractions. Ans. $\frac{y-5x}{y}$ Ans. $\frac{bx+x^2}{b}$ and $\frac{13x+6}{5}$ Ans.

9. Reduce $5+\frac{2x-8}{3x}$, $1-\frac{x-a-1}{a}$, and $1+2x-\frac{x-3}{5x}$, each to improper fractions. Ans. $\frac{17x+8}{3x}$, $\frac{2a-x+1}{a}$, $\frac{10x^2+4x+3}{5x}$

$$1+2x-\frac{x-3}{5}=\frac{5x(1+2x)-(x-3)}{5x}=\frac{5x+10x^2-x+3}{5x}$$

$$=\frac{10x^2+4x+3}{5x} \text{ Ans.}$$

$$1-\frac{x-a-1}{a}=\frac{a-(x-a-1)}{a}=\frac{a-x+a+1}{a}=\frac{2a-x+1}{a} \text{ Ans.}$$

10. Reduce $5\frac{3}{7}$, $1-\frac{3a}{x}$ $2a-\frac{3ax+a^2}{4x}$, $12+\frac{4x-18}{5x}$, to improper fractions.

11. Reduce $x+\frac{1-3a-c}{c}$, and $4+2x-\frac{2x^2-3a}{5a}$ to an improper fraction.

$\frac{(5\times7)+3}{7}=\frac{38}{7}$ Ans. $1-\frac{3a}{x}=\frac{1\times x}{x}-\frac{3a}{x}=\frac{x-3a}{x}$ Ans.

$\frac{2a\times4x}{4x}-\frac{3ax+a^2}{4x}=\frac{8ax-3ax-a^2}{4x}=\frac{5ax-a^2}{4x}$ Ans.

Ans. $\frac{64x-18}{5x}$ Ans. $\frac{cx+1-3a-c}{c}$ Ans. $\frac{20a+10ax-2x^2+3a}{5a}$

CASE II. To reduce an improper fraction to a whole, or mixed quantity. This is the same as Case II. in division.

RULE. Divide the numerator by the nominator for the integral part; and if there be a remainder, place it over the denominator for the fractional part, with the proper sign prefixed.

1. Reduce $\frac{ab-2a^2}{ab}$, $\frac{a^2+x^2}{a\pm x}$, and $\frac{10x^2-5x+3}{5x}$ to whole quantities.

$\frac{ab-2a^2}{ab}=1-\frac{2a}{b}$ Ans. $\frac{10x^2-5x+3}{5x}=2x-1+\frac{3}{5x}$ Ans.

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and $a\pm x)a^2+x^2(a\pm x+\frac{2x^2}{a\pm x}$ Answer as required.

$$\begin{array}{l} a^2\pm ax \\ \hline \mp ax+x^2 \\ \pm ax-x^2 \\ \hline \text{Remainder } 2x^2 \end{array}$$

Here $a\pm x$ is the integral part of the quantity, and $\frac{2x^2}{a\pm x}$ is the fractional part. Therefore $a\pm x+\frac{2x^2}{a\pm x}$ is the mixed quantity required.

2. Reduce $\frac{x^3+y^3}{x+y}$, $\frac{x^3-y^3}{x-y}$, and $\frac{x^2+4ax+4a^2-y^3}{x+2a}$ to their proper terms. Ans. $x^2\pm xy+y^2$, and $x+2a-\frac{y^3}{x+2a}$ Answer.

CASE III. *Fractions reduced to a common denominator.*

RULE. Multiply each numerator separately by every denominator except its own, for the new numerator, and all the denominators together for a common denominator.

1. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$; or $\frac{ax}{3}$ and $\frac{x}{4}$. Also, $\frac{x}{y}$ and $\frac{x+y}{z}$ each pair to equivalent fractions, that shall have common denominators.

$$\left.\begin{array}{l} 2x\times c=2cx \\ a\times b=ab \end{array}\right\}\text{the numerators} \qquad \left\{\begin{array}{l}\text{The new fractions are} \\ \frac{2cx}{ac} \text{ and } \frac{ab}{ac} \text{ Ans.}\end{array}\right.$$
$a\times c=ac$ the common denominator

$$\left.\begin{array}{l} ax\times 4=4ax \\ 3\times x=3x \end{array}\right\}\text{new numerators.} \qquad \left\{\begin{array}{l}\text{The new fractions are} \\ \frac{4ax}{12} \text{ and } \frac{3x}{12}\end{array}\right.$$
$3\times 4=12$ common denominator

$$\left.\begin{array}{l} x\times z=xz \\ (x+y)\times y=xy+y^2 \end{array}\right\}\text{new numerator.} \qquad \left\{\begin{array}{l}\text{The new fractions are} \\ \frac{xy}{yz} \text{ and } \frac{xy+y^2}{yz}\end{array}\right.$$
$y\times z=yz$ common denominator.

2. Reduce $\frac{2x+3}{x}$ and $\frac{5x+1}{3}$; again, $\frac{7x^2-1}{2x}$ and $\frac{4x^2-x+2}{2a^2}$ each pair of fractions to a common denominator.

Ans. $\frac{6x+9}{3x}$ and $\frac{5x^2+x}{3x}$; Ans. $\frac{14a^2x^2-2a^2}{4a^2x}$, and $\frac{8x^3-2x^2+4x}{4a^2x}$

3. Reduce $\frac{a}{2}$, $\frac{3x}{7}$, and $\frac{a+x}{a-x}$: Again, $\frac{3}{4}$, $\frac{3x}{3}$ and $a+\frac{4x}{5}$ each set of fractions to a common denominator.

$$\left.\begin{array}{l} a\times 7\times(a-x)=7a^2-7ax \\ 3x\times 2\times(a-x)=6ax-6x^2 \\ (a+x)\times 2\times 7=14a+14x \end{array}\right\}\text{new numerators.}$$
$2\times 7\times(a-x)=14a-14x$ common denominator.

Hence $\frac{7a^2-7ax}{14a-14x}$, $\frac{6ax-6x^2}{14a-14x}$ and $\frac{14a+14x}{14a-14x}$ are the Ans.

$$\left.\begin{array}{r} a+\frac{4x}{5}=\frac{5a+4x}{5}\text{, and } 3\times3\times5=45 \\ 2x\times4\times5=40x \\ (5a+4x)\times4\times3=60a+48x \end{array}\right\}\text{ new numerators.}$$

$4\times3\times5=60$ common denominator.

The fractions are $\frac{45}{60}$, $\frac{40x}{60}$, and $\frac{60a+48x}{60}$ Answers.

4. Reduce $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ to fractions having a common denominator.

$$\left.\begin{array}{l} a\times d\times f=adf \\ c\times b\times f=cbf \\ e\times b\times d=ebd \end{array}\right\}\text{ the new numerators.}$$

$b\times d\times f=bdf=$ common denominator.

Hence the fractions required are $\frac{adf}{bdf}$, $\frac{cbf}{bdf}$, and $\frac{ebd}{bdf}$.

5. Reduce $\left.\frac{a+x}{a-x} \text{ and } \frac{a-x}{a+x}\right\}$ or $\frac{5x}{a+x}$, $\frac{a-x}{3}$ and $\frac{1}{2x}$ to a common denominator.

$$\left.\begin{array}{l} (a+x)\times(a+x)=a^2+2ax+x^2 \\ (a-x)\times(a-x)=a^2-2ax+x^2 \end{array}\right\}\text{ new numerators.}$$

$$\left.\begin{array}{l} (a-x)\times(a+x)=a^2-x^2 \\ \text{The common denominator.} \end{array}\right\}\text{ Ans. }\frac{a^2+2ax+x^2}{a^2-x^2}\text{ and }\frac{a^2-2ax+x^2}{a^2-x^2}$$

$$\left.\begin{array}{l} \text{Here } 5x\times3\times2x=30x^2 \\ (a-x)\times(a+x)2x=2a^2x-2x^3 \\ 1\times(a+x)\times3=3a+3x \\ \hline (a+x)\times3\times2x=6ax+6x^2 \end{array}\right\}$$

$\therefore$ The new fractions are $\frac{30x^2}{6ax+6x^2}$, $\frac{2a^2x-2x^3}{6ax+6x^2}$ and $\frac{3a+3x}{6ax+6x^2}$

6. Reduce $\frac{a}{3}$, $\frac{2y}{b}$, and $\frac{a+y}{a-y}$; or $\frac{x}{3}$, $\frac{x+1}{5}$ and $\frac{1-x}{1+x}$ to a common denominator.

$$\left.\begin{array}{l} \text{Here } x\times5\times(1+x)=5x^2+5x \\ (x+1)\times3\times(1+x)=3x^2+6x+3 \\ (1-x)\times3\times5=15-15x \\ \hline 3\times5\times(1+x)=15+15x \text{ common denominator.} \end{array}\right\}$$

$\therefore$ the new fractions are $\frac{5x^2+5x}{15+15x}$, $\frac{15-15x}{15+15x}$ and $\frac{3x^2+6x+3}{15+15x}$ Ans.

$$\left.\begin{array}{l} a\times b\times(a-y)=a^2b-aby \\ 2y\times3\times(a-y)=6ay-6y^2 \\ (a+y)\times3\times b=3ab+3by \\ \hline 3\times b\times(a-y)=3ab-3by \\ \text{common denominator.} \end{array}\right\}$$

$\therefore$ the new fractions are $\frac{a^2b-aby}{3ab-3by}$, $\frac{6ay-6y^2}{3ab-3by}$ and $\frac{3ab+3by}{3ab-3by}$ Ans.

7. Reduce $\frac{b+c}{a+b}$ and $\frac{a-4e}{a-b}$ or $a+\frac{x}{2a}-\frac{x^2}{4a^2}$ and $x-\frac{a}{2x}+\frac{a^2}{4x^2}$, each set to a common denominator.

First $\left.\begin{array}{l}(b+c)\ (a-b)=ab+ac-b^2-bc\\(a-4e)\ (a+b)=a^2+ab-4ae-4be\end{array}\right\}$ new numerators.

$(a+b)\ (a-b)=a^2-b^2=$ the common denominator.

Hence, the factors are $\frac{ab+ac-b^2-bc}{a^2-b^2}$ and $\frac{a^2+ab-4ae-4be}{a^2-b^2}$.

Second, $a+\frac{x}{2a}-\frac{x^2}{4a^2}=\frac{2a^2+x}{2a}-\frac{x^2}{4a^2}=\frac{8a^4+4a^2x-2ax^2}{2a\times 4a^2}=\frac{4a^3+2ax-x^2}{4a^2}$; $x-\frac{a}{2x}+\frac{a^2}{4x^2}=\frac{2x^2-a}{2x}+\frac{a^2}{4x^2}=\frac{8x^4-4ax^2+2a^2x}{2x\times 4x^2}=\frac{4x^3-2ax+a^2}{4x^2}$; $\frac{(4a^3+2ax-x^2)4x^2}{4a^2\times 4x^2}=\frac{4a^3x^2+2ax^3-x^4}{4a^2x^2}=\frac{16a^3x^2+8ax^3-4x^4}{16a^2x^2}$; $\frac{(4x^3-2ax+a^2)4a^2}{4a^2\times 4x^2}=\frac{16a^2x^3-8a^3x+4a^4}{16a^2x^2}=\frac{4a^2x^3-2a^3x+a}{4a^2x^2}$. Answers, $\frac{4a^3x^2+2ax^3-x^4}{4a^2x^2}$ and $\frac{4a^2x^3-2a^3x+a^4}{4a^2x^2}$.

8. Reduce $a+x, \frac{a}{a-x}$ and $\frac{a-x}{a}$, or $2a+\frac{3x}{5}, \frac{a}{a-x}$, and $\frac{a-x}{a}$, to a common denominator. Ans. $2a+2+\frac{3a^2x-3ax^2-5x^2}{5a^2-5ax}$.

$\frac{a+x}{1}, +\frac{a}{a-x}$ and $\frac{a-x}{a}$ are $=\frac{(a+x)\ (a-x)a}{a\times 1\times (a-x)}=\frac{a^3-ax^2}{a^2-ax}$, $\frac{a\times 1\times a}{a\times 1\times (a-x)}=\frac{a^2}{a^2-ax}$ and $\frac{(a-x)\ (a-x)\times 1}{a\times 1\times (a-x)}=\frac{a^2-2ax+x^2}{a^2-ax}$.

9. Reduce $ax+\frac{2x+7}{8}$ and $x-\frac{5x-6}{21}$, or $5x+\frac{x-2}{3}$, and $4x-\frac{2x-3}{5x}$ to a common denominator.

The fractions reduced to a common denominator, are $\frac{42x+147}{168}$ and $\frac{40x-48}{168}$; $\therefore\ ax+\frac{42x+147}{168}$ and $x-\frac{40x-48}{168}$.

Here $5x+\frac{x-2}{3}$ and $4x-\frac{2x-3}{5x}$ are $=\frac{15x+x-2}{3\times 5x}, \frac{20x^2-2x+3}{3\times 5x}$ or $\frac{80x^2-10x}{15x}$ and $\frac{60x^2-6x+9}{15x}$.

10. Reduce $\frac{4x}{7}$ and $\frac{x-2}{5}$, or $5x, \frac{2a}{3x^2}$, and $\frac{a+2x}{4x}$ to a common denominator.

Here $\frac{4x\times5}{7\times5}$, $\frac{7(x-2)}{7\times5}$ or $\frac{20x}{35}$, $\frac{7x-14}{35}$, Ans.

Second, $\frac{5x}{1}$, $\frac{2a}{3x^2}$, $\frac{a+2x}{4x}$ or $=\frac{5x\times3x^2\times4x}{1\times3x^2\times4x}$, $\frac{2a\times1\times4x}{1\times3x^2\times4x}$ and $\frac{(a+2x)\times3x^2\times1}{1\times3x^2\times4x}$ or $\frac{60x^4}{12x^3}$, $\frac{8ax}{12x^3}$, and $\frac{3ax^2+6x^3}{12x^3}$.

On the Method of Finding the Greatest Common Measure of two or more Quantities.

43. One quantity is said to *measure* another, when it is contained in that other a certain number of times, without a remainder.

44. A quantity is said to be a *multiple* of another, when it contains that other quantity a certain number of times, without a remainder.

45. A *common measure* of two or more quantities is any quantity which measures them all; and the *greatest* common measure is the greatest quantity which will so measure them. Thus, $2a$ is a common measure of the quantities $24ab^2$, $16a^2bc$, and $12abc^2$, and their *greatest* common measure is $4ab$.

46. If one quantity measures another, it will also measure any *multiple* of that quantity. Thus, let b measure a by the units in m, then $a=mb$; and let na be a multiple (denoted by the units in n) of a, then $na=nmb$; consequently b measures na by the units in nm.

47. If one quantity measures two others, it will also measure their sum and difference. For let c measure a by the units in m, and b by the units in n, then $a=mc$, and $b=nc$; therefore, $a\pm b$[a] $=mc\pm nc=(m\pm n)c$; consequently c measures $a+b$ (their *sum*) by the units in $m+n$, and $a-b$ (their *difference*) by the units in $m-n$. (The quantity $a\pm b$ means a plus or minus b.)

48. The Rule for finding the greatest common measure of two *numbers* may be thus investigated. Let a and b be any two numbers, whereof a is the greater; and let the following operation be performed upon them: viz.

$$\begin{array}{l} b)a(p \\ \quad \underline{pb} \\ \quad c)b(q \\ \qquad \underline{qc} \\ \qquad d)c(r \\ \qquad\quad \underline{rd} \\ \qquad\quad 0 \end{array}$$

Where a divided by b gives the *quotient* p, and *remainder* c; b divided by c, the quotient q, and remainder d; c divided by d, the quotient r, and remainder 0. Then, since in each case the *dividend* is equal to the *divisor* multiplied by the *quotient* plus the *remainder*, we have $c=rd$.

$b = qc + d =$ (since $qc = qrd$) $qrd + d = (qr+1)d$

$a = pb + c = \left\{\begin{matrix}\text{since } pb = (pqr+p)d \\ \text{and } \quad c = \quad rd\end{matrix}\right\} (pqr+p+r)d.$ Hence, since p, q, r are *whole numbers*, d is contained in b as many times as there are units in $qr+1$, and in a as many times as there are units in $pqr+p+r$; consequently the *last divisor* d is a common measure of a and b; and this is evidently the case, whatever be the length of the operation, provided that it be carried on till the remainder is nothing.

The last divisor d is also the *greatest* common measure of a and b. For let x be any common measure of a and b, such that $a = mx$, and $b = nx$, then

$c = a - pb = mx - pnx = (m - pn)x$

$d = b - qc = nx - (qm - pqn)x = (n - qm + pqn)x$; $\therefore$ x measures d by the units in $n - qm + pqn$, that is, *every* common measure of a and b measures d. Now it has been shown that d is a common measure of a and b; and the greatest measure of d is evidently *itself*; consequently d is the greatest common measure of a and b. Hence this rule for finding the greatest common measure of two numbers: "Divide the greater by the lesser, and the preceding divisor by the last remainder, till nothing remains; the last divisor is the greatest common measure."

To find the greatest common measure of *three* numbers, a, b, c; let d be the greatest common measure of a and b, and x the greatest common measure of d and c; then x is the greatest common measure of a, b, and c. For, let $a = md$, $b = nd$, $d = px$; then $a = mpx$, and $b = npx$, therefore x is a common measure of a and b; and, since it also measures c, it will be a common measure of a, b, and c. But, as above, *every* common measure of a and b measures d; therefore, *every* common measure of a, b, and c, measures d and c; and consequently the greatest common measure of d and c, or x, will also be the greatest common measure of a, b, and c.

In general, let there be any set of numbers, a, b, c, d, e, &c.; and let x be the greatest common measure of a and b; y the greatest common measure of x and c; z the greatest common measure of y and d; &c., &c.; then will y be the greatest common measure of a, b, c; z the greatest common measure of a, b, c, d; &c., &c.

49. To find the greatest *simple* common measure of *Algebraic* quantities, the rule is, "to find the greatest common measure of their coefficients, and then annex to it the letters common to all the quantities;" thus the greatest common measure of $24ax^2y^2$, $16bxy$, and $6axy^2$, is $2xy$.

To find the greatest *compound* common measure of two alge-

braic quantities, "first divide each of them by their greatest *simple* common measure (if they have one); arrange their terms according to the dimensions of the same letter, and divide either, or both of them, by the greatest simple factor which it may contain; then perform on them the same operation as that for finding the greatest common measure of two *numbers*, observing only, that the remainders which arise are to be divided by their greatest simple factors, and that the dividends may, if requisite, be multiplied by any simple quantity which will make the first term of the dividend a multiple of the first term of the divisor. Lastly, multiply the compound common measure thus obtained by the *simple* one originally taken out, and the product will be the greatest common measure required."

CASE IV. *To find the greatest common measure, or divisor, of the terms of a fraction, consisting of compound quantities, or to reduce a fraction to its lowest, or most simple terms.*

RULE I. Range the quantities according to the powers, or dimensions of some letter, as in Division, and by inspection expunge the common factor or factors, if any, either in the divisor or dividend; then divide that quantity which has the highest power by the other, whether it be the numerator or denominator; and divide the last divisor by the last remainder, and so on till nothing remains; the last divisor will be the greatest common measure: but if such a divisor cannot be found, the fraction has no common measure. Having found the greatest common measure, divide the terms of the fraction by it, and the resulting fraction will be in its lowest terms.

NOTE. If any of the divisors become negative, they may have their signs changed without altering the truth of the result. Also, if the first term of a divisor be not contained an exact number of times in the first term of the dividend, the latter may be multiplied by any quantity that will make the division complete. And lastly, any quantity which is common to all the terms of the dividend or divisor, may be expunged before the division is commenced.

1. Find the greatest common measure of $\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$ and then reduce the fraction to its lowest terms.

$$
\begin{array}{l}
6a^2+7ax-3x^2)6a^2+11ax+3x^2(1 \\
\qquad\qquad\qquad \underline{6a^2+7ax-3x^2}
\end{array}
$$

Now dividing this $4ax+6x^2$ remainder by $2x$, the quotient is $2a+3x$; and dividing the divisor $6a^2+7ax-3x^2$ by this quotient, we have $2a+3x)6a^2+7ax-3x^2(3a-x$

$$
\begin{array}{l}
\underline{6a^2+9ax} \\
\quad -2ax-3x^2 \\
\quad -2ax-3x^2
\end{array}
$$

, and since there is no remainder,

therefore $2a+3x$, the last divisor, is the greatest common measur required. $2a+3x)\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}=\frac{3a-x}{3a+x}$ Ans.

In this example the dimension of the numerator is the same a that of the denominator; therefore, it makes no difference whethe we divide the numerator by the denominator, or the denominato by the numerator. This will be evident from the following ope ration: $6a^2+11ax+3x^2)6a^2+7ax-3x^2(1$

$$\frac{6a^2+11ax+3x^2}{}$$

Now, dividing this $-4ax-6x^2$ remainder by $-2x$, th quotient is $2a+3x$; or, as the remainder above is now, negative And, by changing all the signs, we obtained $4ax+6x^2$, or $2a+3x$ for the new divisor. $2a+3x)6a^2+11ax+3x^2(3a+x$

$$\frac{6a^2+9ax}{2ax+3x^2}$$
$$2ax+3x^2$$

Hence $2a+3x$ is found to be the greatest common measure, as before.

2. Required to find the greatest common measure of $\frac{x^2-xy-2y^2}{x^2-3xy+2y^2}$ or $\frac{x^2-1}{ax+a}$ or $\frac{7a^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}$ and $\frac{x^3-a^3}{x^4-a^4}$ and then reduce each fraction to their lowest terms.

$x^2-xy-2y^2)x^2-3xy+2y^2(1$

$$\frac{x^2-xy-2y^2}{}$$

Divide this $-2xy+4y^2$ remainder, each term of which can be divided by $2y$, and the signs of the terms being changed, we ob tain $x-2y$ for a new divisor. $x-2y)x^2-xy-2y^2(x+y$

$$\frac{x^2-2xy}{+xy-2y^2}$$
$$+xy-2y^2$$

and since there is no remainder, therefore $x-2y$, the last divisor, is the greatest common measure required.

$$x-2y)\frac{x^2-xy-2y^2}{x^2-2xy+3y^2}=\frac{x+y}{x-y}\text{ Answer.}$$

In this example, the dimensions of the numerator is the sam as that of the denominator; therefore it makes no differenc whether we divide the numerator by the denominator, or the de nominator by the numerator. The same remark will apply to th next example. This will be evident from the following operatio

$x^2-3xy+2y^2)x^2-xy-2y^2(1$

$$\frac{x^2-3xy+2y^2}{}$$

$2xy-4y^2$ the remainder, which is now positiv and divide by $+2y$, and we obtain $x-2y$, for our new divisor, th same as before.

Here the greatest common measure is x^2-8x-3; therefore,

$x^2-8x-3)\dfrac{2x^3-16x-6}{3x^3-24x-9}=\dfrac{2}{3}$ Ans. as required.

Or thus, $\dfrac{2x^3-16x-6}{3x^3-24x-9}=\dfrac{2(x^2-8x-3)}{3(x^2-8x-3)}=\dfrac{2}{3}$ as before.

$$\left.\begin{array}{l}ax+a, \text{ or}\\ x+1\end{array}\right\}\begin{array}{l}x^2-1(x-1, \text{ the least common measure.}\\ \underline{x^2+x}\\ \quad -x-1\\ \quad -x-1\end{array}\qquad x+1)\frac{x^2-1}{ax+a}=\frac{x-1}{a}\text{ Ans.}$$

3d expression solved. Multiply the denominator by 7, we shall have $7a^2-23ab+6b^2)35a^3-126a^2b+77ab^2-42b^3(5a-11b$

$$\begin{array}{l}35a^3-115a^2b+30ab^2\\ \hline -11a^2b+47ab^2-42b^3\\ \text{Multiply by 7}\\ \hline -77a^2b+329ab^2-294b^3\\ -77a^2b+253ab^2-66b^3\\ \hline\end{array}$$

Divide this remainder by $76b^2$ and $76ab^2-228b^3$ we have for the divisor $a-3b)7a^2-23ab+6b^2(7a-2b$

$$\begin{array}{l}7a^2-21ab\\ \hline -2ab+6b^2\\ -2ab+6b^2\end{array}$$

Therefore $a-3b$ is the greatest common measure sought.

$$a-3b)\ \frac{7a^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}=\frac{7a-2b}{5a^2-3ab+2b^2}\text{ Ans.}$$

Now, for the last fraction in example second, the greatest common measure of the fraction is $x-a)\dfrac{x^3-a^3}{x^4-a^4}=\dfrac{x^2+ax+a^2}{x^3+ax^2+a^2x+a^3}$ Ans.

3. Reduce $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ and $\dfrac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$ each to its lowest terms. $x^4+ax^3-a^3x-a^4)x^4+a^2x^2+a^4(1$

$$x^4+ax^3-a^3x-a^4$$

This remainder, $-ax^3+a^2x^2+a^3x+2a^4$ which, being divided by a, and the signs of the terms being changed, becomes $x^3-ax^2-a^2x-2a^3)x^4+ax^3-a^3x-a^4(x+2a$

$$\begin{array}{l}x^4-ax^3-a^2x^2-2a^3x\\ \hline 2ax^3+a^2x^2+a^3x-a^4\\ 2ax^3-2a^2x^2-2a^3x-4a^4\\ \hline\end{array}$$

Divide this remainder, $3a^2x^2+3a^3x+3a^4$ by $3a^2$, we obtain x^2+ax+a^2 for our new divisor.

$x^2+ax+a^2)x^3-ax^2-a^2x-2a^3(x-2a$

$$\begin{array}{l}x^3+ax^2+a^2x\\ \hline -2ax^2-2a^2x-2a^3\end{array}$$

$-2ax^2-2a^2x-2a^3$
$-2ax^2-2a^2x-2a^3$

$$x^2+ax+a^2)\frac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a}=\frac{x^2-ax+a^2}{x^2-a^2} \text{ Ans.}$$

$a^3-a^2x-ax^2+x^3)a^4-x^4(a$
$a^4-a^3x-a^2x^2+ax^3$

Divide this remaind. $a^3x+a^2x^2-ax^3-x^4$ by x
$a^3+a^2x-ax^2-x^3(a^3-a^2x-ax^2+x^3(1$
$a^3-a^2x-ax^2-x^3$

Divide this remainder $-2a^2x+2x^3$ which is now negative, by $-2x$.

$$\frac{-2a^2x+2x^3}{-2x}=a^2-x^2)\,a^3+a^2x-ax^2-x^3\,(a$$
a^3-ax^2

Divide this remainder a^2x-ax^3 by x, and we obtain a^2-x^2 for our new divisor. $a^2-x^2)a^2+x^2(-1$
$-a^2+x^2$

$$a^2-x^2)\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}=\frac{a^2+x^2}{a-x} \text{ the answer}$$

4. Reduce the three following fractions, which are from Simpson's 'Algebra,' to their lowest terms, viz:
$\frac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}$, or $\frac{x^3+ax^2+bx^2-2a^2x+bax-2ba^2}{x^2-bx+2ax-2ab}$ and $\frac{x^4-3ax-8a^2x^2+18a^3x-8a^4}{x^3-ax^2-8a^2x+6a}$. Ans. $\frac{x^2-ax+bx-ab}{ab-b}$

Here, dividing first by the greatest simple divisors, $5a^3$ and b, we have $a^2+2ab+b^2$, and $a^3+2a^2b+2ab^2+b^3$: and if the latter of these be divided by the former, the work will stand thus:

$a^2+2ab+b^2)a^3+2a^2b+2ab^2+b^3(a$
$a^3+2a^2b+ab^2$

where the remainder is $+ab^2+b^3$; which being divided by b^2, its greatest simple divisor, gives $a+b$; by this divide $a^2+2ab+b^2$, and the quotient will come out $a+b$, exactly; therefore, the last divisor, $a+b$, will exactly measure both quantities. ∴ the fraction given will be reduced to $\frac{5a^4+5a^3b}{a^2b+ab^2+b^3}$. Again,

$a^3-a^2x-ax^2+a^3)a^4+0+0+0-x^4(a+x$
$a^4-a^3x-a^2x^2+ax^3$
$a^3x+a^2x^2-ax^3-x^4$
$a^3x-a^2x^2-ax^3+x^4$
$+2a^2x^2+0-2x^4$

$a^2+0-x^2)a^3-a^2x-ax^2+x^3(a-x$
a^3-0-ax^2
$-a^2x+0+x^3$
$-a^2x+0+x^3$. From whence it appears that a^2+0-x^2

or a^2-x^2 will measure both a^4-x^4, and $a^3-a^2x-ax^2+x^3$; and, by dividing thereby, the fraction proposed is reduced to $\frac{a^2+x^2}{a-x}$.

$$x^3-ax^2-8a^2x+6a^3)\,x^4-3ax^3-8a^2x^2+18a^3x-8a^4\,(x-2a$$
$$x^4-ax^3-8a^2x^2+6a^3x$$
$$-2ax^3+0+12a^3x-8a^4$$
$$-2ax^3+2a^2x^2+16a^3x-12a^4$$

remainder $-2a^2x^2-4a^3x+4a^4$; which, divided by $-2a^2$, gives $x^2+2ax-2a^2$ for the next divisor.

$$x^2+2ax-2a^2)\,x^3-ax^2-8a^2x+6a^3\,(x-3a$$
$$x^3+2ax^2-2a^2x$$
$$-3ax^2-6a^2x+6a^3$$
$$-3ax^2-6a^2x+6a^3$$

$$x^2+2ax-2a^2)\,x^4-3ax^3-8a^2x^2+18a^3x-8a^4\,(x^2-5ax+4a^2$$
$$x^4+2ax^3-2a^2x^2$$
$$-5ax^3-6a^2x^2+18a^3x$$
$$-5ax^3-10a^2x^2+10a^3x$$
$$+4a^2x^2+8a^3x-8a^4$$
$$+4a^2x^2+8a^3x-8a^4$$

Now if, by proceeding in this manner, no compound divisor can be found, that is, if the last remainder be only a simple quantity, we may conclude the case proposed does not admit of any, but is already in its lowest terms. Thus, for instance, if the fraction proposed were to be $\frac{a^3+2a^2x+3ax^2+4x^3}{a^2+ax+x^2}$, it is plain by inspection, that it is not reducible by any simple divisor; but to know whether it may not, by a compound one, I proceed as above, and find the last remainder to be the simple quantity $7x^2$; whence I conclude that the fraction is already in its lowest terms.

Another observation may be here made, in relation to fractions that have in them more than two different letters. When one of the letters rises only to a single dimension, either in the numerator or in the denominator, it will be best to divide the said numerator or denominator (whichever it is) into two parts, so that the said letter may be found in every term of the one part, and be totally excluded out of the other; this being done, let the greatest common divisor of these two parts be found; which will evidently be a divisor to the whole; and by which the division of the other quantity is to be tried; as in the following example, where the fraction given is $\frac{x^3+ax^2+bx^2-2a^2x+bax-2ba^2}{x^2-bx+2ax-2ab}$. Here the denominator being the least compounded, and b rising therein to a sin-

gle dimension only, I divide the same into the parts x^2+2ax, and $-bx-2ab$, which, by inspection, appear to be equal to $(x+2a)\times x$, and $(x+2a)\times -b$. Therefore $x+2a$ is a divisor to both the parts, and likewise to the whole, expressed by $(x+2a)\times(x-b)$; so that one of these two factors, if the fraction given can be reduced to lower terms, must also measure the numerator; but the former will be found to succeed, the quotient coming out exactly $x^2-ax+bx-ab$; whence the fraction itself is reduced to $\frac{x^2-ax+bx-ab}{x-b}$; which is not reducible further, by $x-b$, since the division does not terminate without a remainder, as upon trial will be found.

$$a^3-b^3)a^4-x^4(a \quad \text{(See p. 36, ex. 2, last expression.)}$$
$$a^4-ab^3$$

Divide this $+ab^3-b^4$ by b^3, we have $a-b)a^3-b^3(a^2+ab+b^2$

$$a^3-a^2b$$
$$+a^2b-b^3$$
$$a^2b-ab^2$$
$$+ab^2-b^3$$
$$+ab^2-b^3$$

Ans. $a-b$.

$$a^3-a^2x-ax^2+x^3)a^4-x^4(a+x$$
$$a^4-a^3x-a^2x^2+ax^3$$
$$a^3x+a^2x^2-ax^3-x^4$$
$$a^3x-a^2x^2-ax^3+x^4$$

Dividing this remainder, $2a^2x^2-2x^4$ by $2x^2$, or leaving out $2x^2$, which is found in each term of the remainder, the next divisor is

$$a^2-x^2)a^3-a^2x-ax^2+x^3(a-x$$
$$a^3-ax^2$$
$$-a^2x+x^3$$
$$-a^2x+x^3$$

Hence a^2-x^2 is therefore $-a^2x+x^3$, the greatest common measure of the two quantities, and if they be respectively divided by it, the fraction is reduced to $\frac{a^2+x^2}{a-x}$, its lowest terms.

This quantity, $2x^2$, found in every term of one of the divisors, $2a^2x^2-2x^4$, but not in every term of the dividend, $a^3-a^2x-ax^2+x^3$, must be left out; otherwise the quotient will be fractional, which is contrary to the supposition made in the rule: and by omitting this part, $2x^2$, no common measure of the divisor and dividend is left out; because, by the supposition, no part of $2x^2$ is found in all the terms of the dividend.

5. Reduce $\frac{a^6-b^6}{a^4-b^4}$, $\frac{x^2-a^2}{x^4-a^4}$, $\frac{x^4-a^4}{x^5-a^2x^3}$, $\frac{3x^2-22x+39}{x^3-11x^2+39x-45}$ each to its lowest terms. Ans. $\frac{a^4+a^2b^2+b^4}{a^2+b^2}$, $\frac{1}{x^2+a^2}$, and $\frac{3x-13}{x^2-8x+15}$.

$x^2-a^2)x^4-a^4(x^2$
$\quad x^4-a^2x^2$

This divided $+a^2x^2-a^4$ by a^2 gives $x^2-a^2)x^2-a^2(1$
$\quad x^2-a^2$

Hence x^2-a^2 is the simple divisor.

$$x^2-a^2)\frac{x^2-a^2}{x^4-a^4}=\frac{1}{x^2-a^2}$$

$a^4-b^4)a^6-b^6(a^2$
$\quad a^6-a^2b^4$

$+a^2b^4-b^6$ This, divided by b^4, gives $a^2-b^2)a^4-b^4(a^2$
$\quad a^4-a^2b^2$

This, again divided, $a^2b^2-b^4$ by b^2, gives $a^2-b^2)a^2-b^2(1$
$\quad a^2-b^2$

Therefore a^2-b^2 is the simple divisor.

Hence $a^2-b^2)\dfrac{a^6-b^6}{a^4-b^4}(\dfrac{a^4+a^2b^2+b^4}{a^2+b^2}$

6. Reduce $\dfrac{48x^4+16x^2-15}{24x^4-22x^2+5}$, and $\dfrac{20x^4+x^2-1}{25x^4+5x^3-x-1}$ each of these fractions to its lowest terms.

The greatest common measure is $12x^2-5$ and $5x^2-1$, and the reduced fractions are $\dfrac{4x^2+3}{2x^2-1}$ and $\dfrac{4x^2+1}{5x^2+x-1}$.

7. Reduce $\dfrac{6a^5+15a^4b-4a^3c^2-10a^2bc^2}{9a^3b-27a^2bc-6abc^2+18bc^3}$; $\dfrac{a^2b^2-c^2d^2-a^2c^2+c^4}{4a^2d-4acd-2ac^2+2c^3}$, each to its lowest terms.

8. Reduce $\dfrac{9x^5+2x^3+4x^2-x+1}{15x^4-2x^3+10x^2-x+2}$, $\dfrac{c^3-b^3}{c^4-b^3c}$, and $\dfrac{a^2-b^2}{a^4-b^4}$ each to its lowest terms. (For ans. to 7 and 8, see p. 59.)

ADDITION OF FRACTIONS IN ALGEBRA.

RULE. Reduce the fractions to a common denominator, add the numerators together, and under this sum place the common denominator; the result will be the sum of the fractions required.

1. Add $\dfrac{3x}{2a}$ and $\dfrac{x}{5}$; or $\dfrac{x}{2}, \dfrac{x}{3}$ and $\dfrac{x}{4}$; or $\dfrac{4x}{7}$ and $\dfrac{x-2}{5}$ together.

$$\frac{3x\times5+2a\times x}{2a\times5=10a}=\frac{15x+2ax}{10a}, \text{ or } \frac{3\times4\times x}{2\times3\times4}+\frac{2\times x\times4}{2.3.4}+\frac{x\times2\times3}{2.3.4}$$
$$=\frac{13x}{12};\ \frac{4x\times5+7(x-2)}{5\times7}=\frac{20x+7x-14}{35}=\frac{27x-14}{35}.$$

2. Add $2a$, $3a+\dfrac{2x}{5}$ and $a-\dfrac{8x}{9}$; or $2a+\dfrac{3x}{5}$, $\dfrac{a}{a-x}$ and $\dfrac{a-x}{a}$ together.

$2a+3a+a=6a,\ +\frac{2x}{5}-\frac{8x}{9}=\frac{18x-40x}{5\times9}=6a-\frac{22x}{45}.$

3. Add $\frac{2x}{5}$ and $\frac{5x}{7}$, or $15x$ and $\frac{1+2x}{8}$ or $\frac{ax}{b-c}$ and $\frac{ax}{b+c}$ together.

$\frac{2x.7+5.5x}{5\cdot7}=\frac{14x+25x}{35}=\frac{39x}{35}$, or $\frac{15x.8+1+2x}{1\times8}=\frac{120x+1+2x}{8}=\frac{122x+1}{8}$, $\frac{ax(b+c)+ax(b-c)}{(b-c)\times(b+c)}=\frac{abx+acx+abx-acx}{b^2-c^2=(b+c)\,(b-c)}=\frac{2abx}{b^2-c^2}$, Ans.

4. Add $\frac{12x}{7}$ and $\frac{3x}{5}$ or $\frac{5y}{3}$ and $\frac{3y}{8}$ or $\frac{10x-9}{8}$ and $\frac{3x-5}{7}$ together.

$\frac{12x\cdot5+7\cdot3x}{7\cdot5}=\frac{60x+21x}{35}=\frac{81x}{35}$, or $\frac{5y\cdot8+3\cdot3y}{3\cdot8}=\frac{40y+9y}{24}=\frac{49y}{24}$, or $\frac{(10x-9)\times7+(3x-5)\times8}{8\cdot7=8\times7=56}=\frac{70x-63+24x-40}{56}=\frac{94x-103}{56}$, Ans.

5. Add $\frac{a}{b}$, $\frac{2a}{3b}$ and $\frac{5b}{4a}$ or $\frac{2x+3}{5}$, $\frac{3x-1}{2x}$ and $\frac{4x}{7}$ together.

$$\left.\begin{array}{l} a\times3b\times4a=12a^2b \\ 2a\times b\times4a=8a^2b \\ 5b\times b\times3b=15b^3 \end{array}\right\} \therefore \frac{12a^2b+8a^2b+15b^3}{12ab^2}=\frac{20a^2b+15b^3}{12ab^2}=(\text{di-}$$
$$b\times3b\times4a=12ab^2 \quad \text{viding by } b)\ \frac{20a^2+b^2}{12ab}\ \text{Ans.}$$

$$\left.\begin{array}{l} (2x+3)\times2x\times7=28x^2+42x \\ (3x-1)\times5\times7=105x-35 \\ \quad 4x\times5\times2x=40x^2 \\ \hline 5\times2x\times7=70x=\text{com. den.} \end{array}\right\} \begin{array}{l} \therefore\frac{28x^2+42x+105x-35+40x^2}{70x} \\ =\frac{68x^2+147x-35}{70x}\ \text{Ans.} \end{array}$$

6. Add $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ or $\frac{x}{x-3}$ and $\frac{x}{x+3}$ or $\frac{2x-5}{3}$ and $\frac{x-1}{2x}$ together.

Ans., $\frac{2a^2+2b^2}{a^2-b^2}$ or $\frac{2x^2}{x^2-9}$ or $\frac{4x^2-7x-3}{6x}$ as required.

$$\left.\begin{array}{l} (a+b)\,(a+b)=a^2+2ab+b^2 \\ (a-b)\,(a-b)=a^2-2ab+b^2 \\ \hline (a-b)\,(a+b)=a^2-b^2 \end{array}\right\} \begin{array}{l} \therefore\frac{a^2+2ab+b^2+a^2-2ab+b^2}{a^2-b^2}= \\ \frac{2a^2+2b^2}{a^2-b^2}\ \text{Ans.} \end{array}$$

$$\frac{x(x+3)+x(x-3)}{(x-3)\ (x+3)}=\frac{x^2+3x+x^2-3x}{x^2-9}=\frac{2x^2}{x^2-9}\text{ or}$$

$$\frac{(2x-5)2x-3(x-1)}{2x\times3=6x}$$

7. Add $\frac{x-3}{x+3}$ and $\frac{x+3}{x-3}$ or $\frac{a}{a+b}$, $\frac{a-b}{c+d}$ and $\frac{a+b}{a-d}$ or $\frac{3x-7}{8}$ and $\frac{4x}{7}$, or $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d together.

$$\frac{(x-3)\ (x-3)+(x+3)\ (x+3)}{(x+3)\ (x-3)=x^2-9}=\frac{x^2+6x+9+x^2-6x+9}{x^2-9}=\frac{2x^2+18}{x^2-9}$$

$$\frac{(3x-7)7+8\times4x=21x-49+32x}{8\times7=56}=\frac{53x-49}{56}\text{ Ans. to the last.}$$

Here
$$\left.\begin{array}{l} 3x\times3c=9\ cx \\ 2b\times2a=4\ ab \\ d\times2a\times3c=6acd \end{array}\right\}\text{ new nums.}$$
$1\times3c\times2a=6ac$, com. denom.

Hence the fractions required are $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$ and $\frac{6acd}{6ac}$.

Here the three fractions when reduced are $\frac{3}{4}$, $\frac{2x}{3}$ and $a+\frac{4x}{5}=\frac{5a+4x}{5}$;

$$\left.\begin{array}{l} 3\times3\times5=45 \\ 2x\times4\times5=40x \\ (5a+4x)\times4\times3=60a+48x \end{array}\right\}\text{ nums.}$$
$4\times3\times5=60$, com. denom.

$\therefore$ The new fractions are $\frac{45}{60}$, $\frac{40x}{60}$ and $\frac{60a+48x}{60}$

$$\left.\begin{array}{lcl} a\times7\times(a-x) & = & 7a^2-7ax \\ 3x\times2\times(a-x) & = & 6ax-6x^2 \\ (a-x)\times2\times7 & = & 14a+14x \end{array}\right\}\text{ nums.}$$
$2\times(a-x)\times7=14a-14x$ common denominator.

Hence $\frac{7a^2-7ax}{14a-14x}$, $\frac{6ax-6x^2}{14a-14x}$ and $\frac{14a+14x}{14a-14x}$ as required.

8. Reduce $\frac{x}{3}$, $\frac{x+1}{5}$, and $\frac{1-x}{1+x}$ to a common denominator.

$$\left.\begin{array}{lcl} x\times5\times(1+x) & = & 5x+5x^2 \\ 3\times(x+1)\times(1+x)=3(x^2+2x+1) & = & 3x^2+6x+3 \\ 3\times5\times(1-x)=\quad15(1-x) & = & 15-15x \end{array}\right\}\text{ numerators.}$$
$3\times5\times(1+x)=15+15x$, the common denominator.

Hence the new fractions are $\frac{5x+5x^2}{15+15x}$, $\frac{3x^2+6x+3}{15+15x}$, and $\frac{15x-15x}{15+15x}$, as required.

9. Add $\frac{4x}{3a}$, $\frac{2x}{5b}$ or $\frac{a}{3}$, $\frac{a}{4}$, $\frac{a}{5}$ or $\frac{2a-3}{4}$ and $\frac{5a}{8}$, each set of fractions together.

10. Add $2a+\frac{a+3}{5}$ to $4a+\frac{2a-5}{4}$ or $6a$, $\frac{3a^2}{4b}$, and $\frac{a+b}{3b}$ together.

11. Add $\frac{5a}{4}$, $\frac{6a}{5}$, and $\frac{3a+2}{7}$, or $2a$, $\frac{3a}{8}$ and $3+\frac{a}{6}$ each together.

12. Add $8a+\frac{3a}{4}$ and $2a-\frac{5a}{8}$, or $\frac{10a}{9}$ and $\frac{4a}{7}$ together.

SUBTRACTION OF ALGEBRAIC FRACTIONS.

51. **Rule.** Reduce the fractions to a common denominator, if required; then place the difference of their numerators over the common denominator, and it will be the difference required.

1. From $\frac{4x}{5}$ take $\frac{3x+1}{x+1}$, or $\frac{4x+2}{3}$ take $\frac{2x-3}{3x}$, or $\frac{1}{x-y}$ take $\frac{1}{x+y}$.

$\frac{4x}{5}-\frac{3x+1}{x+1}=\frac{4x(x+1)-5(3x+1)}{5(x+1)}=\frac{4x^2+4x-15x-5}{5x+5}=\frac{4x^2-11x-5}{5x+5}$, or $\frac{(4x+2)3x-(2x-3)3}{3\times3x=9x}=\frac{12x^2+6x-6x+9}{9x}=\frac{12x^2+9}{9x}=\frac{4x^2+3}{3x}$, or $\frac{1}{x-y}-\frac{1}{x+y}=\frac{1\times(x+y)-1\times(x-y)}{(x-y)(x+y)=x^2-y^2}=\frac{2y}{x^2-y^2}$, Ans. to the last.

2. From $a+\frac{a-x}{a(a+x)}$ take $\frac{a+x}{a(a-x)}$, or $a+\frac{a-x}{a+x}$ take $a-\frac{a+x}{a-x}$.

Reduced $\frac{a-x}{a(a+x)}$ and $\frac{a+x}{a(a-x)}$ are $\frac{a^2-2ax+x^2}{a(a^2-x^2)}$, and $\frac{a^2+2ax+x^2}{a(a^2-x^2)}$; $\therefore$ $a+\frac{a^2-2ax+x^2}{a(a^2-x^2)}-\frac{a^2+2ax+x^2}{a(a^2-x^2)}=a-\frac{4x}{a(a^2-x^2)}$, Ans.

2d. Reduced are $a+\frac{a^2-2ax+x^2}{a^2-x^2}-(a-\frac{a^2+2ax+x^2}{a^2-x^2})=\frac{2a^2+2x^2}{a^2-x^2}$.

3. From $x+\frac{x}{2b}$ take $x-\frac{x-a}{c}$, or $4x+\frac{3x}{c}$ take $3x-\frac{2x-3b}{c}$.

$x+\frac{x}{2b}-(x-\frac{x-a}{c})=\frac{x}{2b}+\frac{x-a}{c}=\frac{cx}{2bc}+\frac{2bx-2ba}{2bc}$ $=\frac{cx+2bx-2ba}{2bc}$; $4x+\frac{3x}{a}-(3x-\frac{2x-3b}{a})=\frac{(4ax+3x)}{a}$ $-\frac{3ax-2x+3b}{a}=\frac{ax+5x-3b}{a}$.

4. From $\frac{1}{x-y}$ take $\frac{1}{x+y}$, or $5x+\frac{4x-6}{7}$ take $2x+\frac{7x-12}{13}$.

$\frac{13(4x-6)-7(7x-12)}{7\times 13=91}=\frac{52x-78-49x+84}{91}=\frac{3x+6}{91}$, and $5x-2x=3x$; $\therefore\ 3x+\frac{3x+6}{91}$, Answer.

5. From $\frac{a+b}{a-b}$ take $\frac{a-b}{a+b}$, or from $\frac{10x-9}{8}$ take $\frac{3x-5}{7}$.

$$\left.\begin{array}{l}(a+b)(a+b)=a^2+2ab+b^2\\(a-b)(a-b)=a^2-2ab+b^2\\ \hline (a+b)(a-b)=a^2-b^2\end{array}\right\}\therefore\frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2-b^2}=\frac{4ab}{a^2-b^2}$$

as required.

$$\left.\begin{array}{l}(10x-9)7=70x-63\\(3x-5)8=24x-40\\ \hline 8\times 7=7\cdot 8=56\end{array}\right\}\therefore\frac{70x-63-24x+40}{56}=\frac{46x-23}{56}$$

is the fraction required, or answer.

6. From $2x+\frac{2x+7}{8}$ take $x-\frac{5x-6}{21}$, or $3x+\frac{11x-10}{15}$ take $2x+\frac{3x-5}{7}$.

By art. 42, ex. 5, the fractions are $\frac{42x+147}{168}$ and $\frac{40x-48}{168}$; therefore $2x+\frac{42x+147}{168}-(x-\frac{40x-48}{168})=(2-1)x+\frac{82x+99}{168}$.

By art. 42, the fractions are $\frac{45x-75}{105}$ and $\frac{77x-70}{105}$; then $3x+\frac{77x-70}{105}-(2x+\frac{45x-75}{105})=x+\frac{32x+5}{105}$, Ans.

7. From $15y$ take $\frac{1+2y}{8}$, or $\frac{12x}{7}$ take $\frac{3x}{5}$, or $\frac{ax}{b-c}$ take $\frac{ax}{b+c}$.

Answers, $\frac{39x}{35}$, $=x+\frac{4x}{35}$, or $\frac{118y-1}{8}$, or $\frac{2acx}{b^2-c^2}$.

8. Take $\frac{3a}{4}$ from $6a$, or $\frac{5a}{4}-\frac{2a}{3}$, or $\frac{3a+c}{b}-\frac{2b}{c}$.

9. Take $\frac{2a+6}{9}$ from $\frac{4a+8}{5}$, or $2a-\frac{a-3b}{c}$ from $4a+\frac{2a}{c}$.

The fractions reduced to a common denominator.

$$\frac{2x}{5}+\frac{5x}{7}=\frac{2x\times7}{5\times7}+\frac{5x\times5}{5\times7}=\frac{14x}{35}+\frac{25x}{35}=\frac{39x}{35}, \text{ Ans.}$$

Here $\frac{3x}{2a}+\frac{x}{5}=\frac{2a\times x}{2a\times5}+\frac{3x\times5}{2a\times5}=\frac{2ax}{10a}+\frac{15x}{10a}=\frac{15x+2ax}{10a}$, Ans.

Here $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=\frac{12x}{24}+\frac{8x}{24}+\frac{6x}{24}=\frac{13x}{12}$, Ans.

$$\frac{4x}{7}+\frac{x-2}{5}=\frac{20x}{35}+\frac{7(x-2)}{35}=\frac{20x+7x-14}{35}=\frac{27x-14}{35}, \text{ Ans.}$$

Here these fractions may be written thus; $2a+3a+a+\frac{2x}{5}-\frac{8x}{9}=6a+\frac{2x}{5}-\frac{8x}{9}=6a+\frac{18x}{45}-\frac{40x}{45}=6a-\frac{22x}{45}$, Ans.

Here the fractions reduced to a common denominator are $\frac{3a^2x-3ax^2}{5a^2-5ax}$, $\frac{5a^2}{5a^2-5ax}$, and $\frac{5a^2-10ax+5x^2}{5a^2-5ax}$; therefore the sum is $2a+\frac{3a^2x-3ax^2+10a^2-10ax+5x^2}{5a^2-5ax}=2a+2+\frac{3a^2-3ax^2+5x^2}{5a^2-5ax}$ Ans.

The fractions reduced are $\frac{5x^2-10x}{15x}$, and $\frac{6x-9}{15x}$; $\therefore 5x+\frac{5x^2-10x}{15x}+4x-\frac{6x-9}{15x}=9x+\frac{5x^2-16x+9}{15x}$, Ans.

But if the fractions were $5x$, $\frac{x+2}{3}$ and $4x-\frac{2x-3}{5x}$ the final result would be $=9x+\frac{x+2}{3}-\frac{2x-3}{5x}=9x+\frac{5x^2+10x}{15x}-\frac{6x-9}{15x}=9x+\frac{5x^2+10x-6x+9}{15x}=9x+\frac{5x^2+4x+9}{15x}$, Ans.

The fractions reduced to a common denominator are $\frac{8a}{12x^2}$, and $\frac{3ax+6x^2}{12x^2}$; $\therefore 5x+\frac{8a}{12x^2}+\frac{3ax+6x^2}{12x^2}=5x+\frac{8a+3ax+6x^2}{12x^2}$, Ans.

Add $a+x$, $\frac{a}{a-x}$, and $\frac{a-x}{a}$ together.

MULTIPLICATION OF ALGEBRAIC FRACTIONS.

52. Rule. Multiply all the numerators together for the numerator of the product, and all the denominators together for its denominator; the former placed over the latter will give the product required.

1. Multiply $\frac{2x}{5}$ by $\frac{3x}{2}$, or $\frac{3x}{2}$ by $\frac{5x}{3b}$, or $\frac{2x}{x-1}$ by $\frac{3x}{7}$.

$\frac{2x\times3x}{5\times2a}=\frac{6x^2}{10a}=\frac{3x^2}{5a}$, or $\frac{5x\times3x}{2\times3b}=\frac{15x^2}{2b}$, or $\frac{2x\times3x}{7(x-1)}=\frac{6x^2}{7x-7}$.

2. Multiply $\frac{3x^2-5x}{14}$ by $\frac{7a}{2x^3-3x}$, or $\frac{3x^2}{5x-10}$ by $\frac{15x-30}{2x}$.

$\frac{3ax-5a}{4x^2-6}$; or $\frac{3x^2(15x-30)}{2x(5x-10)}=\frac{3x^2\times3}{2x}=\frac{9x^2}{2x}=\frac{9x}{2}$, Ans.

3. Multiply $\frac{x-2}{5}$ by $\frac{4x}{7}$, or $2x$, $\frac{3x}{4}$ by $\frac{5x}{7}$, or $7x$, $\frac{2x}{5}$ by $\frac{3x^2}{2a}$.

$\frac{(x-2)4x}{5\times7}=\frac{4x^2-8x}{35}$, or $\frac{2x\times3x\times5x}{1\times4\times7}=\frac{30x^3}{28}=\frac{15x^3}{14}$, Ans.

Lastly, $\frac{7x\times2x\times3x^2}{1\times5\times2a}=\frac{7x\times x\times3x^2}{1\times5\times a}=\frac{21x^4}{5a}$, Ans.

4. Multiply $\frac{2x}{3}$, $\frac{4x^2}{7}$ by $\frac{a}{a+x}$, or $3x$, $\frac{x+1}{2a}$ by $\frac{x-1}{a+x}$.

5. Multiply $2a+\frac{bx}{a}$ by $3a-\frac{b}{ax}$, or $5a+\frac{b}{3x}$ by $3a-\frac{b}{5x}$.

6. Multiply $a+\frac{ax}{a-x}$ by $\frac{a^2-x^2}{a-x}$, or $3x$, $\frac{x+1}{2a}$ by $\frac{x-1}{a+b}$.

7. Multiply $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$, and $a+\frac{ax}{a-x}$ together.

$a+\frac{ax}{a-x}=\frac{a^2}{a-x}$; $\therefore \frac{a^2-x^2}{a+b}\times\frac{a^2-b^2}{ax+x^2}\times\frac{a^2}{a-x}=\frac{a-b}{x}\times a^2=\frac{a^3-a^2b}{x}$.

8. Multiply $a+\frac{x}{2a}-\frac{x^2}{4a^2}$ by $x-\frac{a}{2x}+\frac{a^2}{4x^2}$.

By art. 42, ex. 7, last set, $(\frac{4a^3+2ax-x^2}{4a^2})\times(\frac{4x^3-2ax+a^2}{4x^2})=$
$\frac{4a^3-8a^4x+16a^3x^3-5a^2x^2+2a^2x+2ax^3+8ax^4-4x^5}{16a^2x^2}$, Ans.

9. Multiply $\frac{4a}{3}$ by $\frac{6a}{5c}$ or $\frac{3a}{4}$ by $\frac{4b^2}{3a}$ or $\frac{3a}{b} \times \frac{8ac}{b}$ by $\frac{4ab}{3c}$.

10. Multiply $2a+\frac{ab}{2c}$ by $\frac{3a^2}{b}$ or $\frac{2a^2-2b^2}{3bc}$ by $\frac{4a^2+2b^2}{a+b}$ together.

11. Multiply $3a$ and $\frac{2a+1}{a}$ by $\frac{2a}{2a+b}$ or $\frac{2x}{a}$, $\frac{3ab}{c}$ by $\frac{5ac}{2b}$.

Ans., $15ax$.

$\frac{4a}{3} \times \frac{6a}{5c} = \frac{24a^2}{15c} = \frac{8a^2}{5c}$ or $\frac{3a}{4} \times \frac{4b^2}{3a} = b^2$ or $\frac{3a}{b} \times \frac{8ac}{b} \times \frac{4ab}{3c} = \frac{32a^3}{b}$, Ans.

$\frac{12a^3c+3a^3b}{2bc}$, or $\frac{8a^4-4a^2b^2-4b^4}{3abc+3b^3c}$ or $\frac{12a^2-3}{2a+b}$, Ans.

DIVISION OF FRACTIONS IN ALGEBRA.

53. RULE. Invert the divisor, and then proceed as in Multiplication.

EXAMPLES.

1. Divide $\frac{x^2-9}{5}$ by $\frac{x+3}{4}$ or $\frac{4x^2}{7}$ by $5x$, or $\frac{9y^2-3y}{7}$ by $\frac{3y^2}{7}$, or $\frac{x+1}{6}$ by $\frac{2x}{3}$.

$\frac{x^2-9}{5} \div \frac{x+3}{4} = \frac{(x^2-9)4}{(x+3)5} = \frac{4x-12}{5}$ Ans., or $\frac{4x^2}{7} \div 5 = \frac{4x^2}{7} \times \frac{1}{5x} = \frac{4x}{35}$ Ans. $\frac{9y^2-3y}{7} \div \frac{3y^2}{7} = \frac{9y^2-3y}{7} \times \frac{7}{3y^2} = \frac{3y-1}{y}$ Ans., or $\frac{x+1}{6} \div \frac{2x}{3} = \frac{3x+3}{6 \times 2x} = \frac{x+1}{4x}$.

2. Divide $\frac{x}{1-x}$ by $\frac{x}{5}$ or $\frac{9y^2}{2y}$ by $\frac{3y^2}{5y-10}$ or $\frac{7x}{5}$ by $\frac{3}{x}$.

$\frac{x}{1-x} \div \frac{x}{5} = \frac{x}{1-x} \times \frac{5}{x} = \frac{5x}{x-x^2} = \frac{5}{1-x}$ Ans. $\frac{9y^2}{2y} \div \frac{3y^2}{5y-10} = \frac{9y^2}{2y} \times \frac{5y-10}{3y^2} = \frac{3}{2y} \times \frac{5y-10}{1} = \frac{15y-30}{2y}$ Ans., or $\frac{7x}{5} \div \frac{3}{x} = \frac{7x}{5} \times \frac{x}{3} = \frac{7x^2}{15}$, Ans.

3. Divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$ or $\frac{x^4-b^4}{x^2-2b+b^2}$ by $\frac{x^2+bx}{x-b}$

$\frac{(2a+x)x}{c^3-x^3} \div \frac{x}{c-x} = \frac{2ax+x^2}{c^3-x^3} \times \frac{c-x}{x} = \frac{2a+x}{c^2+cx+x^2}$, Ans.

Again, $\frac{x^4-b^4}{x^2-2bx+b^2} \div \frac{x^2+bx}{x-b} = \frac{x^4-b^4}{(x-b)^2} \times \frac{x-b}{x^2+bx} =$
$\frac{(x^2+b^2)(x^2-b^2)(x-b)}{(x-b)(x-b)(x+b)\times x} = \frac{(x^2+b^2)(x^2-b^2)}{(x^2-b^2)\times x} = \frac{x^2+b^2}{x}$, Ans.

4. Divide $\frac{x^2+xy}{x-y}$ by $\frac{x^4-y^4}{(x-y)^2}$ or $\frac{5a^4-5b^4}{2a^2-4ab+2b^2}$ by $\frac{6a^2+5ab}{4a-4b}$.

$\frac{(x^2+xy)(x^2-2xy+y^2)}{(x-y)(x^4-y^4)} = \frac{(x^2+xy)(x-y)}{x^4-y^4} = \frac{x^2+xy=x(x+y)}{x^3+x^2y+xy^2+y^3}$.

$\frac{5a^4-5b^4}{2a^2-4ab+2b^2} \div \frac{6a^2+5ab}{4a-4b} = \frac{(5a^4-5b^4)\times(4a-4b)}{(2a^2-4ab-2b^2)(6a^2+5ab)}$.
$\frac{(5\times2)(a^4-b^4)}{(a-b)(6a^2+5ab)} = \frac{5.2.(a^3+a^2b+ab^2+b^3}{1\times(6a^2+5ab)} = \frac{10(a^3+a^2b+ab^2+b^3}{6a^2+5ab}$.

5. Divide $\frac{3x}{4}$ by $\frac{11}{12}$ or $\frac{6x^2}{5}$ by $3x$, or $\frac{3x+1}{9}$ by $\frac{4x}{3}$ or $\frac{4x}{2x-1}$ by $\frac{x}{3}$.

$\frac{3x}{4} \div \frac{11}{12} = \frac{3x}{4} \times \frac{12}{11} = \frac{9x}{11}$ or $\frac{6x^2}{5} \div \frac{3x}{1} = \frac{6x^2}{5} \times \frac{1}{3x} = \frac{2x}{5}$, A.

6. Divide $\frac{4x}{5}$ by $\frac{3a}{5b}$ or $\frac{2a-b}{4cd}$ by $\frac{5ac}{6d}$ or $\frac{5a^4-5b^4}{2a^2-4ab+2b^2}$ by $\frac{6a^2+5ab}{4a-4b}$.

$\frac{2a-b}{4cd} \div \frac{5ac}{6d} = \frac{(2a-b)}{4cd} \times \frac{6d}{5ac} = \frac{6a-3b}{10ac^2}$, Ans.

$\left(\frac{5a^4-5b^4}{2a^2-4ab+2b^2}\right) \times \left(\frac{4a-4b}{6a^2+5ab}\right) = \frac{5(a^4-b^4)}{a-b} \times \frac{2}{6a^2+5ab} =$
$\frac{5(a^3+a^2b+ab^2+b^3)}{1} \times \left(\frac{2}{6a^2+5ab}\right) = \frac{10(a^3+a^2b+ab^2+b^3)}{6a^2+5ab}$, A.

7. Divide $\frac{7x}{5}$ by $\frac{3}{x}$ or $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$ or $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$

8. Divide $\frac{x^2-a^2}{x^3+a^3}$ by $\frac{x-a}{x+a}$ or $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$.

$\frac{3x+1}{9} \div \frac{4x}{3} = \frac{(3x+1)}{9} \times \frac{3}{4x} = \frac{1}{4} + \frac{1}{12x}$ or $\frac{4x}{2x-1} \div \frac{x}{3}$
$= \frac{4x}{(2x+1)} \times \frac{3}{x} = \frac{12}{2x-1}$ or $\frac{4x}{5} \div \frac{3a}{5b} = \frac{4x}{5} \times \frac{5b}{3a} = \frac{4bx}{3a}$, Ans.

5 7

INVOLUTION.

54. Involution is the raising of any given quantity to any proposed power; such as the square, cube, &c.

If a quantity be continually multiplied by itself, it is said to be involved, or raised; and the power to which it is raised is expressed by the number of times the quantity has been employed in the multiplication; thus $a\times a$, or a^2, is called the second power of a; $a\times a\times a$, or a^3, the cube, or third power; $a\times a\ldots.(n)$, or a^n, the nth power.

Rule. When the quantity or root has no index, its power will be represented by placing the index of the required power above it; thus the fourth power of x is x^4; the fourth power of $x+y$ is $(x+y)^4$. If the quantity proposed be a compound one, the involution may either be represented by the proper index, or it may actually take place. If the quantity to be involved be negative, the signs of the even powers will be positive, and the signs of the odd powers negative; for $-a\times-a=a^2$; $-a\times-a\times-a=-a^3$. If the quantity be a fraction, raise both the numerator and denominator to the same power. When the quantity to be raised is itself a power, multiply the index of the quantity by the index of the power; thus the cube of a^2 is $a^6=a^{2\times3}$: or generally let m or n represent any powers whatever; then x^m, raised to the nth power is $x^{m\times n}$, or x^{mn}; and $-x^m$, raised to any power, n, will give $+$, plus, or $-$, minus, according as n is an odd or even number. If n be used for an uneven number, the sign will be $+$, if odd $-$; then $-x^m$ to the nth power, will be represented by $\pm x^{mn}=\pm x^{m\times n}$.

Roots and Powers of Numbers.

Root	1	2	3	4	5	6	7	8	9	10
Square	1	4	9	16	25	36	49	64	81	100
Cube	1	8	27	64	125	216	343	512	729	1000
4th power	1	16	81	256	625	1296	2401	4096	6561	10000
5th power	1	32	243	1024	3125	7776	16807	32768	59049	100000

The operation is performed in the same manner for simple algebraic quantities, except that in this case it must be observed, that the powers of negative quantities are alternately $+$ and $-$; the even powers being positive, and the odd powers negative. Thus the square of $+2a$ is $+2a\times+2a$, or $4a^2$; the square of

$-2a$ is $-2a\times-2a$, or $+4a^2$; but the cube of $-2a=-2a\times-2a\times-2a=+4a^2\times-2a=-8a^3$.

The several powers of $\frac{a}{b}$ are, square $=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}$, cube $=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^3}{b^3}$, 4th power $=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^4}{b^4}$.

Upon this principle the powers of the several roots in the following table are calculated.

Roots and Powers of Simple Algebraic Quantities.

Root	a	$-b$	$2b^2$	$\frac{a}{2b}$	$-\frac{3x^2}{y}$	$\frac{2a}{3b}$	a^2b	$-\frac{a^2}{b^3}$	$-\frac{3x}{5}$	$\frac{x}{4y}$
Square	a^2	$+b^2$	$4b^4$	$\frac{a^2}{4b^2}$	$+\frac{9x^4}{y^2}$	$\frac{4a^2}{9b^2}$	a^4b^2	$+\frac{a^4}{b^6}$	$+\frac{9x^2}{25}$	$\frac{x^2}{16y^2}$
Cube	a^3	$-b^3$	$8b^6$	$\frac{a^3}{8b^3}$	$-\frac{27x^6}{y^3}$	$\frac{8a^3}{27b^3}$	a^6b^3	$-\frac{a^6}{b^9}$	$-\frac{27x^3}{125}$	$\frac{x^3}{64y^3}$
4th power	a^4	$+b^4$	$16b^8$	$\frac{a^4}{16b^4}$	$+\frac{81x^8}{y^4}$	$\frac{16a^4}{81b^4}$	a^8b^4	$+\frac{a^8}{b^{12}}$	$+\frac{81x^4}{625}$	$\frac{x^4}{256y^4}$
5th power	a^5	$-b^5$	$32b^{10}$	$\frac{a^5}{32b^5}$	$-\frac{243x^{10}}{y^5}$	$\frac{32a^5}{243b^5}$	$a^{10}b^5$	$-\frac{a^{10}}{b^{15}}$	$-\frac{243x^5}{3125}$	$\frac{x^5}{1024y^5}$

What is the square of $a+2b$?

$$\begin{array}{rl}
 & a+2b \\
 & a+2b \\
\hline
 & a^2+2ab \\
 & \quad\;\; +2ab+4b^2 \\
\hline
\text{Sq.}= & a^2+4ab+4b^2 \\
 & a+2b \\
\hline
 & a^3+4a^2b+\;4ab^2 \\
 & \quad\;\; +2a^2b+\;8ab^2+8b \\
\hline
\text{Cube} & a^3+6a^2b+12ab^2+8b^3
\end{array}$$

What is the cube of a^2-x?

$$\begin{array}{rl}
 & a^2-x \\
 & a^2-x \\
\hline
 & a^4-\;a^2x \\
 & \quad\;\; -\;a^2x+x^2 \\
\hline
\text{Square}= & a^4-2a^2x+x^2 \\
 & a^2-x \\
\hline
 & a^6-2a^4x+\;a^2x^2 \\
 & \quad\;\; -\;a^4x+2a^2x^2-x^3 \\
\hline
\text{Cube}= & a^6-3a^4x+3a^2x^2-x^3
\end{array}$$

It is very well known that the value of the figures in the common arithmetical scale increases in a tenfold proportion from the right to the left; a number, therefore, may be expressed by the addition of the units, tens, hundreds, &c. of which it consists.

A number of 2 figures may be expressed by $10a+b$.

............3 figures................by $100a+10b+c$.

............4 figures............by $1000a+100b+10c+d$.

EVOLUTION.

55. Evolution is the reverse of Involution, and consists in finding the square, cube, &c. roots of any given quantity.

Case I. *To extract the roots of a simple quantity, or powers.*

Rule. Extract the root of the numerical coefficient, if it have any, as in common arithmetic; then divide the index of the given power by 2, for the square root, 3 for the cube root, 4 for the biquadrate root, and depending on the index of the root required; thus the square root $9x^2 = 3x^{\frac{2}{2}} = 3x^1 = 3x$; and the cube root of $8x^6 = 2x^{\frac{6}{3}} = 2x^2$.

If the coefficient be a fraction, extract the root both of its numerator and denominator; thus, the square $\frac{4}{9}x^2 = \frac{2}{3}x$, or $\frac{1}{2}x^2 = \frac{1}{4}x$.

1. Find the square root of $16a^2x^2 = 16 \times a^2 \times x^2$, or
$\sqrt{16a^2x^2} = \sqrt{16} \times \sqrt{a^2} \times \sqrt{x^2} = \pm 4 \times a \times x = \pm 4ax$, Ans.
2. Find the cube root of $8a^3x^3$, or $(8 \times a^3 \times x^3)$.
$\therefore \sqrt[3]{8a^3x^3} = \sqrt[3]{8} \times \sqrt[3]{a^3} \times \sqrt[3]{x^3} = 2 \times a \times x = 2ax$, Ans.
3. Find the cube root of $-125a^3x^6$, or $-125 \times a^3 \times x^6$.
$\therefore \sqrt[3]{-125a^3x^6} = \sqrt[3]{-125} \times \sqrt[3]{a^3} \times \sqrt[3]{x^6} = -5 \times a \times x^2 = -5ax^2$ Ans.
4. Find the 5th root of $-\frac{32x^{10}y^5}{243a^{15}}$.

$$\sqrt[5]{-\frac{32x^{10}y^5}{243a^{15}}} = -\frac{2x^{\frac{10}{5}}y^{\frac{5}{5}}}{3a^{\frac{15}{5}}} = -\frac{2x^2y}{3a^3}, \text{ Ans.}$$

5. Find the square root of $2a^2b^4$, or $\frac{8a^2b^2}{3c^3}$, or $4a^2x^6$, or $\frac{4a^4}{9x^2y^2}$.
6. Find the cube root of $-64a^3b^6$, or $-125a^3x^6$, or $\frac{8a^3}{125x^6}$.
7. Find the fourth root of $81a^4b^6$, or $256a^4x^8$. $3ab\sqrt{b}$, or $4ax^2$.
8. Find the fifth root of $-32a^5b^6$, or $-\frac{32a^5x^{10}}{243}$.

$\sqrt{2a^2b^4} = \sqrt{(a^2b^4 \times 2)} = ab^2\sqrt{2}$, or $\sqrt{(\frac{8a^2b^2}{3c^3}} = \sqrt{(\frac{4a^2b^2}{c^2} \times \frac{2}{3c}) =}$ $\frac{2ab}{c}\sqrt{\frac{2}{3c}}$, or $\sqrt{4a^2x^6} = \sqrt{4}\sqrt{a^2}\sqrt{x^6} = 2ax^3$, or $\sqrt{\frac{4a^4}{9x^2y^2}} = \frac{\sqrt{4}\sqrt{a^4}}{\sqrt{9}\sqrt{x^2}\sqrt{y^2}} = \frac{2a^2}{3xy}$, Ans.
$\sqrt[3]{-64a^3b^6} = -4ab^2$, or $\sqrt[3]{-125a^3x^6} = \sqrt[3]{125}\sqrt[3]{a^3}\sqrt[3]{x^6} = -5ax^2$ or

$\sqrt[3]{\frac{8a^3}{125x^6}} = \frac{\sqrt[3]{8}\sqrt[3]{a^3}}{\sqrt[3]{125}\sqrt[3]{x^6}} = \frac{2a}{5x^2}$, Ans. $\sqrt[4]{81a^4b^6} = \sqrt[4]{(81a^4b^4\times b^2)} = 3ab\sqrt{b}$, or $\sqrt[4]{256a^4x^8} = \sqrt[4]{256}\sqrt[4]{a^4}\sqrt[4]{x^8} = 4ax^2$, Ans. $\sqrt[5]{-32a^5b^6} = \sqrt[5]{(-32a^5b^5\times b)} = -2ab\sqrt[5]{b}$, Ans. or $\sqrt[5]{\frac{-32a^5x^{10}}{243}}$

$$= \frac{\sqrt[5]{-32}\sqrt[5]{a^5}\sqrt[5]{x^{10}}}{\sqrt[5]{243}} = \frac{-2ax^2}{3}, \text{ Ans.}$$

CASE II. *To extract the square root of a compound quantity.*

RULE. Arrange the terms according to the power of some letter, as in division. Find the root of the first term, and set it in the quotient. Subtract the square of the root thus found from the first term, and there will be no remainder. Then bring down the next two terms for a dividend, and take double (or two times) the root already found for a divisor. Divide the dividend by the divisor, and put the result both in the quotient and divisor. Then proceed as in common arithmetic.

$$\begin{array}{lll}
 & a^2+2ab+b^2+2ac+2bc+c^2 & (a+b+c \\
 & a^2 & \\ \hline
2a+b) & 2ab+b^2 & \\
 & 2ab+b^2 & \\ \hline
2a+2b+c) & 2ac+2bc+c^2 & \\
 & 2ac+2bc+c^2 & \\ \hline
\end{array}$$

$$\begin{array}{lll}
1. & 9x^4-12x^3+16x^2-8x+4 & (3x^2-2x+2 \\
 & 9x^4 & \quad 2 \\ \hline
6x^2-2x & -12x^3+16x^2 & 6x^2 \text{ constant trial divisor.} \\
\text{Divisor.} & -12x^3+4x^2 & \\ \hline
6x^2-4x+2 & 12x^2-8x+4 & \\ \hline
\end{array}$$

First, we extract the square root of $9x^4$, the first term. This gives $3x^2$ for the first term of the root required. This we place on the right hand of the dividend, in the manner of the quotient, as in division. Squaring this term, and subtracting it from the dividend, we get for a remainder $-12x^3+16x^2-8x+4$: we now double $3x^2$, and place it as a divisor on the left of this remainder, and dividing by it, $6x^2$, the first term of the remainder, we get or obtain the quotient $-2x$, the second term of the root sought, which we annex, with the proper sign, to the double root $6x^2$; multiply the whole of this quantity, $6x^2-2x$, by $-2x$, and subtracting the product from the first remainder, we obtain for a second remainder $12x^2-8x+4$; then by doubling $3x^2-2x$, the two terms of the root thus found, and dividing $12x^2$, the first term of the new remainder, by $6x^2$, the first term of the double root, we obtain $+2$ for a quotient, which is the third term of the root sought; and by annexing it to the double root $6x^2-4x$, and then multiplying the whole of this quotient, $6x^2-4x+2$, by 2, and subtracting the product from the second remainder, we find 0 for a new remainder, which shows that the root required is $3x^2-2x+2$.

2. $4x^6+12x^5+5x^4-2x^3+7x^2-2x+1(2x^3+3x^2-x+1$

Mult. $4x^6$ $\qquad$ 2

$4x^3$ constant trial divis.

$4x^3+3x^2 \mid 12x^5+5x^4$

Mult. $\mid 12x^5+9x^4$

$4x^3+6x^2-x \mid -4x^4-2x^3+7x^2$

$\mid -4x^4-6x^3+x^2$

$4x^3+6x^2-2x+1 \mid 4x^3+6x^2-2x+1$

$\mid 4x^3+6x^2-2x+1$

Root.

3. $9x^6-12x^5+10x^4-10x^3+5x^2-2x+1(3x^3-2x^2+x-1$

$9x^6$ $\qquad$ $2,$

Constant $6x^3$ trial divisor.

$6x^3-2x^2)-12x^5+10x^4$

1st divis. $-12x^5+4x^4$

$6x^3-4x^2+x)6x^4-10x^3+5x^2$

2d divisor. $6x^4-4x^3+x^2$

$6x^3-4x^2+2x-1)-6x^3+4x^2-2x+1$

3d divisor. $-6x^3+4x^2-2x+1$

Dividend.

4. $4x^3-20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}}+24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}}$

$4x^3$

$2x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}}-6y^{\frac{5}{6}}b^{\frac{1}{2}}$ root.

$4x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}})-20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}}$

1st divisor. $-20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}}$

$4x^{\frac{3}{2}}-10x^{\frac{3}{4}}a^{\frac{2}{3}}+6xy^{\frac{5}{6}}b^{\frac{1}{2}})+24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}}$

2d divisor. $24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}}$

5. What is the square root of $1+x$?

$1+x(1+\frac{1}{2}x-\frac{1}{4}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+$ &c.

1

$2+\frac{1}{2}(x$

$x+\frac{1}{4}x^2$

$2+x-\frac{1}{8}x^2)-\frac{1}{4}x^2$

$-\frac{1}{4}x^2-\frac{1}{8}x^3+\frac{1}{64}x^4$

$2+x-\frac{1}{4}x^2+\frac{1}{16}x^3)\frac{1}{8}x^3-\frac{1}{64}x^4$

$\frac{1}{8}x^3+\frac{1}{16}x^4-\frac{1}{64}x^5+\frac{1}{256}x^6$

$2+x-\frac{1}{4}x^2+\frac{1}{8}x^3+$, &c.$)-\frac{5}{64}x^4+\frac{1}{64}x^5-\frac{1}{256}x^6$

Ex. 6.

$a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6(a^3-3a^2x+3ax^2-x^3$
a^6

$2a^3-3a^2x)-6a^5x+15a^4x^2$
$-6a^5x+9a^4x^2$

$2a^3-6a^2x+3ax^2)\ 6a^4x^2-20a^3x^3+15a^2x^4$
$6a^4x^2-18a^3x^3+9a^2x^4$

$2a^3-6a^2x+6ax^2-x^3)-2a^3x^3+6a^2x^4-6ax^5+x^6$
$-2a^3x^3+6a^2x^4-6ax^5+x^6$

7. Extract the square root of $1\pm x$.

$1\pm x)1\pm\frac{1}{2}x-\frac{1}{8}x^2\pm\frac{1}{16}x^3-\frac{5}{128}x^4\pm$, &c.
1

$2\pm\frac{1}{2}x)\pm x$
$\pm x+\frac{1}{4}x^2$

$2\pm x-\frac{1}{8}x^2)-\frac{1}{4}x^2$
$-\frac{1}{4}x^2\mp\frac{1}{8}x^3+\frac{1}{64}x^4$

$2+x-\frac{1}{4}x^2+\frac{1}{16}x^3)\pm\frac{1}{8}x^3-\frac{1}{64}x^4$
$\pm\frac{1}{8}x^3+\frac{1}{16}x^4\mp\frac{x^5}{64}+\frac{1}{256}x^6$

$2\pm x-\frac{1}{4}x^2\pm\frac{1}{8}x^3-\frac{5}{128}x^4)^*-\frac{5}{64}x^4\pm\frac{1}{64}x^5-\frac{1}{256}x^6$

$(-\frac{1}{16}x^4\times4=-\frac{4}{64}x^4$
$+\frac{1}{64}x^4$
$^*\ -\frac{5}{64}x^4$

8. Extract the square root of $a^2\pm x^2$.

$a^2\pm x^2\left|a\pm\frac{x^2}{2a}-\frac{x^4}{8a^3}\pm\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\pm\right.$, &c. answer.
a^2

$2a\pm\frac{x^2}{2a}\left|\pm x^2\right.$
$\pm x^2+\frac{x^4}{4a^2}$

$2a\pm\frac{x^2}{a}-\frac{x^4}{8a^3}\left|-\frac{x^4}{4a^2}\right.$
$-\frac{x^4}{4a^2}\mp\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$

$2a\pm\frac{x^2}{a}-\frac{x^4}{4a^3}\pm\frac{x^6}{16a^4}\left|\pm\frac{x^6}{8a^4}-\frac{x^8}{64a^6}\right.$
$\pm\frac{x^6}{8a^4}+\frac{x^8}{16a^6}\mp\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}}$

$2a\pm\frac{x^2}{a}-\frac{x^4}{4a^3}\pm\frac{x^6}{8a^5}-\frac{5x^8}{128a^7}\left|-\frac{5x^8}{64a^6}\pm\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}}\right.$, &c. and so on to an infinite number of terms.

$1+1)1\pm\frac{1}{2}-\frac{1}{8}\pm\frac{1}{16}-\frac{5}{128}\pm$, &c.
1

$2\pm\frac{1}{2})\pm1$
$\pm1+\frac{1}{4}$

$2\pm1-\frac{1}{8})-\frac{1}{4}$
$-\frac{1}{4}\mp\frac{1}{8}+\frac{1}{64}$
$\pm\frac{1}{8}-\frac{1}{64}$, &c.

$$a^2+x\left(a+\frac{x}{2a}-\frac{x^2}{8a^3}+\frac{x^3}{16a^5},\right. \text{ \&c. root.}$$
$$a^2$$
$$2a+\frac{x}{2a})x$$
$$+x+\frac{x^2}{4a^2}$$
$$2a+\frac{x}{a}-\frac{x^2}{8a^3})-\frac{x^2}{4a^2}$$
$$-\frac{x^2}{4a^2}-\frac{x^3}{8a^4}+\frac{x^4}{64a^6}$$
$$\text{Rem. }+\frac{x^3}{8a^4}-\frac{x^4}{64a^6}$$

$$a^2-2bx-x^2\left(a-\frac{2bx+x^2}{2a}-\frac{4b^2x^2+4bx^3+x^4}{8a^3}\right.$$
$$a^2$$
$$2a-\frac{2bx+x^2}{2a})-2bx-x^2$$
$$-2bx-x^2+\frac{4b^2x^2+4bx^3+x^4}{4a^2}$$
$$\text{Rem. }-\frac{4b^2x^2+4bx^3+x^4}{4a^2}$$

$$a^4+4a^3+6a^2+4a+1(a^2+2a+1 \text{ root.}$$
$$a^4$$
$$2a^2+4a)4a^3+6a^2$$
$$4a^3+4a^2$$
$$2a^2+4a+1)+2a^2+4a+1$$
$$2a^2+4a+1$$

$$a^4-2a^3+2a^2-a+\tfrac{1}{4}(a^2-a+\tfrac{1}{2}$$
$$a^4$$
$$2a^2-a)-2a^3+2a^2$$
$$-2a^3+a^2$$
$$2a^2-2a+\tfrac{1}{2})a^2-a+\tfrac{1}{4}$$
$$a^2-a+\tfrac{1}{4}$$

$$a^2-ab\left(a-\frac{b}{2}-\frac{b^2}{8a}-\frac{b^3}{16a^2},\right. \text{ \&c. the root.}$$
$$a^2$$
$$2a-\frac{b}{2})-ab$$
$$-ab+\frac{b^2}{4}$$
$$2a-b-\frac{b^2}{8a})-\tfrac{1}{4}b^2$$
$$-\tfrac{1}{4}b^2+\frac{b^3}{8a}+\frac{b^4}{64a^2}$$
$$2a-b-\frac{b^2}{4a}-\frac{b^3}{16a^2})-\frac{b^3}{8a}-\frac{b^4}{64a^2}$$

$$1-x^2(1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^4-\tfrac{1}{16}\text{—, \&c. root.}$$
$$1$$
$$2-\tfrac{1}{2}x^2)-x^2$$
$$-x^2+\tfrac{1}{4}x^4$$
$$2-x^2-\tfrac{1}{8}x^4)-\tfrac{1}{4}x^4$$
$$-\tfrac{1}{4}x^4+\tfrac{1}{8}x^6+\tfrac{1}{64}x^8$$
$$\text{Remainder, }-\tfrac{1}{8}x^6-\tfrac{1}{64}x^8$$

16. Extract the square root of $4x^4-16x^3+24x^2-16x+4$.
Ans. $2x^2-4x+2$.

17. Extract the square root of $16x^4+24x^3+89x^2+60x+100$.
Ans. $4x^2+3x+10$.

18. Extract the square root of $1\pm y$.

Ans. $1\pm\frac{y}{2}-\frac{y^2}{8}\pm\frac{y^3}{16}-\frac{5y^4}{128}\pm$, &c.

19. Extract the square root of $9x^6-12x^5+10x^4-28x^3+17x^2-8x+16$. Ans. $3x^3-2x^2+x-4$.

To extract the Cube Root of a Compound Quantity.

56. RULE. Arrange the terms according to the dimensions of some letter, as in division, and extract the root of the first term, which must be a cube. Place this root in the quotient, subtract its cube from the first term, and there will be no remainder. Bring down the three next terms for a dividend, and put three times the square just found in the divisor's place, and see how often it is contained in the first term of the dividend, and the quotient is the next term of the root. Add three times the product of the two terms of the root, plus the square of the last term, to the terms already in the divisor's place, and the divisor will be complete. Multiply the divisor by the last term of the root, subtract the product from the dividend, and bring down the next three terms for a dividend, and proceed as before.

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3(a+b \\
 & a^3 \\
3a^2+3ab+b^2) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3
\end{array}$$

$\{(a\times a)3+3(a\times b)+(b\times b)\}=3a^2+3ab+b^2=$ the first divisor.
$\{(a+b)^2\times 3+3(a+b)c+c\times c\}=3a^2+6ab+3b^2+3ca+3cb+c^2$, the divisor for the third letter in the root.
$3(a+b+c)^2+3d(a+b+c)+(d\times d)=$ divisor for the fourth term of the root, &c. &c.

1. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$\begin{array}{r|l}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1(x^2-2x+1. \\
 & x^6 \\
3x^4-6x^3+4x^2 & -6x^5+15x^4-20x^3 \\
 & -6x^5+12x^4-8x^3 \\
3x^4-12x^3+15x^2-6x+1 & 3x^4-12x^3+15x^2-6x+1 \\
 & 3x^4-12x^3+15x^2-6x+1
\end{array}$$

2. Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

$$\begin{array}{r|l}
 & x^6+6x^5-40x^3+96x-64(x^2+2x-4 \\
 & x^6 \\
3x^4+6x^3+4x^2 & 6x^5-40x^3 \\
 & 6x^5+12x^4+8x^3 \\
3x^4+12x^3-24x+16 & -12x^4-48x^3+96x-64 \\
 & -12x^4-48x^3+96x-64
\end{array}$$

3. Extract the cube root of $27x^9-54x^8+63x^7-71x^6+57x^5-36x^4+22x^3-9x^2+3x-1$.

Dividend.

$$27x^9-54x^8+63x^7-71x^6+57x^5-36x^4+22x^3-9x^2+3x-1$$

$$27x^9 \qquad \text{Root or quotient } 3x^3-2x^2+x+1$$

$$27x^6-18x^5+4x^4 \,\Big|\, -54x^8+63x^7-71x^6$$

$$\qquad\qquad -54x^8+36x^7-8x^6$$

$$27x^6-36x^5+21x^4-6x^3+x^2 \,\Big|\, 27x^7-63x^6+57x^5-36x^4+22x^3$$

$$\qquad\qquad 27x^7-36x^6+21x^5-6x^4+x^3$$

$$27x^6-36x^5+30x^4-21x^3+9x^2-3x+1 \,\Big|\, -27x^6+36x^5-30x^4+21x^3-9x^2+3x-1$$

$$\qquad\qquad -27x^6+36x^5-30x^4+21x^3-9x^2+3x-1$$

4. Extract the cube root of $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$.

Ans. $a+b+c$.

5. $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6(x^2+2xy+y^2$

$$x^6$$

$$3x^4+6x^3y+4x^2y^2)6x^5y+15x^4y^2+20x^3y^3$$

$$6x^5y+12x^4y^2+8x^3y^3$$

$$3x^4y^2+12x^3y^3+15x^2y^4+6xy^5+y^6$$

$$3x^4y^2+12x^3y^3+15x^2y^4+6xy^5+y^6$$

$$3(x^2+2xy)^2=3x^4+12x^3y+12x^2y^2$$

$$3(x^2+2xy)\times y^2=3x^2y^2+6xy^3$$

$(3x^4+12x^3y+15x^2y^2+6xy^3+y^4)$ last divisor.

If the root consists of three terms, a, b, c, they may be obtained by first finding a and b, as above, and then deriving c from $(a+b)$ in the same manner that b was derived from a.

$$(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3(a+b+c$$

$$(a+b)^3$$

$$3(a+b)^2+3(a+b)c+c^2)3(a+b)^2c+3(a+b)c^2+c^3$$

$$3(a+b)^2c+3(a+b)c^2+c^3$$

6. Extract the cube root of $8x^3+36x^2+54x+27$. Ans. $2x+3$.

7. Extract the cube root of $27x^6-54x^5+63x^4-44x^3+21x^2-6x+1$. Ans. $3x^2-2x+1$.

[From p. 41.] Here the numerator being the least compounded, and b rising therein to a single dimension only, I divide the same into the parts $6a^5 - 4a^3c^2$, and $15a^4b - 10a^2bc^2$, which, by inspection, appear to be equal $2a^3(3a^2 - 2c^2)$, and $5a^2b(3a^2 - 2c^2)$; $\therefore 3a^2-2c^2$ is a divisor to both parts, and likewise to the whole, expressed by $(3a^2-2c^2)\times(2a^3+5a^2b)$; so that one of these two factors, if the fraction given can be reduced to lower terms, must also measure the denominator; but the former will be found to succeed: thus $3a^2-2c^2)9a^3b-27a^2bc-6abc^2+18bc^3(3ab-9bc$ $\therefore 3a^2-2c^2$.

Here, the denominator being the least compounded, and d rising therein to a single dimension only, I divide the same into the parts $4a^2d-4acd$, and $-2ac^2+2c^3$: which, by inspection, appear to be equal to $4ad(a-c)$, and $-2c^2(a-c)$. $\therefore a-c$ is a divisor to both the parts, and likewise to the whole, expressed by $(4a^2d-2c)\times(a-c)$; so that one of these two factors, if the fraction given can be reduced to lower terms, must also measure the numerator; but the latter will be found to succeed: thus,
$a-c)a^2d^2-c^2d^2-a^2c^2+c^4(ad^2+cd^2-ac^2-c^3$; $\therefore a-c$ is the greatest
$a-c\left(\dfrac{a^2d^2-c^2d^2-a^2c^2+c^4}{4a^2d-4acd-2ac^2+2c^3}\right(=\dfrac{ad^2+cd^2-ac^2-c^3}{4ad-2c^2}$, the Answer.

Solution to the 8th. Multiplying the numeator by 5 and I have.

$$15x^4-2x^3+10x^2-x+2)\,45x^5+10x^3+20x^2-5x+40(3x$$
$$45x^5-6x^4+30x^3-3x^2+6x$$
$$\times \text{ by } 5,\ 6x^4-20x^3+23x^2-11x+5$$
$$15x^4-2x^3+10x^2-x+2)30x^4-100x^3+115x^2-55x+25(+2$$
$$30x^4-4x^3+20x^2-2x+4$$
$$-96x^3+95x^2-53x+21$$

Multiply the last divisor by 32, and we shall have

$$-96x^3+95x^2-53x+21(480x^4-64x^3+320x^2-32x+64(-5x$$
$$480x^4-475x^3+265x^2-105x$$
$$\text{mult by } 32,\ 411x^3+55x^2+73x+64$$
$$-96x^3+75x^2-53x+21)13152x^3+1760x^2+2336x+2048(-137$$
$$13152x^3+13015x^2+7261x-2877$$

Divide this by 4925, it becomes $3x^3-x^2+1$; which by another operation exactly divides $-96x^3+95x^2-51x+21$; and $\therefore$ in the greatest common measure and the reduced form, is $\dfrac{3x^3+x^2+1}{5x^2+x+2}$

SIMPLE EQUATIONS.

1. An equation is a proposition which declares the equality of 2 quantities expressed algebraically. This is done by connecting these quantities by the sign (=): thus, $x-4=6-x$ is an equa-

tion expressing the equality of the quantities $x-4$, and $6-x$. Also, $x-5=0$ is an equation which asserts that $x-5$ is equal to nothing, and therefore that the positive part of the expression is equal to the negative part.

(2.) A simple equation is one which, when cleared of fractions and surds, contains only the first power of the unknown quantity. (See Definitions 3 and 4, page 1; and also Definitions 32 to 36, page 4.)

(6.) A cubic equation, or an equation of three dimensions, is one into which the cube of the unknown quantity enters, with the simple and quadratic powers.

(7.) A pure quadratic is one into which only the square of the unknown quantity enters.

(8.) An adfected quadratic is one which involves the square of the unknown quantity, and also the simple power and unknown quantities. Thus, $ax^2+b=0$ is a pure quadratic, and $ax^2+bx+c=0$ is an adfected quadratic.

(9.) The resolution of equations is the determining from some quantities given the values of others which are unknown, so that these latter may answer certain conditions proposed.

(10.) And these values are called roots of the equation.

(11.) If equal quantities be added to equal quantities, the sums will be equal.

(12.) If equal quantities be taken from equal quantities, the remainders will be equal.

(13.) If equal quantities be multiplied by the same or equal quantities, the products will be equal.

(14.) If equal quantities be divided by the same or equal quantities, the quotients will be equal.

(15.) If the same quantity be added to and subtracted from another, the value of the latter will not be altered.

(16.) If a quantity be both multiplied and divided by another, its value will not be altered.

(17.) Any quantity may be transposed from one side of an equation to the other, by changing its sign:

Because, in this transposition, the same quantity is merely subtracted from each side of the equation; and if equals be taken from equals, the remainders are equal. Thus, if $x+9=15$, and 9 be subtracted from each side, $x=15-9$, or 6. Also, if $x+b=a$, and b be subtracted from each side, $x=a-b$. And if $x-c=d$, and c be added to each side, $x=d+c$.

Also, if $5x-7=2x+2$, and $2x$ be taken from each side, $5x-$

$2x-7=2$, or $3x-7=2$; and if -7 be subtracted, or (which is the same thing) if $+7$ be added to each side, $3x=2+7=9$.

Also, if $x-a+b=c-3x$, then, by subtracting $-a+b-3x$ from each side, we have $x+3x=a-b+c$.

Cor. 1. Hence, if the signs of all the terms on each side of an equation be changed, the two sides still remain equal; because in this change every term is transposed.

Cor. 2. Hence, when the known and unknown quantities are connected in an equation by the signs $+$ or $-$, they may be separated by transposing the known quantities to one side, and the unknown to the other.

Cor. 3. Hence, also, if any quantity be found on both sides of an equation, it may be taken away from each; thus, if $x+y=5+y$, then $x=5$. If $a-b=c+d-b$, then $a=c+d$.

(18.) If every term on each side of an equation be multiplied by the same quantity, the results will be equal:

Because, in multiplying every term on each side by any quantity, the value of the whole side is multiplied by that quantity; and (13) if equals be multiplied by the same quantity, the products will be equal. Thus, if $x=5+a$, then $6x=30+6a$, by multiplying every term by 6.

Cor. 1. Hence an equation, of which any part is fractional, may be reduced to an equation expressed in integers, by multiplying every term by the denominator of the fraction. If there be more fractions than one in the given equation, it may be so reduced by multiplying every term either by the product of the denominators, or by a common multiple of them; and if the least common multiple be used, the equation will be in its lowest terms. Thus, if $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=13$; if every term be multiplied by 12, which is the least common multiple of 2, 3, 4; $6x+4x+3x=156$.

Cor. 2. Hence, also, if every term on both sides have a common multiplier or divisor, that common multiplier or divisor may be taken away; thus, if $ax^2+abx=cdx$; each term being divided by the common multiplier x, $ax+ab=cd$. Also, if $\frac{5}{4}x+\frac{a+6}{4}=\frac{4x-7}{4}$, then also $5x+a+6=4x-7$.

Also, if $\frac{ax+ab}{c}=\frac{ad}{c}+\frac{4ax}{c}$, then, multiplying by $\frac{c}{a}$, $x+b=d+4x$.

Cor. 3. Also, if each member of the equation have a common divisor, the equation may be reduced by dividing both sides by that common divisor;

Thus, if $ax^2-a^2x=abx-a^2b$, each side is divisible by $ax-a^2$, whence $x=b$.

Cor. 4. Hence also any term of an equation may be made a square, by multiplying all the terms of the equation by the quantities necessary; as, if $ax^2+bcx=cd^2$, the first term may be made a square by multiplying each term by a, and $a^2x^2+abcx=acd^2$.

(19.) If each side of an equation be raised to the same power, the results are equal; thus, if $x=6$, $x^2=36$;

If $x+a=y-b$, then $x^2+2ax+a^2=y^2-2by+b^2$.

And if the same roots be extracted on each side, the results are equal; thus, if $x^2=49$, $x=7$.

If $x^3=a^3b^3$, then $x=ab$; if $x^2+2x+1=y^2-y+\frac{1}{4}$, then $x+1=y-\frac{1}{2}$, and if $x^2-4ax+4a^2=y^2+6by+9b^2$, then $x-2a=y+3b$.

For (13 and 14) when equal quantities on each side of an equation are multiplied or divided by equal quantities, the results will be equal.

Cor. Hence, if that side of the equation which contains the unknown quantity be a perfect square, cube, or other power, by extracting the square root, cube root, &c. of both sides, the equation will be reduced to one of lower dimensions: thus, if $x^2+8x+16=36$, $x+4=6$;

If $x^3+3x^2+3x+1=27$, $x+1=3$;
If $x^4+2x^3+x^2=100$, $x^2+x=10$.

(20.) Any equation may be cleared of a single radical quantity by transposing all the other terms to the contrary side, and raising each side to the power denominated by the surd. If there are more than one surd, the operation must be repeated. Thus, if $x=\sqrt{(ax+b^2)}$, by squaring each side $x^2=ax+b^2$, which is free from surds. Also, if $\sqrt{(x^2+7)}+x=7$, then, (17) by transposition, $\sqrt{(x^2+7)}=7-x$; and (19) by squaring each side, $x^2+7=49-14x+x^2$, which is free from surds. Also, if $x+\sqrt[3]{(a^2x)}=b$, then, (17) by transposition, $\sqrt[3]{(a^2x)}=b-x$; and, (19) by cubing each side, $a^2x=b^3-3b^2x+3bx^2-x^3$, which is free from surds. Also, if $\sqrt{\{x^2+\sqrt{(x^2+21)}\}}-1=x$, then, (17) by transposition, $\sqrt{\{x^2+\sqrt{(x^2+21)}\}}=x+1$, and (19) by squaring each side, $x^2+\sqrt{(x^2+21)}=x^2+2x+1$; therefore, (17, Cor. 3,) $\sqrt{(x^2+21)}=2x+1$, and, (19) by squaring each side, $x^2+21=4x^2+4x+1$, which is free from surds. And, if $\sqrt[3]{\{a^2x+\sqrt{(a^3x^3)}\}}=c$, (19) by cubing each side, $a^2x+\sqrt{(a^3x^3)}=c^3$, and (17) by transposition, $\sqrt{(a^3x^3)}=c^3-a^2x$; therefore, (19) by squaring each side, $a^3x^3=c^6-2a^2c^3x+a^4x^2$, which is free from surds.

(21.) Any proportion may be converted into an equation; for the product of the extremes is equal to the product of the means.

Let $a : b :: c : d$, by the nature of proportion $\frac{a}{b}=\frac{c}{d}$; therefore, (18, Cor. 1,) $ad=bc$.

SIMPLE EQUATIONS.

RULE I. Any term may be transposed or transferred from one side of an equation to the other, by changing its signs.

Thus, if $x+3=7$; then will $x=7-3$, or $x=4$.
And, if $x-4+6=8$; then will $x=8+4-6=6$.
Also, if $x-a+b=c-d$; then will $x=a-b+c-d$.
And, if $4x-8=3x+20$; then $4x-3x=20+8$, and consequently $x=28$.

From this rule it also follows, that if a quantity be found on each side of an equation, with the same sign, it may be left out of both of them; and that the signs of all the terms of any equation may be changed from $+$ to $-$, or from $-$ to $+$, without altering its value.

Thus, if $x+5=7+5$; then, by cancelling, $x=7$.
And if $a-x=b-c$; then, by changing the signs, $x-a=c-b$, or $x=a+c-b$.

1. Given $2x+3=x+17$ to find x. Ans. $x=14$.
By transposing, gives $2x-x=17-3=14$. Whence $x=14$, Ans.

2. Given $5x-9=4x+7$ to find x. Ans. $x=16$.
By transposing, gives $5x-4x=7+9=16$. Whence $x=16$, Ans.

3. Given $x+9-2=4$ to find x. Ans. $x=-3$.
By transposing, gives $x=4+2-9=-3$, Ans.

4. Given $9x-8=8x-5$ to find x. Ans. $x=3$.
By transposing, gives $9x-8x=8-5=3$. Whence $x=3$, Ans.

5. Given $7x+8-3=6x+4$ to find x. Ans. $x=-1$.
By transposing, gives $7x-6x=4+3-8=-1$. Whence $x=-1$, Ans.

6. What number is that, to the double of which if 18 be added the sum will be 82?

Let $x=$ the number required. Then $2x+18=82$; $\therefore$ by transposition, $2x=64$, and $x=32$.

7. What number is that, to the double of which if 44 be added, the sum is equal to four times the required number?

Let $x=$ the number. Then $2x+44=4x$; $\therefore$ by transposition, $44=2x$, and $22=x$.

8. What number is that, the double of which exceeds its half by 6?

Let $x=$ the number. Then $2x-\frac{1}{2}x=6$; $\therefore$ $4x-x=12$, or $3x=12$, $\therefore$ $x=4$.

9. From two towns which are 187 miles distant, two travellers set out at the same time with an intention of meeting. One

of them goes 8 miles and the other 9 miles a day. In how many days will they meet?

Let $x=$ the number of days required; then $8x=$ the number of miles one travelled, and $9x=$ the number the other travelled; and since they meet, they must together have travelled the whole distance, consequently $8x+9x=187$, or $17x=187$, $\therefore x=11$.

RULE II. If an unknown quantity has a coefficient, its value may be found by dividing both sides of the equation by that coefficient, because if two equal quantities be divided by the same or equal quantities, the quotients will be equal.

Thus, if $ax=3ab-c$; then will $x=3b-\frac{c}{a}$.

And, if $2x+4=16$; then will $x+2=8$, or $x=8-2=6$.

Also, if $\frac{x}{2}=5+3$; then will $x=10+6=16$.

And, if $\frac{2x}{3}-2=4$; then $2x-6=12$, or, by division, $x-3=6$, or $x=9$.

1. Given $16x+2=34$ to find x. Ans. $x=2$.

By transposing, gives $16x=34-2=32$, and by division $x=\frac{32}{16}=2$, Ans.

2. Given $4x-8=-3x+13$ to find x. Ans. $x=3$.

Here $4x-8=-3x+13$, by transposing, gives $4x+3x=13+8$, or $7x=21$, and, by division, $x=\frac{21}{7}=3$, Ans.

3. Given $10x-19=7x+17$ to find x. Ans. $x=12$.

Here $10x-19=7x+17$, by transposition, $10x-7x=17+19$, or $3x=36$, and, by division, $x=\frac{36}{3}=12$, Ans.

4. Given $8x-3+9=-7x+9+27$ to find x. Ans. $x=2$.

Here $8x-3+9=-7x+9+27$, by transposing, $8x+7x=27+9-9+3$, or $15x=30$, and, by division, $x=\frac{30}{15}=2$, Ans.

5. Given $3ax-3ab=12d$. Ans. $x=b+\frac{4d}{a}$.

Here $3ax-3ab=12d$, by transposition, $3ax=12d+3ab$, and, by division, $x=\frac{12d+3ab}{3a}=b+\frac{4d}{a}$, Ans.

6. A cask which held 146 gallons was filled with a mixture of brandy, wine, and water. In it there were 15 gallons of wine more than there were of brandy, and as much water as both wine and brandy. What quantity was there of each?

Let $x=$ the number of gallons of brandy, $\therefore x+15=$ number of gallons of wine, and $2x+15=$ number of gallons of water.

$\therefore x+x+15+2x+15=146$, $\therefore$ by transposition, $4x=116$, and $x=29$. $\therefore$ there were 29, 44, and 73 gallons respectively of brandy, wine, and water.

7. A person employed 4 workmen; to the first of whom he gave 2 shillings more than to the second; to the second 3 shillings more than to the third; and to the third 4 shillings more than to the fourth. Their wages amounted to 32 shillings. What did each receive?

Let $x=$ the sum received by the fourth,
$\therefore x+4=$ - - - - - - third,
$x+7=$ - - - - - - second,
and $x+9=$ - - - - - - first.

$$\therefore x+x+4+x+7+x+9=32,$$
and, by transposition, $4x=12$,
consequently $x=3$.

$\therefore$ they received 12, 10, 7, and 3 shillings respectively.

8. A father, taking his 4 sons to school, divided a certain sum amongst them. Now the third had 9 shillings more than the youngest; the second 12 shillings more than the third; and the eldest 18 shillings more than the second; and the whole sum was 6 shillings more than 7 times the sum which the youngest received. How much had each?

Suppose the youngest received x shillings,
then the third received $x+9$
the second, - - $x+21$
and the eldest, - - $x+39$

$\therefore x+x+9+x+21+x+39=7x+6$; $\therefore$ by transposition, $63=3x$, and $\therefore 21=x$. Consequently they received 21, 30, 42, and 60 shillings respectively.

9. A sum of money was to be divided amongst six poor persons; the second received $10d.$, the third $14d.$, the fourth $25d.$, the fifth $28d.$, and the sixth $33d.$ less than the first. Now the sum distributed was $10d.$ more than treble of what the first received. What did each receive?

Let $x=$ what the first rec'd.,
$\therefore x-10=$ - - second, -
$x-14=$ - - third, -
$x-25=$ - - fourth, -
$x-28=$ - - fifth, -
$x-33=$ - - sixth, -

The sum of which $=6x-110=3x+10$ by supposition. $\therefore$ by transposition, $3x=120$ and $x=40$. $\therefore$ they received 40, 30, 26, 15, 12, 7 pence respectively.

10. It is required to divide the number 99 into five such parts, that the first may exceed the second by 3; be less than the third by 10; greater than the fourth

by 9; and less than the fifth by 16.

Let $x=$ the first part,

$\therefore x-3=$ second,

$x+10=$ third,

$x-9=$ fourth,

$x+16=$ fifth.

$\therefore x+x-3+x+10+x-9+x+16=99$, or $5x+14=99$.

$\therefore$ by transposition, $5x=85$, and $x=17$. $\therefore$ the parts are 17, 14, 27, 8, and 33.

11. What two numbers are those whose sum is 59, and difference 17?

Let $x=$ the less,

$\therefore x+17=$ the greater,

and $\therefore x+x+17=59$;

by transposition, $2x=42$,

and $x=21$, the less,

$\therefore$ the greater $=38$.

12. What number is that, the treble of which increased by 12 shall as much exceed 54 as that treble is below 144?

Let $x=$ the number. $\therefore 3x+12-54=144-3x$; $\therefore 6x=186$, and $x=31$.

13. Two persons began to play with equal sums of money: the first lost 14 dollars, the other won 24 dollars, and then the second had twice as many dollars as the first. What sum had each at first?

Let $x=$ the sum;

$\left.\begin{array}{l}\text{Then } x-14 \\ \text{and } x+24\end{array}\right\}=$ the sums each had after playing;

$\therefore 2x-28=x+24$; $\therefore x=52$.

14. At a certain election 943 men voted, and the candidate chosen had a majority of 65. How many voted for each?

Let $x=$ the number of votes the unsuccessful candidate had;

$\therefore x+65=$ the number the successful one had. $\therefore x+x+65=943$; $2x=878$, and $x=439$. $\therefore$ the numbers were 439 and 504.

RULE III. Any equation may be cleared of fractions by multiplying each of its terms, successively, by the denominators of those fractions, or by multiplying both sides by the products of all the denominators, or by any quantity that is divisible by each of the denominators.

Thus, if $\frac{x}{3}+\frac{x}{4}=5$, then, multiplying by 3, we have $x+\frac{3x}{4}=15$; and this, multiplied by 4, gives $4x+3x=60$; whence, by addition, $7x=60$, or $x=\frac{60}{7}=8\frac{4}{7}$.

And, if $\frac{x}{4}+\frac{x}{6}=10$; then, multiplying by 12, (which is a multiple of 4 and 6,) $3x+2x=120$, or $5x=120$, or $x=\frac{120}{5}=24$.

It also appears, from this rule, that if the same number, or quantity, be found in each of the terms of an equation, either as a multiplier or divisor, it may be expunged from all of them, without altering the result.

Thus, if $ax=ab+ac$; then, by cancelling, $x=b+c$.

And if $\frac{x}{a}+\frac{b}{a}=\frac{c}{a}$; then, $x+b=c$, or $x=c-b$.

1. Given $\frac{3x}{2}=\frac{x}{4}+24$ to find x. Ans. $x=19\frac{1}{5}$.

Multiplying by 4, (which is a multiple of 4 and 2,) $6x=x+96$, or $6x-x=96$, or $5x=96$, and by division $x=\frac{96}{5}=19\frac{1}{5}$, Ans.

2. Given $\frac{x}{3}+\frac{x}{5}+\frac{x}{2}=62$ to find x. Ans. $x=60$.

Here $\frac{x}{3}+\frac{x}{5}+\frac{x}{2}=62$; then, multiplying by 3, we have $x+\frac{3x}{5}+\frac{3x}{2}=186$, and multiplying by 5, gives $5x+3x+\frac{15x}{2}=930$; and this, multiplied by 2, gives $10x+6x+15x=1860$, or $31x=1860$, and, by division, $x=\frac{1860}{31}=60$, Ans.

3. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x+19}{2}$ to find x. Ans. $x=9$.

Multiplying by 6, (which is a multiple of 2, 3, and 2,) $3x-9+2x=120-3x-57$, and, by transposition and division, $x=\frac{72}{8}=9$, Ans.

4. Given $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$ to find x. Ans. $x=13$.

Multiplying by 12, we get $6x+6+4x+8=192-3x-9$, or $13x=169$; and, by division, $x=\frac{169}{13}=13$, Ans.

5. Given $\frac{x+a}{b}+\frac{x}{c}=\frac{2x}{a}+\frac{a+b}{d}$ to find x.

Ans. $x=\frac{a^2cb+acb^2-a^2cd}{acd+abd-2cbd}$.

Here $\frac{x+a}{b}+\frac{x}{c}=\frac{2x}{a}+\frac{a+b}{d}$; then, multiplying by b, $x+a+\frac{bx}{c}=\frac{2bx}{a}+\frac{ab+b^2}{d}$, mult. by c, $cx+ca+bx=\frac{2bcx}{a}+\frac{abc+cb^2}{d}$, and, by multiplying by a and d successively, we have $adcx+a^2cd+abdx=2bcdx+a^2bc+acb^2$, or, by transposition, $adcx+abdx=2bcdx+a^2bc+acb^2-a^2cd$, and, by division, $x=\frac{a^2bc+acb^2-a^2cd}{adc+abd-2bcd}$, A.

RULE IV. If the unknown quantity be a surd, transpose the rest of the terms, and let the surd quantity stand alone on one side of the equation; then take away the radical sign from it, and raise the other side of the equation to the power denoted by the index of the surd.

Thus, if $\sqrt{x}-2=3$; then will $\sqrt{x}=3+2=5$, or, by squaring, $x=5^2=25$.

And if $\sqrt{(3x+4)}=5$; then will $3x+4=25$, or $3x=25-4=21$, or $x=\frac{21}{3}=7$.

Also, if $\sqrt[3]{(2x+3)}+4=8$; then will $\sqrt[3]{(2x+3)}=8-4=4$, or $2x+3=4^3=64$, and $\therefore 2x=64-3=61$, or $x=\frac{61}{2}=30\frac{1}{2}$.

1. Given $2\sqrt{x}+3=9$ to find x. Ans. $x=9$.

Here $2\sqrt{x}+3=9$; then, $2\sqrt{x}=9-3=6$ by transposition, and $4x=36$ by squaring, or $x=\frac{36}{4}=9$, Ans.

2. Given $\sqrt{(x+1)}-2=3$ to find x. Ans. $x=24$.

Here $\sqrt{(x+1)}-2=3$; then $\sqrt{(x+1)}=3+2=5$, or $x+1=25$ by squaring, and consequently $x=25-1=24$, Ans.

3. Given $\sqrt[3]{(3x+4)}+3=6$ to find x. Ans. $x=7\frac{2}{3}$.

Here $\sqrt[3]{(3x+4)}+3=6$; then $\sqrt[3]{(3x+4)}=6-3=3$, or $3x+4=27$ by cubing, and $3x=27-4=23$, or $x=\frac{23}{3}=7\frac{2}{3}$, Ans.

4. Given $\sqrt{(4+x)}=4-\sqrt{x}$ to find x. Ans. $x=2\frac{1}{4}$.

Here $\sqrt{(4+x)}=4-\sqrt{x}$; then $4+x=16-8\sqrt{x}+x$ by squaring, and, by trans. $8\sqrt{x}=12$, or $2\sqrt{x}=3$; and $4x=9$ by squaring, or $x=\frac{9}{4}=2\frac{1}{4}$, Ans.

5. Given $\sqrt{(4a^2+x^2)}=\sqrt[4]{(4b^4+x^4)}$ to find x.

Here $\sqrt{(4a^2+x^2)}=\sqrt[4]{(4b^4+x^4)}$; then, by squaring $4a^2+x^2=\sqrt{(4b^4+x^4)}$, and squaring again, $16a^4+8a^2x^2+x^4=4b^4+x^4$, by transposition and division $x^2=\frac{b^4-4a^4}{2a^2}$, and $\therefore x=\sqrt{(\frac{b^4-4a^4}{2a^2})}$, An.

RULE V. If that side of the equation which contains the unknown quantity be a complete power, the equation may be reduced to a lower dimension, by extracting the root of the said power from both sides of the equation.

Thus, if $x^2=81$; then $x=\sqrt{81}=9$; and if $x^3=27$, then $x=\sqrt[3]{27}=3$. Also, if $3x^2-9=24$; then $3x^2=24+9=33$, or $x^2=\frac{33}{3}=11$, and $\therefore x=\sqrt{11}$. And, if $x^2+6x+9=27$; then, since the left hand side of the equation is a complete square, we shall have, by extracting the roots, $x+3=\sqrt{27}=\sqrt{(9\times3)}=3\sqrt{3}$, or $x=3\sqrt{3}-3$.

1. Given $9x^2-6=30$ to find x. Ans. $x=2$.

Then $9x^2=30+6=36$, or $x^2=\frac{36}{9}=4$, and $\therefore x=\sqrt{4}=2$, Ans.

2. Given $x^3+9=36$ to find x. Ans. $x=3$.

Then $x^3=36-9=27$, or $x=\sqrt[3]{27}=3$, Ans.

3. Given $x^2+x+\frac{1}{4}=\frac{81}{4}$ to find x. Ans. $x=4$.

By extracting the square of both sides $x+\frac{1}{2}=\frac{9}{2}=4\frac{1}{2}$, or $x=4\frac{1}{2}-\frac{1}{2}=4$, Ans.

4. Given $x^2+ax+\frac{a^2}{4}=b^2$ to find x. Ans. $x=b-\frac{a}{2}$.

By extracting the square root of both sides $x+\frac{a}{2}=b$, or $x=b-\frac{a}{2}$, Ans.

5. Given $x^2+14x+49=121$ to find x. Ans. $x=4$.
By extracting the square root of both sides of the equation $x+7=11$, or $x=11-7=4$, Ans.

RULE VI. Any proportion may be converted into an equation by making the products of the extremes and means equal to each other.

Thus, if $3x : 16 :: 5 : 6$; then $3x\times 6=16\times 5$, or $18x=80$, or $x=\frac{80}{18}=\frac{40}{9}=4\frac{4}{9}$.

And if $\frac{2x}{3} : a :: b : c$; then will $\frac{2cx}{3}=ab$, or $2cx=3ab$; or, by division, $x=\frac{3ab}{2c}$.

Also, if $12-x : \frac{x}{2} :: 4 : 1$; then $12-x=\frac{4x}{2}=2x$, or $2x+x=12$, and consequently $x=\frac{12}{3}=4$.

1. Given $\frac{3}{4}x : a :: 5bc : cd$ to find x. Ans. $x=\frac{20ab}{3d}$.

$\frac{3}{4}x\times cd=a\times 5bc$, or $3cdx=20abc$, and consequently $x=\frac{20abc}{3cd}=\frac{20ab}{3d}$, Ans.

2. Given $10-x : \frac{2}{3}x :: 3 : 1$ to find x. Ans. $x=3\frac{1}{3}$.
Then $10-x=2x$ by mult. ext. and means, and $3x=10$, or $x=\frac{10}{3}=3\frac{1}{3}$, Ans.

3. Given $8+8x : 4x :: 8 : 2$ to find x. Ans. $x=1$.
Here $8+8x : 4x :: 8 : 2$; then $16+16x=32x$ by mult. ext. and means, and $32x-16x=16$, or $16x=16$; $\therefore x=\frac{16}{16}=1$, Ans.

4. Given $x : 6-x :: 2 : 4$ to find x. Ans. $x=2$.
Here $x : 6-x :: 2 : 4$; then $4x=12-2x$ by mult. ext. and means, and $4x+2x=12$, or $6x=12$; $x=\frac{12}{6}=2$, Ans.

5. Given $4x : a :: 9\sqrt{x} : 9$ to find x. Ans. $x=\frac{a^2}{16}$.

Then $36x=9a\sqrt{x}$ by mult. ext. and means, or $4x=a\sqrt{x}$, and by squaring $16x^2=a^2x$; therefore, by division, $x=\frac{1}{16}a^2$, Ans.

MISCELLANEOUS EXAMPLES.

1. Given $5x-15=2x+6$ to find the value of x.

Here $5x-2x=6+15$, or $3x=6+15=21$; and therefore $x=\frac{21}{3}=7$.

2. Given $40-6x-16=120-14x$, to find the value of x.

Here $14x-6x=120-40+16$; or $8x=136-40=96$; and therefore $x=\frac{96}{8}=12$.

3. Given $3x^2-10x=8x+x^2$, to find the value of x.

Here $3x-10=8+x$, by dividing by x; or $3x-x=8+10=18$ by transposition.

And consequently $2x=18$, or $x=\frac{18}{2}=9$.

4. Given $6ax^3-12abx^2=2ax^3+6ax^2$, to find the value of x.

Here $2x-4b=x+2$, by dividing by $3ax^2$; or $2x-x=2+4b$; and therefore $x=4b+2$.

5. Given $x^2+2x+1=16$, to find the value of x.

Here $x+1=4$, by extracting the square root of each side.

And therefore, by transposition, $x=4-1=3$.

6. Given $5ax-3b=2dx+c$, to find the value of x.

Here $5ax-2dx=c+3b$; or $(5a-2d)x=c+3b$; and therefore, by division, $x=\dfrac{c+3b}{5a-2d}$.

7. Given $\frac{1}{2}x-\frac{1}{3}x+\frac{1}{4}x=10$, to find the value of x.

Here $x-\frac{2}{3}x+\frac{2}{4}x=20$; and $3x-2x+\frac{6}{4}x=60$; or $12x-8x+6x=240$; whence $10x=240$, or $x=24$.

8. Given $\dfrac{x-3}{2}+\dfrac{x}{3}=20-\dfrac{x-19}{2}$, to find the value of x.

Here $x-3+\frac{2}{3}x=40-x+19$; or $3x-9+2x=120-3x+57$; whence $3x+2x+3x=120+57+9$; that is, $8x=186$, or $x=23\frac{1}{4}$.

9. Given $\sqrt{\frac{2}{3}x}+5=7$, to find the value of x.

Here $\sqrt{\frac{2}{3}x}=7-5=2$; whence, by squaring, $\frac{2}{3}x=2^2=4$ and $2x=12$, or $x=6$.

10. Given $x+\sqrt{(a^2+x^2)}=\dfrac{2a^2}{\sqrt{(a^2+x^2)}}$, to find the value of x.

Here $x\sqrt{(a^2+x^2)}+a^2+x^2=2a^2$; or $x\sqrt{(a^2+x^2)}=a^2-x^2$, and $x^2.(a^2+x^2)=a^4-2a^2x^2+x^4$; whence $a^2x^2+x^4=a^4-2a^2x^2+x^4$, and $a^2x^2=a^4-2a^2x^2$; therefore $3a^2x^2=a^4$, or $x^2=\dfrac{a^4}{3a^2}=\dfrac{a^2}{3}$; and consequently $x=\sqrt{\dfrac{a^2}{3}}=a\sqrt{\dfrac{1}{3}}=a\sqrt{\frac{3}{9}}=\frac{1}{3}a\sqrt{3}$, the answer.

EXAMPLES FOR PRACTICE.

1. Given $3x-2+24=31$, to find the value of x.

By transposing gives $3x=31+2-24=9 \therefore x=\frac{9}{3}=3$, Ans.

2. Given $4-9y=14-11y$, to find the value of y.

Here $4-9y=14-11y$, or $11y-9y=14-4$, or $2y=10$, whence $y=\frac{10}{2}=5$, Ans.

3. Given $x+18=3x-5$, to find the value of x.

$3x-x=18+5$, or $2x=23$, whence $x=11\frac{1}{2}$.

4. Given $x+\frac{1}{2}x+\frac{1}{3}x=11$, to determine the value of x.

Mult. by 6. $6x+3x+2x=66$, or $11x=66$, whence $x=6$

5. Given $2x-\frac{1}{2}x+1=5x-2$, to find the value of x.

Multiply the given equation by 2, and we have $4x-x+2=10x-4$; whence $10x+x-4x=4+2$, or $7x=6$, whence $x=\frac{6}{7}$.

6. Given $\frac{1}{2}x+\frac{1}{3}x-\frac{1}{4}x=\frac{7}{10}$, to determine the value of x.

Mult. by 60, gives $30x+20x-15x=42$, or $35x=42$, or $x=\frac{42}{35}=\frac{6}{5}=1\frac{1}{5}$, Ans.

7. Given $\frac{x+3}{2}+\frac{1}{3}x=4-\frac{x-5}{4}$, to find the value of x.

Multiply by 12, gives $6x+18+4x=48-3x+15$, or $13x=45$, whence $x=\frac{45}{13}=3\frac{6}{13}$, Ans.

8. Given $2+\sqrt{3x}=\sqrt{(4+5x)}$, to find the value of x.

Being squared gives $4+4\sqrt{3x}+3x=4+5x$, whence $4\sqrt{3x}=5x-3x=2x$. Squaring $48x=4x^2$, or dividing by $4x$, we have $x=12$.

9. Given $x+a=\frac{x^2}{a+x}$, to find the value of x.

$x^2+2ax+a^2=x^2$, whence $2ax=-a^2$, or $2x=-a$, or $x=-\frac{1}{2}a$.

10. Given $\sqrt{x}+\sqrt{(a+x)}=\frac{2a}{\sqrt{(a+x)}}$, to find the value of x.

Multiplying by $\sqrt{(a+x)}$, gives $\sqrt{(ax+x^2)}+a+x=2a$, or $\sqrt{(ax+x^2)}=a-x$.

Hence by squaring $ax+x^2=a^2-2ax+x^2$.

Consequently $3ax=a^2$, or $3x=a$, or $x=\frac{a}{3}$, Ans.

11. Given $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$, to find the value of x.

Multiplying by 12 gives $3ax-3b+4a=6bx-4bx+4a$.

Whence $3ax-2bx=3b$, or $x(3a-2b)=3b$. $\therefore x=\frac{3b}{3a-2b}$.

12. Given $\sqrt{(a^2+x^2)}=\sqrt[4]{(b^4+x^4)}$, to find the value of x.

By squaring, gives $a^2+x^2=\sqrt{(b^4+x^4)}$, squaring again gives $a^4+2a^2x^2+x^4=b^4+x^4$; whence $2a^2x^2=b^4-a^4$, or $x=\sqrt{\frac{b^4-a^4}{2a^2}}$.

13. Given $\sqrt{(a+x)}+\sqrt{(a-x)}=\sqrt{ax}$, to find the value of x.

Here $\sqrt{(a+x)}+\sqrt{(a-x)}=\sqrt{ax}$, by squaring, gives $2a+2\sqrt{(a^2-x^2)}=ax$, whence $\sqrt{(a^2-x^2)}=\frac{ax-2a}{2}$; squaring again, $a^2-x^2=\frac{a^2x^2-4a^2x+4a^2}{4}$. $\therefore -4x^2=a^2x^2-4a^2x$, or, dividing by x, $-4x=a^2x-4a^2$, or $x=\frac{4a^2}{a^2+4}$.

14. Given $\frac{a}{1+x}+\frac{a}{1-x}=b$, to determine the value of x.

By reduction $\frac{a-ax+a+ax}{1-x^2}=b$, or $\frac{2a}{1-x^2}=b$, whence $2a=b-bx^2$, or $bx^2=b-2a$. $\therefore x=\sqrt{\frac{b-2a}{b}}$.

15. Given $a+x=\sqrt{\{a^2+x\sqrt{(b^2+x^2)}\}}$, to find the value of x.

By squaring $a+x=\sqrt{\{a^2+x\sqrt{(b^2+x^2)}\}}$, we have $a^2+2ax+x^2=a^2+x\sqrt{(b^2+x^2)}$; transposing, $2ax+x^2=x\sqrt{(b^2+x^2)}$; divide by x, $2a+x=\sqrt{(b^2+x^2)}$; by squaring, $4a^2+4ax+x^2=b^2+x^2$; whence $4ax=b^2-4a^2$, or $x=\frac{b^2}{4a}-a$, the answer required.

16. Given $\frac{1}{2}\sqrt{(x^2+3a^2)}-\frac{1}{2}\sqrt{(x^2-3a^2)}=x\sqrt{a}$, to find the value of x.

Multiplying the given equation by 2, we have $\sqrt{(x^2+3a^2)}-\sqrt{(x^2-3a^2)}=2x\sqrt{a}$; $\therefore$ by squaring $2x^2-2\sqrt{(x^4-9a^4)}=4ax^2$, or $-\sqrt{(x^4-9a^4)}=2ax^2-x^2$; squaring again, $x^4-9a^4=4a^2x^4-4ax^4+x^4$, or $(4a-4a^2)x^4=9a^4$, or $x=\sqrt[4]{\frac{9a^3}{4-4a}}$.

17. Given $\sqrt{(a+x)}+\sqrt{(a-x)}=b$, to find the value of x.

Here $\sqrt{(a+x)}+\sqrt{(a-x)}=b$, by squaring, $2a+2\sqrt{(a^2-x^2)}=b^2$, or $\sqrt{(a^2-x^2)}=\frac{1}{2}b^2-a$; whence $a^2-x^2=\frac{1}{4}b^4-ab^2+a^2$, and $x=\sqrt{(ab^2-\frac{1}{4}b^4)}=\frac{1}{2}b\sqrt{(4a-b^2)}$.

18. Given $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$, to find the value of x.

By cubing both sides after the form of Ex. 7, Case x, Surds, we have $2a+3\sqrt[3]{(a^2-x^2)}\{\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}\}=b^3$. But since $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$, this becomes $2a+3\sqrt[3]{(a^2-x^2)}\times b=b^3$, or $3\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{b}$, or $\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{3b}$; whence $a^2-x^2=\frac{(b^3-2a)^3}{27b^3}=(\frac{b^3-2a}{3b})^3$; therefore $x=\sqrt{\{a^2-(\frac{b^3-2a}{3b})^3\}}$.

19. Given $\sqrt{a}+\sqrt{x}=\sqrt{ax}$, to find the value of x.

Which, divided by $\sqrt{x}$, gives $\frac{\sqrt{a}}{\sqrt{x}}+1=\sqrt{a}$, or $\frac{\sqrt{a}}{\sqrt{x}}=-1+$

$\sqrt{a}$, or $\sqrt{a}-1$; whence $\sqrt{x}(\sqrt{a}-1)=\sqrt{a}$, or $\sqrt{x}=\frac{\sqrt{a}}{\sqrt{a}-1}$;

therefore $x=\frac{a}{(\sqrt{a}-1)^2}$.

20. Given $\sqrt{\left\{\frac{x+1}{x-1}\right\}}+\sqrt{\left\{\frac{x-1}{x+1}\right\}}=a$, to determine the value of x.

By squaring, gives $\frac{x+1}{x-1}+\frac{x-1}{x+1}+2\sqrt{\frac{x^2-1}{x^2-1}}=a^2$, or $\frac{2x^2+2}{x^2-1}$ $+2=a^2$; or, by multiplying, $2x^2+2+2x^2-2=x^2a^2-a^2$; whence $4x^2-a^2x^2=-a^2$, or $(a^2-4)x^2=a^2$; therefore $x=\frac{a}{\sqrt{(a^2-4)}}$, Ans.

21. Given $\sqrt{(a^2+ax)}=a-\sqrt{(a^2-ax)}$, to find the value of x.

This equation transposed is $\sqrt{(a^2+ax)}+\sqrt{(a^2-ax)}=a$, squaring $2a^2+2\sqrt{(a^4-a^2x^2)}=a^2$; whence $\sqrt{(a^4-a^2x^2)}=-\frac{1}{2}a^2$; squaring again, $a^4-a^2x^2=\frac{1}{4}a^4$, or $a^2x^2=\frac{3}{4}a^4$, whence $x=\sqrt{\frac{3}{4}a^2}=\frac{a}{2}\sqrt{3}$.

22. Given $\sqrt{(a^2-x^2)}+x\sqrt{(a^2-1)}=a^2\sqrt{(1-x^2)}$, to find the value of x.

By transposition, $\sqrt{(a^2-x^2)}=a^2\sqrt{(1-x^2)}-x\sqrt{(a^2-1)}$; squaring, $a^2-x^2=a^4-a^4x^2-2xa^2\sqrt{(1-x^2)(a^2-1)}+a^2x^2-x^2$. Again, by transposition and division, $2x\sqrt{(1-x^2)(a^2-1)}=a^2-a^2x^2+x^2-1=(1-x^2)(a^2-1)$; by squaring, $4x^2(1-x^2)(a^2-1)=(1-x^2)^2(a^2-1)^2$, and $4x^2=(1-x^2)(a^2-1)$, dividing the whole by $(1-x^2)(a^2-1)$; therefore, by actually multiplying, $4x^2=a^2-a^2x^2+x^2-1$, or $3x^2+a^2x^2=a^2-1$; whence $x^2=\frac{a^2-1}{a^2+3}$; and, consequently, $x=\sqrt{(\frac{a^2-1}{a^2+3})}$, Ans.

23. Given $\sqrt{(x+a)}=c-\sqrt{(x+b)}$, to find the value of x.

Squaring $x+a=c^2-2c\sqrt{(x+b)}+x+b$, and transposing $2c\sqrt{(x+b)}=c^2+b-a$, or $\sqrt{(x+a)}=\frac{c^2+b-a}{2c}$; whence $x+b=(\frac{c^2+b-x}{2c})^2$, and $x=(\frac{c^2+b-a}{2c})^2-b$, the answer.

24. Given $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt{\frac{4bc}{a^2-x^2}}$, to find the value of x.

By squaring gives $\frac{b}{a+x}+\frac{c}{a-x}+\sqrt{\frac{4bc}{a^2-x^2}}=\sqrt{\frac{4bc}{a^2-x^2}}$; whence $\frac{b}{a+x}+\frac{c}{a-x}=0$, or $\frac{ab-bx+ac+cx}{a^2-x^2}=0$; therefore $ab-$

$bx + ac + cx = 0$, and $bx - cx = ab + ac$; consequently $x = \frac{ab+ac}{b-c} = a(\frac{b+c}{b-c})$.

Of the resolution of Simple Equations, containing Two Unknown Quantities.

When there are two unknown quantities, and two independent simple equations involving them, they may be reduced to one, by any of the three following rules:

RULE I. Observe which of the unknown quantities is the least involved, and find its value in each of the equations, as if the other were known, by the methods already explained; then let the two values, thus found, be put equal to each other, and there will arise a new equation, with only one unknown quantity in it, the value of which may be found as before.

1. Given $\left\{\begin{matrix} 2x+3y=23 \\ 5x-2y=10 \end{matrix}\right\}$ to find the values of x and y.

Here, from the first equation, $x = \frac{23-3y}{2}$, and $x = \frac{10+2y}{5}$; whence we have $\frac{23-3y}{2} = \frac{10+2y}{5}$, or $115 - 15y = 20 + 4y$, or $19y = 115 - 20 = 95$; that is, $y = \frac{95}{19} = 5$, and $x = \frac{23-15}{2} = 4$.

2. Given $\left\{\begin{matrix} x+y=a \\ x-y=b \end{matrix}\right\}$ to find the values of x and y.

Here, from the first equation, $x = a - y$, and from the second, $x = b + y$, whence $a - y = b + y$, or $2y = a - b$; and therefore $y = \frac{a-b}{2}$, and $x = a - y$; or, by substitution, $x = a - \frac{a-b}{2} = \frac{a+b}{2}$.

3. Given $\left\{\begin{matrix} \frac{1}{2}x+\frac{1}{3}y=7 \\ \frac{1}{3}x+\frac{1}{2}y=8 \end{matrix}\right\}$ to find the values of x and y.

Here, from the first equation, $x = 14 - \frac{2}{3}y$, and from the second, $x = 24 - \frac{3}{2}y$; therefore, by equality, $14 - \frac{2}{3}y = 24 - \frac{3}{2}y$, and, consequently, $42 - 2y = 72 - \frac{9}{2}y$; or, by multiplication, $84 - 4y = 144 - 9y$; and, therefore, also $5y = 144 - 84 = 60$, or, by division, $y = \frac{60}{5} = 12$, and $x = 14 - \frac{24}{3} = 6$.

4. Given $4x + y = 34$, and $4y + x = 16$, to find the values of x and y. Ans. $x = 8$, $y = 2$.

From 1st $x = \frac{34-y}{4}$, and from 2d, $x = 16 - 4y$; therefore, by equality, $\frac{34-y}{4} = 16 - 4y$, and, consequently, $34 - y = 64 - 16y$, or $16y - y = 64 - 34$, or $15y = 30$; whence $y = \frac{30}{15} = 2$, and $x = 16 - 8 = 8$.

5. Given $2x+3y=16$, and $3x-2y=11$, to find the values of x and y. Ans. $x=5$, $y=2$.

Here $2x+3y=16$, and $3x-2y=11$; then from 1st $x=\frac{16-3y}{2}$, and from 2d, $x=\frac{11+2y}{3}$; therefore, by equality, $\frac{11+2y}{3}=\frac{16-3y}{2}$, and, consequently, $22+4y=48-9y$, or $13y=48-22=26$; whence $y=\frac{26}{13}=2$, and $x=\frac{11+4}{3}=\frac{15}{3}=5$.

6. Given $\frac{2}{5}x+\frac{3}{4}y=\frac{9}{20}$, and $\frac{3}{4}x+\frac{2}{5}y=\frac{61}{120}$, to find the values of x and y. Ans. $x=\frac{1}{2}$, $y=\frac{1}{3}$.

By multiplying the 1st by 20, and the second by 20, we shall have $8x+15y=9$, and $15x+8y=10\frac{1}{6}$; whence, from the former equation, $x=\frac{9-15y}{8}$, and from the latter $x=\frac{10\frac{1}{6}-8y}{15}$; $\therefore$ by equality, $\frac{10\frac{1}{6}-8y}{15}=\frac{9-15y}{8}$, or $81\frac{1}{3}-64y=135-225y$; hence $161y=53\frac{2}{3}$, or $y=53\frac{2}{3}\div161=\frac{1}{3}$; also $x=\frac{9-15y}{8}=\frac{9-5}{8}=\frac{1}{2}$.

7. Given $\begin{cases}\frac{1}{2}x+2y=a\\ \frac{1}{2}x-2y=b\end{cases}$ to find x and y. Ans. $x=a+b$, and $y=\frac{1}{4}a-\frac{1}{4}b$.

From the 1st $x=2a-4y$, and from the 2d, $x=2b+4y$; hence $2b+4y=2a-4y$, or $8y=2a-2b$; $y=\frac{1}{4}a-\frac{1}{4}b$, and $x=2b+4y=2b+a-b=a+b$.

8. Given $\begin{cases}\frac{1}{2}x+\frac{1}{3}y=8\\ \frac{1}{3}x-\frac{1}{2}y=1\end{cases}$ to find x and y. Ans. $x=12$, and $y=6$.

From the 1st $x=16-\frac{2}{3}y$, and from the 2d, $x=3+\frac{3}{2}y$; hence $3+\frac{3}{2}y=16-\frac{2}{3}y$, or $18+9y=96-4y$; whence $y=\frac{78}{13}=6$, and $x=16-\frac{2}{3}y=16-\frac{12}{3}=12$.

9. Given $\begin{cases}\frac{1}{2}x+\frac{1}{3}y=9\\ x:y::4:3\end{cases}$ to find x and y. Ans. $x=12$, and $y=9$.

From the 1st $x=18-\frac{2}{3}y$, and from the 2d $x=\frac{4}{3}y$, hence $\frac{4}{3}y=18-\frac{2}{3}y$, or $4y=54-2y$; whence $y=\frac{54}{6}=9$, and $x=\frac{4}{3}y=\frac{36}{3}=12$.

10. Given $x+y=80$, and $\frac{3}{8}x=\frac{1}{3}y$, to find x and y.

Here $x+y=80$, and $\frac{3}{8}x=\frac{1}{3}y$; then, from the 1st, $x=80-y$, and from the 2d, $x=\frac{8}{9}y$; hence $\frac{8}{9}y=80-y$, or $9y+8y=640$, whence $y=\frac{640}{17}=37\frac{11}{17}$, and $x=80-y=80-37\frac{11}{17}=42\frac{6}{17}$.

11. Given $y-6=\frac{1}{2}x$, and $x=y+6$, to find x and y. Ans. $x=24$, and $y=18$.

From the 1st, $x=2y-12$, and $2y-12=y+6$; whence $y=18$, and $x=y+6=18+6=24$.

RULE II. Find the values of either of the unknown quantities in that equation in which it is the least involved; then substitute this value in the place of its equal in the other equation, and there will arise a new equation with only one unknown quantity in it, the value of which may be found as before.

1. Given $\left\{\begin{array}{l} x+2y=17 \\ 3x-\ y=\ 2 \end{array}\right\}$ to find the values of x and y

From the first equation, $x=17-2y$; which value, being substituted for x, in the second, gives $3(17-2y)-y=2$, or $51-6y-y=2$, or $7y=51-2=49$, whence $y=\frac{49}{7}=7$, and $x=17-2y=3$.

2. Given $\left\{\begin{array}{l} x+y=13 \\ x-y=\ 3 \end{array}\right\}$ to find the values of x and y.

From the first equation, $x=13-y$; which value being substituted for x, in the second, gives $13-y-y=3$, or $2y=13-3=10$, whence $y=\frac{10}{2}=5$, and $x=13-y=8$.

3. Given $\left\{\begin{array}{l} x:y::a:b \\ x^2+y^2=c \end{array}\right\}$ to find the values of x and y.

Here the analogy in the first, turned into an equation, gives $bx=ay$, or $x=\frac{ay}{b}$, and this value, substituted for x in the second, gives $(\frac{ay}{b})^2+y^2=c$, or $\frac{a^2y^2}{b^2}+y^2=c$, whence we have $a^2y^2+b^2y^2=b^2c$, or $y^2=\frac{b^2c}{a^2+b^2}$; and consequently $y=b\sqrt{\frac{c}{a^2+b^2}}$, and $x=a\sqrt{\frac{c}{a^2+b^2}}$.

4. Given $\frac{1}{7}x+7y=99$, and $\frac{1}{7}y+7x=51$, to find the values of x and y. Ans. $x=7$, and $y=14$.

From the 1st, $x=693-49y$; which value, being substituted for x, in the second, gives $\frac{1}{7}y+7(693-49y)=51$, or $2400y=33600$, whence $y=14$, and $x=693-49y=693-686=7$.

5. Given $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$, to find the values of x and y. Ans. $x=60$, $y=40$.

From the second, $x=\frac{18y+2100}{47}$; which value, being substituted for x, in the 1st, gives $\frac{9y+1050}{47}-12=\frac{y}{4}+8$, or clearing of fractions and transposing, $11y=440$, or $y=\frac{440}{11}=40$; whence $x=\frac{18y+2100}{47}=\frac{720+2100}{47}=60$.

6. Given $x+y=s$, and $x^2-y^2=d$, to find the values of x and y.

Ans. $x=\frac{s^2+d}{2s}$, $y=\frac{s^2-d}{2s}$.

From the 1st, $x=s-y$, or $x^2=s^2-2sy+y$; which value being substituted for x, in the 2d, gives $s^2-2sy+y^2-y^2=d$, or $y=\frac{s^2-d}{2s}$, and $x=s-y=s-\frac{s^2-d}{2s}=\frac{s^2+d}{2s}$.

7. Given $5x-3y=150$, and $10x+15y=825$, to find x and y.

Ans. $x=45$, and $y=25$.

From the 1st, $x=\frac{150+3y}{5}$; which value being substituted for x, in the 2d, we shall have $\frac{1500+30y}{5}+15y=825$, or $21y=525$, whence $y=\frac{525}{21}=25$, and $x=\frac{150+3y}{5}=\frac{150+75}{5}=45$.

8. Given $x+y=16$, and $x:y::3:1$, to find x and y.

Ans. $x=12$, and $y=4$.

Here $x+y=16$, and $x:y::3:1$; then, from the 2d, $x=3y$; which value of x, being substituted in the 1st, we shall have $3y+y=16$, or $y=4$, and consequently $x=12$.

9. Given $x+\frac{1}{2}y=12$, and $y+\frac{1}{2}x=9$, to find x and y.

Ans. $x=10$, and $y=4$.

Here $x+\frac{1}{2}y=12$, and $y+\frac{1}{2}x=9$; then, from the 2d, $x=18-2y$; which value, being substituted in the 1st, we shall have $18-2y+\frac{1}{2}y=12$; hence $y=4$, and $x=10$.

10. Given $x:y::3:2$, and $x^2-y^2=20$, to find x and y.

Ans. $x=6$, and $y=4$.

Here $x:y::3:2$, and $x^2-y^2=20$; then, from the 1st, $x=\frac{3}{2}y$, or $x^2=\frac{9y^2}{4}$; which value being substituted in the 2d, gives $\frac{9y^2}{4}-y^2=20$, or $y^2=16$, hence $y=\sqrt{16}=4$, and $x=\frac{3}{2}y=\frac{12}{2}=6$.

11. Given $\frac{x}{2}-12=\frac{y}{4}+13$ and $\frac{x+y}{5}+\frac{x}{3}+16=\frac{2x-y}{4}+27$, to find x and y.

Ans. $x=60$, and $y=20$.

Here $\frac{x}{2}-12=\frac{y}{4}+13$, and $\frac{x+y}{5}+\frac{x}{3}+16=\frac{2x-y}{4}+27$; then, from the 2d, $x=\frac{660-27y}{2}$; which value, being substituted in the 1st, we shall have $\frac{660-27y}{4}=\frac{y}{4}+25$; hence, by transposing and reducing, $28y=560$, or $y=\frac{560}{28}=20$, and $x=\frac{660-27y}{2}=\frac{660-540}{2}=60$.

RULE III. Let one or both of the given equations be multiplied, or divided, by such numbers, or quantities, as will make the term that contains one of the unknown quantities the same in each of them; then, by adding, or subtracting, the two equations thus obtained, as the case may require, there will arise a new equation, with only one unknown quantity in it, which may be resolved as before.

1. Given $\left\{ \begin{array}{c} 3x+5y=40 \\ x+2y=14 \end{array} \right\}$ to find the values of x and y.

First, multiply the second equation by 3, and it will give $3x+6y=42$.

Then subtract the first equation from this, and it will give $6y-5y=42-40$, or $y=2$.

Whence, also, $x=14-2y=14-4=10$.

2. Given $\left\{ \begin{array}{c} 5x-3y=9 \\ 2x+5y=16 \end{array} \right\}$ to find the values of x and y.

Multiply the first equation by 2, and the second by 5; then $10x-6y=18$, and $10x+25y=80$.

And if the former of these be subtracted from the latter, there will arise $31y=62$, or $y=\frac{62}{31}=2$.

Whence, by the first equation, $x=\dfrac{9+3y}{5}=\dfrac{15}{5}=3$.

3. Given $\dfrac{x+8}{4}+6y=21$, and $\dfrac{y+6}{3}=23-5x$, to find x and y.

Here $\dfrac{x+8}{4}+6y=21$, and $\dfrac{y+6}{3}=23-5x$; then, from the 1st, $x+24y=76$, and from the 2d, $y+15x=63$. Multiply the 1st by 15, and we have $15x+360y=1140$, then subtract the 2d equation from this, and we shall have $359y=1077$, or $y=\frac{1077}{359}=3$, and from the 1st, $x=76-24y=76-72=4$.

4. Given $3x+7y=79$, and $2y=9+\frac{1}{2}x$, to find x and y.

Here $3x+7y=79$, and $2y=9+\frac{1}{2}x$; then, from the 2d, $4y-x=18$. Multiply this equation by 3, and we have $12y-3x=54$, to which add the 1st, and we shall have $19y=133$, or $y=\frac{133}{19}=7$, and hence, from the 2d equation, $x=4y-18=28-18=10$.

5. Given $30x+40y=270$, and $50x+30y=340$, to find x and y.

Ans. $x=5$, and $y=3$.

Here $30x+40y=270$, and $50x+30y=340$; then, multiplying the 1st by 5, and the 2d by 3, we shall have $150x+200y=1350$, and $150x+90y=1020$; then, by subtracting the latter from the former, gives $110y=330$, or $y=\dfrac{330}{110}=3$, and from the 1st $x=\dfrac{270-40y}{30}=\dfrac{270-120}{30}=5$.

6. Given $3x-3y=2x+2y$, and $x+y : xy :: 3 : 5$, to find x and y. Ans. $x=10$, and $y=2$.

Here $3x-3y=2x+2y$, and $x+y : xy :: 3 : 5$; then, from the 1st, $x=5y$, and from the 2d, $5x+5y=3xy$, or by substituting the value of x, found from the 1st, we shall have $25y+5y=15y^2$, and dividing by y and transposing $y=\frac{30}{15}=2$, and $x=5y=10$.

7. Given $x^2y+xy^2=30$, and $x^3+y^3=35$, to find x and y.

Here $x^2y+xy^2=30$, and $x^3+y^3=35$; then, adding 3 times the 1st equation to the 2d, gives $x^3+3x^2y+3xy^2+y^3=125$, and by extracting the cube root $x+y=5$, and, from the 1st, $(x+y)xy=30$; hence, by substitution, $5xy=30$, or $xy=6$, and $x^2+2xy+y^2=25$, from which subtract 4 times the last equation, and we shall have $x^2-2xy+y^2=1$, and by evolution $x-y=1$; adding this equation to $x+y=5$, and we shall have $2x=6$, or $x=3$, whence $y=2$.

8. Given $\frac{3x-5y}{2}=\frac{2x+y}{5}-3$, and $8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}$ to find x and y. Ans. $x=12$, and $y=6$.

Here $\frac{3x-5y}{2}=\frac{2x+y}{5}-3$, and $8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}$; then from the 1st, $27y-11x=30$, and, from the 2d, $9x-2y=96$. Multiplying the 1st of these equations by 2, and the second by 27, we shall have $54y-22x=60$, and $243x-54y=2592$; then, by adding the two last equations together, we have $221x=2652$, or $x=\frac{2652}{221}=12$, and $9x-2y=96$; hence $2y=9x-96=108-96=12$, hence $y=\frac{12}{2}=6$.

9. Given $x+y : a :: x-y : b$, and $x^2-y^2=c$, to find the values of x and y.

In the first of the two equations $(x+y) : a :: (x-y) : b$, and $x^2-y^2=c$; $\therefore$ $b(x+y)=a(x-y)$, or $(b+a)y=(a-b)x$.

Whence $y=\frac{a-b}{a+b}x$, and $y^2=\frac{(a-b)^2}{(a+b)^2}x^2$. Substitute this for y^2 in the 2d equa., and we have $x^2-\frac{(a-b)^2}{(a+b)^2}x^2=c$, or

$$x^2(a+b)^2-(a-b)^2x^2=c(a+b)^2, \text{ or}$$
$$x^2\{(a+b)^2-(a-b)^2\}=c(a+b)^2, \text{ or}$$

$x^2(4ab)=c(a+b)^2$; whence $x^2=\frac{c(a+b)^2}{4ab}$, or $x=\frac{a+b}{2}\sqrt{\frac{c}{ab}}$, $y^2=x^2-c=\frac{c(a+b)^2}{4ab}-c=\frac{c(a-b)^2}{4ab}$, or $y=\frac{a-b}{2}\sqrt{\frac{c}{ab}}$, the answer.

10. Given $ax+by=c$, and $dx+ey=f$, to find the values of x and y. Ans. $x=\frac{ce-bf}{ae-bd}$, $y=\frac{af-dc}{ae-bd}$.

11. Given $\left.\begin{array}{l}ax+by=c\\ dx+ey=f\end{array}\right\}$

Mult. 1st by d, $dax+dby=dc$

Mult. 2d by a, $dax+aey=af$

By subt. $dby-aey=dc-af$

Whence $y=\frac{dc-af}{db-ae}=\frac{af-dc}{ae-db}$

Mult. 1st by e, $eax+eby=ec$

Mult. 2d by b, $bdx+eby=bf$

By subtrac. $eax-bdx=ec-bf$

Whence $x=\frac{ec-bf}{ea-bd}=\frac{bf-ce}{bd-ea}$.

12. Given $x+y=a$, and $x^2-y^2=b$. Here divide $x^2-y^2=b$ by $x+y=a$; and we have $x-y=\frac{b}{a}$

Whence by add. $2x=a+\frac{b}{a}$, $x=\frac{a^2+b}{2a}$.

And by subt. $2y=a-\frac{b}{a}$, or $y=\frac{a^2-b}{2a}$.

13. Given $x^2+xy=a$, and $y^2+xy=b$, to find the values of x and y.

Here $\left.\begin{array}{l}x^2+xy=a\\ y^2+xy=b\end{array}\right\}$

By addition, $x^2+2xy+y^2=a+b$, or $(x+y)^2=a+b$, or $x+y=\sqrt{(a+b)}$. Now the two proposed equations may be put under the form $x(x+y)=a$, or $x\sqrt{(a+b)}=a$
$y(x+y)=b$, or $y\sqrt{(a+b)}=b$.

Whence, by division, $x=\frac{a}{\sqrt{(a+b)}}$; and $y=\frac{b}{\sqrt{(a+b)}}$.

Of the resolution of simple equations, containing three or more unknown quantities.

When there are three unknown quantities, and three independent simple equations containing them, they may be reduced to one by the following method.*

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* The necessity for observing that the given equations in this and other similar cases are so proposed as to be independent of each other, will be obvious from the following example:

$$x-2y+z=5;\ 2x+y-z=7;\ x+3y-2z=2;$$

where, if it were required to determine the value of x, y, and z, it will be found, by eliminating x from each of them, and then equating the results, that $5y-3z=-3$, and $5y-3z=-3$; which equations, being identical, or both the same, furnish no determinate answer. And, in effect, if the three equations be properly examined, it will be found, that the third is merely the difference of the first and second, and consequently involves no condition but what is contained in the other two.

Rule. Find the values of one of the unknown quantities in each of the three given equations, as if all the rest were known; then put the first of these values equal to the second, and either the first or second equal to the third, and there will arise two new equations with only two unknown quantities in them, the values of which may be found as in the former case; and thence the value of the third.

Or, multiply each of the equations by such numbers, or quantities, as will make one of their terms the same in them all; then having subtracted any two of these resulting equations from the third, or added them together, as the case may require, there will remain only two equations, which may be resolved by the former rules.

And in nearly the same way may four, five, &c. unknown quantities be exterminated from the same number of independent simple equations; but, in cases of this kind, there are frequently shorter and more commodious methods of operation, which can only be learnt from practice.*

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=29 \\ x+2y+3z=62 \\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=10 \end{array}\right\}$ to find x, y, and z.

Here, from the first equation, $x=29-y-z$.

From the second, $x=62-2y-3z$.

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* The values of the unknown quantities in the three literal equations $ax+by+cz=d$; $a'x+b'y+c'z=d'$; $a''x+b''y+c''z=d''$, may be exhibited in general terms, like those before mentioned, as follows:

$$x=\frac{db'c''-dc'b''+cd'b''-bd'c''+bc'd''-cb'd''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}$$

$$y=\frac{ad'c''-ac'd''+ca'd''-da'c''+dc'a''-cd'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}$$

$$z=\frac{ab'd''-ad'b''+da'b''-ba'd''+bd'a''-db'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}$$

which formulæ, by substitution, may be employed for the resolution of any numeral case of this kind, as in the instance of two equations before given. The numerator of any of these equations, such as z, consists of all the different products, which can be made of three opposite coefficients taken from the orders in which z is not found; and the denominator consists of all the products that can be made of the three opposite coefficients taken from the orders which involve the three unknown quantities.

And from the third, $x=20-\frac{2}{3}y-\frac{1}{2}z$.
Whence $29-y-z=62-2y-3z$,
And, also, $29-y-z=20-\frac{2}{3}y-\frac{1}{2}z$,
From the first of which, $y=33-2z$,
And from the second, $y=27-\frac{3}{2}z$,
Therefore, $33-2z=27-\frac{3}{2}z$, or $z=12$;
Whence, also, $y=33-2z=9$,
And $x=29-y-z=8$.

2. Given $\left\{\begin{array}{l} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-\ z=63 \end{array}\right\}$ to find x, y, and z.

Here, multiplying the first equation by 6, the second by 3, and the third by 2, we shall have $\left\{\begin{array}{l} 12x+24y-18z=132, \\ 12x-\ 6y+15z=\ 54, \\ 12x+14y-\ 2z=126. \end{array}\right\}$ And, subtracting the second of these equations successively from the first and third, there will arise $\left\{\begin{array}{l} 30y-33z=78, \\ 20y-17z=72. \end{array}\right\}$ Or, by dividing the first of these two equations by 3, and then multiplying the result by 2, $\left\{\begin{array}{l} 20y-22z=52, \\ 20y-17z=72. \end{array}\right\}$

Whence, by subtracting the former of these from the latter, we have $5z=20$, or $z=4$. And, consequently, by substitution and reduction, $y=7$, and $x=3$. Otherwise,

Multiplying the first equat. by 2,	$4x+8y-6z=44$
and subtracting the second,	$4x-2y+5z=18$
there results (A)	$10y-11z=26$
Again, mult. the first equation by 3,	$6x+12y-9z=66$
and subtracting the third,	$6x+\ 7y-\ z=63$
there results	$5y-\ 8z=\ 3$
and mult. this result by 2,	$10y-16z=\ 6$
which subtracted from equation (A),	$10y-11z=26$
gives	$5z=20$

$\therefore z=4,\ y=\left(\frac{3+8z}{5}\right)=7$, and $x=\left(\frac{22-4y+3z}{2}\right)=3$.

3. Given $x+y+z=53$, $x+2y+3z=105$, and $x+3y+4z=134$, to find the values of x, y, and z.

Subtract 1st from 2d, $y+2z=52$
Subtract 2d from 3d, $y+\ z=29$ } Now, subtracting the latter from the preceding one, we have $z=23$.
Also from the last, $y=29-z=29-23=6$,
And from the first, $x=53-y-z=53-29=24$,
That is, $x=24$, $y=6$, and $z=23$.

4. Given $x+\frac{1}{2}y+\frac{1}{3}z=32$, $\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15$, and $\frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12$, to find the values of x, y and z.

Multiplying the first by six, and the second and third by 60, we have

$$\left.\begin{array}{r}6x+3y+2z=192\\20x+15y+12z=900\\15x+12y+10z=720\end{array}\right\}$$

Again, multiply the first by 10, the second by 3, and the third by 4, give

$$\begin{array}{r}60x+30y+20z=1920\\60x+45y+36z=2700\\60x+48y+40z=2880.\end{array}$$

Subtracting now the first of these from the 2d, and the second from the 3d, we have

$$\left\{\begin{array}{r}15y+16z=780\\3y+4z=180\end{array}\right\}$$

Mult. the latter by 5, $15y+20z=900$
Subtract............ $15y+16z=780$

$$4z=120, \text{ or } z=30.$$

But $y=\dfrac{180-4z}{3}=20$, and $x=\dfrac{192-3y-2z}{6}=12$; therefore, $x=12$, $y=20$, and $z=30$.

5. Given $7x+5y+2z=79$, $8x+7y+9z=122$, and $x+4y+5z=55$, to find the values of x, y and z.

Multiply the first by 8, the second by 7, and the third by 56, and we have

$$\left\{\begin{array}{r}56x+40y+16z=632\\56x+49y+63z=854\\56x+224y+280z=3080\end{array}\right\}$$

Subtract the first from the 2d, and the second from the 3d, and we obtain

$$\left.\begin{array}{r}9y+47z=222\\175y+217z=2226\end{array}\right\}$$

Multiply the first of these by 175, and the second by 9, and we have

$$\begin{array}{r}1575y+8225z=38850\\1575y+1953z=20034\end{array}$$

By subtraction $6272z=18816$, or $z=3$.

But $y=\dfrac{222-47z}{9}=9$, and $x=\dfrac{55-4y-5z}{1}=4$.

That is, $x=4$, $y=9$, and $z=3$.

6. Given $\left.\begin{array}{l}x+y=a\\x+z=b\\y+z=c\end{array}\right\}$ to find the values of x, y and z.

By addition, $2x+2y+2z=a+b+c$, or
$x+y+z=\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c$; from which, subtracting each of the three given equations respectively, we have

$$\left\{\begin{array}{l}z=-\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c\\y=\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}c\\x=\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}c\end{array}\right\} \text{ the values sought.}$$

7. Given $\frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62$, $\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47$, and $\frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38$, to find x, y and z.

By clearing of fractions, these become $6x+4y+3z=744$, $20x+15y+12z=2820$, and $15x+12y+10z=2280$.

Multiplying the 1st by 20, the second by 6, and the 3d by 8, we shall have

$$120x+80y+60z=14880$$
$$120x+90y+72z=16920$$
$$120x+96y+80z=18240$$

And, subtracting the 1st of these equations from the second, and the second from the third, there will arise $10y+12z=2040$, and $6y+8z=1320$. Or, multiplying the 1st of these two equations by 3, and 2d by 5, we shall have $30y+36z=6120$, and $30y+40z=6600$; whence, by subtracting the former of these from the latter, we have $4z=480$, or $z=120$; and, consequently, by substitution and reduction, $x=24$, and $y=60$.

8. Given $z+y=x+100$, $y-2x=2z-100$, and $z+100=3x+3y$, to find x, y and z.

Then, by transposing, $z+y-x=100$, $y-2x-2z=-100$, and $z-3x-3y=-100$. And, by adding twice the 1st to the second, and subtracting the third from the 1st, we shall have $3y-4x=100$, and $2x+4y=200$; multiplying the 2d of these two equations by 2, and adding the result to the first, we have $11y=500$, or $y=45\frac{5}{11}$; and, consequently, by substitution and reduction, $x=9\frac{1}{11}$, and $z=63\frac{7}{11}$.

9. Given $x+y+z=7$, $2x-3=y+3z$, and $5x+5z=3y+19$, to find x, y and z.

Then, by adding the 1st to the second, and 3 times the 1st to the third, and transposing the terms of the second and third, we shall have $3x-2z=10$, and $8x+8z=40$. Or, dividing the latter of these two equations by 4, we have $2x+2z=10$, and, adding this to the former, $5x=20$, or $x=4$; and, consequently, $z=1$, and $y=2$.

10. Given $3x+5y-4z=25$, $5x-2y+3z=46$, and $3y+5z-x=62$, to find x, y and z.

Then, adding 3 times the 3d to the 1st, and 5 times the 3d to the 1st, we shall have

$$14y+11z=211, \text{ and } 13y+28z=356.$$

Or, multiplying the 1st of these two equations by 13, and the second by 14, there will result,

$$182y+143z=2743, \text{ and } 182y+392z=4984.$$

And, subtracting the former from the latter,

$$249z=2241, \text{ or } z=\tfrac{2241}{249}=9.$$

And, consequently, by substitution and reduction, $y=8$, and $x=7$.

11. * Given $x+y+z=13$, and $x+y+u=17$, $x+z+u=18$, and $y+z+u=21$, to find x, y, z, and u.

Then, assume $S=x+y+z+u$, and the above equations will be transformed into the following ones: $S-u=13$, $S-z=17$, $S-y=18$, and $S-x=21$. Add all these equations together, and we have $4S-x-y-z-u=69$, that is, $4S-(x+y+z+u)=69$.

But $x+y+z+u=S$, therefore $4S-S=69$, or $3S=69$, and $S=23$, and, consequently, by substituting for S its value in the four transformed equations, we shall have

$$23-u=13,\ 23-z=17,\ 23-y=18, \text{ and } 23-x=21.$$

And, consequently, $x=2$, $y=5$, $z=6$, and $u=10$.

MISCELLANEOUS QUESTIONS,

PRODUCING SIMPLE EQUATIONS.

The usual method of resolving algebraic questions, is first to denote the quantities that are to be found by x and y, or some of the other final letters of the alphabet; then, having properly examined the state of the question, perform with these letters, and the known quantities, by means of the common signs, the same operations and reasonings that it would be necessary to make if the quantities were known, and it was required to verify them, and the conclusion will give the result sought.

Or, it is generally best, when it can be done, to denote only one of the unknown quantities by x, y, or z, and then to determine the expression for the others from the nature of the question; after which the same method of reasoning may be followed, as above. And, in some cases, other methods of proceeding may be used; which practice and observation alone can suggest.

1. What number is that whose third part exceeds its fourth part by 16?

Let $x=$ the number required.

Then its $\frac{1}{3}$ part will be $\frac{1}{3}x$, and its $\frac{1}{4}$ part $\frac{1}{4}x$.

And therefore $\frac{1}{3}x-\frac{1}{4}x=16$, by the question.

That is, $x-\frac{3}{4}x=48$, or $4x-3x=192$.

Hence $x=192$, the number required.

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* This can be resolved by proceeding after the same manner as equations involving three unknown quantities; but the resolution of it may be greatly facilitated, by introducing into the calculation, besides the principal unknown quantities, a new unknown quantity arbitrarily assumed; such as, for example, the sum of all the rest: and, when a little practised in such calculations, they become easy.

2. It is required to find two numbers such that their sum shall be 40, and their difference 16.

Let x denote the least of the two numbers required,

Then will $x+16$ = the greater number,

And $x+x+16=40$, by the question.

That is, $2x=40-16$, or $x=\frac{24}{2}=12=$ least number.

3. Divide \$1000 between A, B, and C, so that A shall have \$72 more than B, and C \$100 more than A.

Let x = B's share of the given sum; then will $x+72$ = A's share, and $x+172$ = C's share. Hence their sum is $x+x+72+x+172$, or $3x+244=1000$, by the question; that is, $3x=1000-244=756$: or, $x=\frac{756}{3}=\$252=$ B's share.

Hence $x+\ \ 72=\$324=$ A's share.

And $x+172=\$424=$ C's share.

Also, as above, $\$252=$ B's share.

Sum of all \$1000, the proof.

4. It is required to divide \$1000 between two persons, so that their shares of it shall be in the proportion of 7 to 9.

Let x = the first person's share.

Then will $1000-x$ = second person's share,

And $x : 1000-x :: 7 : 9$, by the question.

That is, $9x=(1000-x)\times7=7000-7x$,

Or $9x+7x = 7000$, or $x=\frac{7000}{16}=\$437\frac{1}{2}=$ 1st share, and $1000-x = 1000 - \$437\frac{1}{2} = \$562\frac{1}{2} =$ 2d share.

5. The paving of a square court with stones, at 2*s.* a yard, will cost as much as the enclosing it with pallisades at 5*s.* a yard; required the side of the square.

Let x = length of the side of the square sought;

Then $4x$ = number of yards of enclosure,

And x^2 = number of yards of pavement.

Hence $4x\times5=20x$ = price of enclosing it,

And $x^2\times2=2x^2$ = the price of the paving.

Therefore $2x^2=20x$, by the question,

Or $2x=20$, and $x=10$, the length of the side required.

6. Out of a cask of wine, which had leaked away a third part, 21 gallons were afterwards drawn, and the cask being then gauged, appeared to be half full; how much did it hold?

Let x = the number of gallons the cask is supposed to have held. Then would it have leaked away $\frac{1}{3}x$ gallons; whence there had been taken out of it, altogether, $21+\frac{1}{3}x$ gallons, and therefore $21+\frac{1}{3}x=\frac{1}{2}x$, by the question; that is, $63+x=\frac{3}{2}x$, or $126+2x=3x$: consequently, $3x-2x=126$, or $x=126$, the number of gallons required.

7. What fraction is that, to the numerator of which if 1 be added, its value will be $\frac{1}{3}$, but if one be added to the denominator, its value will be $\frac{1}{4}$?

Let the fraction required be represented $\frac{x}{y}$.

Then $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$, by the question.

Hence $3x+3=y$, and $4x=y+1$, or $x=\frac{y+1}{4}$.

Therefore $3\left\{\frac{y+1}{4}\right\}+3=y$, or $3y+3+12=4y$;

That is, $y=15$, and $x=\frac{y+1}{4}=\frac{15+1}{4}=\frac{16}{4}=4$;

Whence the fraction that was to be found is $\frac{4}{15}$.

8. A market woman bought in a certain number of eggs at 2 a penny, and as many others at 3 a penny, and having sold them out again, altogether, at the rate of 5 for 2*d.*, found she had lost 4*d.*; how many eggs had she?

Let x = the number of eggs of each sort;
Then will $\frac{1}{2}x$ = the price of the first sort,
And $\frac{1}{3}x$ = the price of the second sort.
But $5 : 2 :: 2x$ (the whole number of eggs) : $\frac{4}{5}x$;
Whence $\frac{4}{5}x$ = the price of both sorts, when mixed together, at the rate of 5 for 2*d.*
And consequently $\frac{1}{2}x+\frac{1}{3}x-\frac{4}{5}x = 4$, by the question;
That is, $15x+10x-24x = 120$, or $x = 120$, the number of eggs of each sort, as required.

9. If A can perform a piece of work in 10 days, and B in 13, in what time will they finish it, if they are both set about it together?

Let the time sought be denoted by x.

Then $\frac{x}{10}$ = the part done by A in one day,

And $\frac{x}{13}$ = the part done by B in one day;

Consequently $\frac{x}{10}+\frac{x}{13}=1$, (the whole work;)

That is, $13x+10x = 130$, or $23x = 130$:
Whence $x = \frac{130}{23} = 5\frac{15}{23}$ days, the time required.

10. If one agent, A, alone, can produce an effect, e, in the time a, and another agent, B, alone in the time b; in what time will both of them together produce the same effect?

Let the time sought be denoted by x.

Then $a : e :: x : \frac{ex}{a}$ = part of the effect produced by A,

And $b : e :: x : \frac{ex}{b}$ = part of the effect produced by B.

Hence $\frac{ex}{a} + \frac{ex}{b} = e$, (the whole effect,) by the question.

Or, $\frac{x}{a} + \frac{x}{b} = 1$, by dividing each side by e. Therefore $x + \frac{ax}{b} = a$, or $bx + ax = ab$; consequently, $x = \frac{ab}{a+b}$ = time required.

11. How much rye, at 4*s*. 6*d*. a bushel, must be mixed with 50 bushels of wheat, at 6*s*. a bushel, so that the mixture may be worth 5*s*. a bushel?

Let x = the number of bushels required.
Then $9x$ is the price of the rye in sixpences,
And 600 the price of the wheat in ditto;
Also, $(50+x) \times 10$ the price of both in ditto;
Whence $9x+600=500+10x$, by the question,
Or, by transposition, $10x-9x=600-500$:
Consequently, $x=100$, the number of bushels required.

12. A laborer engaged to serve for 40 days, on condition that for every day he worked he should receive 20*d*., but for every day he was absent he should forfeit 8*d*.; now, at the end of the time, he had to receive 1*l*. 11*s*. 8*d*.; how many days did he work, and how many was he idle?

Let the number of days that he worked be denoted by x; then will $40-x$ be the number of days he was idle; also $20x$ the sum earned, and $(40-x) \times 8$, or $320-8x$, the sum forfeited; whence $20x-(320-8x)=380d.$ (=1*l*. 11*s*. 8*d*.), by the question: that is, $20x-320+8x=380$, or $28x=380+320=700$; consequently, $x=\frac{700}{28}=25$, the number of days he worked, and $40-x=40-25=15$, the number of days he was idle.

13. It is required to divide a line, of 15 inches in length, into two such parts, that one may be three-fourths of the other.

Let one of the parts $= x$. Then the other will be $= 15-x$,
And by the question, $15-x=\frac{3}{4}x$, or $60-4x=3x$, or $7x=60$; whence $x=\frac{60}{7}=8\frac{4}{7}$, the one part; and $15-x=6\frac{3}{7}$, the other.

14. My purse and money together are worth 20*s*., and the money is worth 7 times as much as the purse. How much is there in it?

Let the value of the purse be x; then the money $= 7x$; and by the question $7x+x=20$, or $8x=20$, or $x=\frac{20}{8}$, 2*s*. 6*d*., the value of the purse, and consequently 17*s*. 6*d*., the money contained in it.

15. A shepherd being asked how many sheep he had in his flock, said, 'If I had as many more, half as many more, and 7 sheep and a half, I should have just 500.' How many had he?

Let the number of sheep $=x$; then, by the question, $x+x+\frac{1}{2}x+7\frac{1}{2}=500$; or $2\frac{1}{2}x=492\frac{1}{2}$; mult. by 2, $5x=985$; Whence $x=\frac{985}{5}=197$, the number sought.

16. A post is one-fourth of its length in the mud, one-third in the water, and 10 feet above the water. What is its whole length?

Let the length of the post $=x$; then by the question $\frac{1}{4}x+\frac{1}{3}x+10=x$; mult. by 12, $3x+4x+120=12x$;

Transposing, $5x=120$, or $x=\frac{120}{5}=24$ feet, the answer required.

17. After paying away $\frac{1}{4}$ of my money, and then $\frac{1}{5}$ of the remainder, I had 72 guineas left. What had I at first?

Let the number of guineas $=x$; then $x-\frac{1}{4}x-\frac{3}{20}x=72$; or mult. by 20, $20x-5x-3x=1440$; whence $12x=1440$, or $x=\frac{1440}{12}=120$ guineas.

18. It is required to divide \$300 between A, B and C, so that A may have twice as much as B, and C as much as A and B together.

Let B's share $=x$; then, by the question,
A's share $=2x$,
C's share $=3x$;

Consequently, $x+2x+3x=300$, or $6x=300$; whence $x=\frac{300}{6}=\$50$, B's share; $2x=\$100$, A's share, and $3x=\$150$, C's share.

19. A person, at the time he was married, was 3 times as old as his wife; but, after they had lived together 15 years, he was only twice as old. What were their ages on the wedding day?

Let the age of the wife at the time of the marriage $=x$, and that of the husband $3x$;

Then after 15 years their ages will be $x+15$, and $3x+15$; and by the question, $3x+15=2(x+15)$, or $3x+15=2x+30$; whence, by transposing, $x=15$, the age of the wife; and $3x=45$, the age of the husband.

20. What number is that from which if 5 be subtracted, two-thirds of the remainder will be 40?

Let the number sought $=x$; then by the question $x-5$ will be the remainder, and $\frac{2(x-5)}{3}=40$; whence $2x-10=120$, or $2x=130$; and consequently $\frac{130}{2}=65$, the number sought.

21. At a certain election, 1296 persons voted, and the successful candidate had a majority of 120. How many voted for each?

Let the less number of voters equal x; then the greater number $=x+120$, and by the question,

$$x+x+120=1296;$$

Whence $2x=1296-120=1176$; conseq. $x=588$ for one candidate, and $x+120=708$ for the other.

22. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140. What is the age of each?

Let x represent the age of C; then

$3x$ is the age of B, and

$6x$ the age of A.

Now by the question, $x+3x+6x=10x=140$; whence $x=\frac{140}{10}=14$, C's age; $3x=42$, B's age, and $6x=84=$A's age.

23. Two persons, A and B, lay out equal sums of money in trade; A gains \$126, and B loses \$87, and A's money is now double of B's. What did each lay out?

Let the equal sum laid out by each be x; then A leaves off with $x+126$, and B with $x-87$; and by the question $x+126=2(x-87)$; or $x+126=2x-174$;

Whence $x=\$300$, the first stock of each.

24. A person bought a chaise, horse and harness for 60*l.*; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness. What did he give for each?

Let the price of the harness $=x$; then the price of the horse $=2x$, and the price of the chaise $=6x$; and by the question $x+2x+6x=60l.$, or $9x=60$;

Whence $x=\frac{60}{9}=6l.\ 13s.\ 4d.$, the value of the harness; $2x=13l.\ 6s.\ 8d.$, the horse, and $6x=40l.$, the chaise.

25. A person was desirous of giving 3*d.* apiece to some beggars, but found he had not money enough in his pocket by 8*d.*; he therefore gave them each 2*d.*, and had then 3*d.* remaining. Required the number of beggars.

Let x denote the number of beggars; then, by the question, $3x-8$ was the number of pence he had about him;

Which, from the other part of the question, may also be denoted by $2x+3$;

Whence $3x-8=2x+3$, or, by transposing, $x=11$, the number of beggars.

26. A servant agreed to live with his master for 8*l.* a year, and a livery, but was turned away at the end of seven months, and received only 2*l.* 13*s.* 4*d.* and his livery. What was its value?

Let x denote the value of the livery; then $x+8$ is the whole amount of his hire for the year, or for 12 months.

$$12 : 8 :: 7 : 4\tfrac{2}{3}.$$

$$5 : 4\tfrac{3}{5}-2\tfrac{3}{5}=2 : : 12 : 4\tfrac{4}{5}.$$

Hence, as $12 : 7 : : x+8 : \frac{7x+56}{12}$, the hire for 7 months; but, by the question, the servant received $x+2\tfrac{2}{3}$; whence $\frac{7x+56}{12}=x+2\tfrac{2}{3}$, or $7x+56=12x+32$,

And by transposition, $5x=24$, or $x=\frac{24}{5}=4\tfrac{4}{5}l.=4l.\ 16s.$

In the preceding examples only one unknown quantity has been employed; but it will be more convenient in many of the following questions to use two or more unknown letters, according to the nature of the equation.

27. A person left 560*l.* between his son and daughter, in such a manner, that for every half crown the son should have, the daughter was to have a shilling. What were their respective shares?

Here let x denote the son's share, and y the daughter's;

Then the value of their shares will obviously have to each other the same ratio as half a crown to a shilling; that is, as 5 to 2. Hence, then, we have $\left.\begin{array}{r} x : y : : 5 : 2 \\ \text{And } x+y=560 \end{array}\right\}$ From the first $2x=5y$, or $x=\frac{5}{2}y$, whence, from the second, $\frac{5}{2}y+y=560$; or $5y+2y=7y=1120$; where $y=\frac{1120}{7}=160l.$, the daughter's share, and $x=\frac{5}{2}y=400l.$, the son's share.

28. There is a certain number, consisting of two places of figures, which is equal to four times the sum of its digits; and if eighteen be added to it, the digits will be inverted. What is the number?

Here it may be observed, that every number consisting of two digits is equal to ten times the digits in the tens place, plus that in the units.

If, therefore, x be put for the former, and y for the latter, the number itself will be denoted by $10x+y$; and the number with the digits inverted by $10y+x$.

Hence, by question, $\left\{\begin{array}{l} 10x+y=4x+4y \\ 10x+y+18=10y+x \end{array}\right\}$

Where the second equation gives $9x-9y=-18$,

or $x-y=-2$, or $x=y-2$;

Which, substituted in the 1st, gives $10(y-2)+y=4(y-2)+4y$.

And this, by multiplication and transposition, becomes

$10y+y-4y-4y=20-8$, or $3y=12$;

Whence $y=\frac{12}{3}=4$, and $x=y-2=2$;

Therefore the number sought is 24.

29. Two persons, A and B, have both the same income; A saves a fifth of his yearly, but B, by spending \$50 per annum more than A, at the end of four years finds himself **\$100** in debt. What was their income?

Let x represent the income of each; then, by the question, A's yearly expenditure is $\frac{4}{5}x$, and that of B $\frac{4}{5}x+50$; in 4 years $\therefore$ B spends $\frac{16}{5}x+200$, which exceeds his income in the same time (viz. $4x$) by 100; hence we have the equation $\frac{16}{5}x+200=4x+100$, or $16x+1000=20x+500$, whence $4x=500$, or $x=\frac{500}{4}=\$125$.

30. When a company at a tavern came to pay their reckoning, they found, that if there had been three persons more, they would have had a shilling apiece less to pay, and if there had been two less, they would had a shilling apiece more to pay. Required the number of persons, and the quota of each.

Let the number of persons in company be x, and the number of shillings each paid $=y$; then xy will be the whole reckoning.

Now had there been three persons more in company, viz. $(x+3)$, each would have paid $(y-1)$ shillings; whence we have $(x+3)(y-1)=xy$; and from the other conditions of the question, $(x-2)(y+1)=xy$; whence, from actual multiplication, these become $\left.\begin{array}{l}xy+3y-x-3=xy\\ xy-2y+x-2=xy\end{array}\right\}$; and, consequently, by addition, we have $2xy+y-5=2xy$.

Or, cancelling the $2xy$ on both sides $y-5=0$, or $y=5$, the number of shillings each paid. And, by subtracting the second equation from the first, $5y-2x-1=0$; whence $2x=5y-1$, or $x=\dfrac{5y-1}{2}=\frac{24}{2}=12$, the number of persons in company.

31. A person at a tavern borrowed as much money as he had about him, and out of the whole spent 1*s.*; he then went to a second tavern, where he also borrowed as much as he had now about him, and out of the whole spent 1*s.*; and going on, in this manner, to a third and fourth tavern, he found, after spending his shilling at the latter, that he had nothing left. How much money had he at first?

Let x denote the money he had about him; then, by borrowing x and spending one shilling, he had left $2x-1$.

Also, at the second tavern, after borrowing $2x-1$, he had $4x-2$; but spending one shilling, he had left $4x-3$.

At the third tavern he borrowed $4x-3$, and then had $8x-6$; and, after spending one shilling, he had left $8x-7$.

At the fourth tavern, borrowing $8x-7$, he had $16x-14$; but, after spending another shilling, he had left $16x-15$; which, by the question, is equal to nothing.

Whence $16x-15=0$, or $x=\frac{15}{16}=0s.\ 11\frac{1}{4}d.$ the money he had at first.

32. It is required to divide the number 75 into two such parts, that three times the greater shall exceed seven times the less by 15.

Let x and y denote the two parts; then, by the question,

$$\left.\begin{array}{l} x+y=75, \text{ and} \\ 3x-7y=15 \end{array}\right\} \left\{\begin{array}{l} \text{Mult. the first by } 3, \ldots\ldots 3x+3y=225 \\ \text{Subtract the 2d,} \ldots\ldots\ldots\ldots 3x-7y=\ \ 15 \end{array}\right.$$

$$\text{And we have} \ldots\ldots\ldots\ldots\ 10y=210$$

Whence $y=\frac{210}{10}=21$, the least,

And $x=75-y=54$, the greatest number.

33. In a mixture of British spirits and water, $\frac{1}{2}$ of the whole plus 25 gallons was spirits, and $\frac{1}{3}$ part minus 5 gallons was water. How many gallons were there in each?

Let $x=$ the whole quantity of the mixture;

then $\frac{1}{2}x+25$ was the quantity of spirits, and

$\frac{1}{3}x-\ 5$ the quantity of water,

Which, altogether, made the whole x; therefore, by addition,

$\frac{1}{2}x+\frac{1}{3}x+20=x$; or, by mult. by 6,

$3x+2x+120=6x$;

Whence, by transposing, $x=120$, and conseq.

$\frac{1}{2}x+25=85$ gallons of spirits,

$\frac{1}{3}x-\ 5=35$ gallons of water.

34. A bill of 120*l.* was paid in guineas and moidores, and the number of pieces of both sorts that were used were just 100; how many were there of each, reckoning the guinea at 21*s.*, and the moidores at 27*s.*?

Let $x=$ the number of guineas, and

$y=$ the number of moidores;

Then $21x=$ the shillings paid in guineas,

And $27y=$ the shillings paid in moidores.

Now, the whole number of pieces used being 100, and the number of shillings paid being 2400, we have,

$$\left.\begin{array}{l} x+\ \ y=\ 100 \\ 21x+27y=2400 \end{array}\right\} \begin{array}{l} \text{Multiply the first by} \\ \text{27, and we shall have} \end{array} \left\{\begin{array}{l} 27x+27y=2700 \\ 21x+27y=2400 \end{array}\right.$$

Hence, by subtract. $6x=300$, or $x=50$, the number of guineas; and consequently $y=100-x$, or $y=50$, the number of moidores.

35. Two travellers set out at the same time from London and York, whose distance is 197 miles. One of them goes 14 miles a day, and the other 16. In what time will they meet?

Let x be the number of days; then $14x$ miles will be travelled by one, and $16x$ miles by the other.

Hence $30x=197$, or $x=\frac{197}{30}=6^{d}.\ 13^{h}\frac{3}{5}$, the time required.

36. There is a fish whose tail weighs 9*lb.*; his head weighs as much as his tail and half his body, and his body weighs as

much as his head and his tail. What is the whole weight of the fish?

Let x to denote the weight of the body; then $\frac{1}{2}x+9$ is the weight of the head, and, by the question, $x=\frac{1}{2}x+9+9$, or $\frac{1}{2}x=18$;

Whence $x=36$, the weight of the body, $\frac{1}{2}x+9=27$, weight of the head, and 9 the weight of the tail; consequently $36+27+9=72lbs.$, the weight of the fish.

Otherwise,

Let $2x=$ the weight of the body; then $9+x=$ the weight of the tail. $\therefore$ $9+9+x=2x$; by transposition, $x=18$; $\therefore$ the fish weighed $36+27+9=72lbs.$

37. It is required to divide the number 10 into three such parts, that if the first be multiplied by 2, the second by 3, and the third by 4, the three products shall be all equal.

Let $\frac{1}{2}x$, $\frac{1}{3}x$ and $\frac{1}{4}x$ represent the three parts required, and the three latter conditions of the question will be answered:

For the first multiplied by 2, the second by 3, and the third by 4, will obviously be all equal to x, and therefore equal to each other. Hence, then, it only remains to fulfil the equation. $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=10$; or, multiplying by 12, to clear it of fractions, $6x+4x+3x=120$, whence $13x=120$, or $x=\frac{120}{13}$.

Therefore, $\frac{120}{2\times13}=4\frac{8}{13}$; $\frac{120}{3\times13}=3\frac{1}{13}$, and $\frac{120}{4\times13}=2\frac{4}{13}$, are the parts sought.

38. It is required to divide the number 36 into three such parts, that $\frac{1}{2}$ the first, $\frac{1}{3}$d of the second, and $\frac{1}{4}$ of the third, shall be all equal to each other.

Let $2x$, $3x$ and $4x$ be the three parts; then it is obvious that $\frac{1}{2}$ the first, $\frac{1}{3}$d of the second, and $\frac{1}{4}$th of the third, are equal to each other;

Wherefore, there only remains the equation

$$2x+3x+4x=36,$$

Whence $9x=36$, or $x=\frac{36}{9}=4$;

Conseq. $2x=8$, $3x=12$, and $4x=16$, are the parts required.

39. A person has two horses, and a saddle, which of itself is worth \$50. Now, if the saddle be put on the back of the first horse, it will make his value double that of the second, and if it be put on the back of the second, it will make his value triple that of the first. What is the value of each horse?

Let the value of the first horse be x, and of the second y; then by the question,

$$\left.\begin{array}{l} x+50=2y \\ y+50=3x \end{array}\right\}, \text{ or, by transposition, } \left\{\begin{array}{l} 2y-x=50 \\ -y+3x=50 \end{array}\right.$$

Multiply the latter by 2, and we shall have $-2y+6x=100$;

to which adding $2y-x=50$, we have $5x=150$, or $x=\$30$, the value of the first horse; and, from the second equation, $y=3x-50=\$40$, the value of the second.

40. If A gives B 5*s*. of his money, B will have twice as much as the other has left; and if B gives A 5*s*. of his money, A will have three times as much as the other has left. How much has each?

Let $x=$ A's money, and $y=$ B's. Then, when B gives A 5*s*. the latter will have $x+5$, and the former $y-5$; and if A gives B 5*s*., then A will have $x-5$, and B $y+5$.

Now, by the question,

$$\left.\begin{array}{l} y+5=2(x-5) \\ x+5=3(y-5) \end{array}\right\} \text{ or, by mult. and transposition, } \left\{\begin{array}{l} 2x-y=15 \\ -x+3y=20 \end{array}\right.$$

Multiply the latter by 2, and we shall have $-2x+6y=40$; to which adding $2x-y=15$, the sum gives $5y=55$, or $y=11$, B's money, and $2x=15+y$, or $x=\dfrac{15+y}{2}=\dfrac{15+11}{2}=13$, B's money.

41. What two numbers are those whose difference, sum, and product are to each other as the numbers 2, 3 and 5, respectively?

Let x and y be the two numbers. Then, by the question, the difference is to the sum as $2:3$, and the sum to the product as $3:5$; that is, $\left\{\begin{array}{l} x-y : x+y :: 2 : 3 \\ x+y : xy :: 3 : 5 \end{array}\right\}$ which, by multiplying extremes and means, gives $\left\{\begin{array}{l} 3x-3y=2x+2y \\ 5x+5y=3xy \end{array}\right\}$ From the first $x=5y$; which, substituted in the second, gives $25y+5y=15y^2$; hence, dividing by y, and transposing, $15y=30$; or $y=2$, one number, and $x=5y=10$, the other.

42. A person in play lost a fourth of his money, and then won back 3*s*., after which he lost a third of what he now had, and then won back 2*s*.; lastly, he lost a seventh of what he then had, and after this found he had but 12*s*. remaining. What had he at first?

Let $x=$ the number of shillings he had at first. Then, by the question, he lost $\frac{1}{4}x$, and therefore had $\frac{3}{4}x$ left, to which he won 3*s*. After this, he had $\frac{3}{4}x+3=\dfrac{3x+12}{4}$, of which last he lost one-third, and had then two-thirds of it remaining, viz. $\dfrac{6x+24}{12}$; to which adding 2*s*. won, he had $\dfrac{6x+48}{12}$. Then, losing one-seventh of this, he had six-sevenths of it, viz. $\dfrac{36x+288}{84}$ left; which, by

the question, was 12*s*.; whence $\frac{36x+288}{84}=12$, or $36x+288=1008$; and, consequently, $36x=720$, or $x=20s.$, the money he had at the beginning.

43. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3, but 2 of the greyhound's leaps are as much as 3 of the hare's. How many leaps must the greyhound take to catch the hare? Ans. 300.

Let x be the number of leaps taken by the dog; then will $\frac{4}{3}x$ be the number taken by the hare after the starting of the dog; and consequently, $\frac{4}{3}x+50$ will be the whole number of leaps taken by the hare, and which must be equal in extent to the distance run by the dog; that is, $3x=2(\frac{4}{3}x+50)$, or $9x=8x+300$, and $x=300$, the answer required.

44. It is required to divide the number 90 into four such parts that if the first part be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient shall be all equal.

Ans. The parts are 18, 22, 10, and 40.

Let $x-2$, $x+2$, $\frac{1}{2}x$, and $2x$ be the parts required; then, if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the results will be equal; and, by the question, $x-2+x+2+\frac{1}{2}x+2x=90$, or $4x+\frac{1}{2}x=90$; that is, $8x+x=180$, or $x=\frac{180}{9}=20$; hence $x-2=18$, $x+2=22$, $\frac{1}{2}x=10$, and $2x=40$, the answer.

45. There are three numbers whose differences are equal, (that is, the second exceeds the first as much as the third exceeds the second,) and the first is to the third as 5 to 7; also, the sum of the three numbers is 324. What are those numbers?

Let x, y and z be the three numbers; then, by the question, $x+y+z=324$, $x:z::5:7$, and $x+z=2y$, or $x+z-2y=0$. Now, by subtracting the last equation from the first, we have $3y=324$, or $y=\frac{324}{3}=108$; and, consequently, by substitution and reduction, $x=90$, and $z=126$.

46. A man and his wife usually drank out a cask of beer in 12 days, but when the man was from home it lasted the woman 30 days. How many days would the man alone be in drinking it?

Let x be the number of days the man would be in drinking it by himself. Then $\frac{1}{x}$ will be the quantity he drinks in a day, and $\frac{1}{30}$ is the quantity the woman drinks in a day;

Therefore, the quantity both drink together is $\frac{1}{x}+\frac{1}{30}$.

But, by the question, $12(\frac{1}{x}+\frac{1}{30})=1$, or, multiplying by $30x$,

$360+12x=30x$; whence, by transposing, $18x=360$, or $x=20$, the number of days sought.

47. A general, ranging his army in the form of a solid square, finds he has 284 men to spare; but, increasing the side by one man, he wants 25 to fill up the square. How many soldiers had he?

Let x be the number of men in the side of the less square, and $x+1$ the number in the side of the greater. Then x^2 will be the whole number of men in the former, and $(x+1)^2=x^2+2x+1$, in the latter; whence x^2+284, and $x^2+2x+1-25$ will each express the whole number of men; from which we have this equation, $x^2+2x-24=x^2+284$, or $2x=284+24=308$, whence $x=154$; and, consequently, $x^2+284=24000$, the whole number of men.

48. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days, how many days will it take each person to perform the same work alone? Ans. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

Let x, y, and z, be the number of days in which A, B, and C, respectively, would finish the work; then A will do $\frac{1}{x}$ part of it in one day, B will do $\frac{1}{y}$ part, and C $\frac{1}{z}$ part. Then, by the question, we shall have $\frac{1}{x}+\frac{1}{y}=\frac{1}{8}$; $\frac{1}{x}+\frac{1}{z}=\frac{1}{9}$; $\frac{1}{y}+\frac{1}{z}=\frac{1}{10}$. And consequently by addition $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{8}+\frac{1}{9}+\frac{1}{10}=\frac{121}{360}$. Or, by division, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{121}{720}$. From this subtracting each of the three first equations, we have

$\frac{1}{z}=\frac{31}{720}$, or $z=\frac{720}{31}=23\frac{7}{31}$ days for C.

$\frac{1}{y}=\frac{41}{720}$, or $y=\frac{720}{41}=17\frac{23}{41}=$ days for B.

$\frac{1}{x}=\frac{49}{720}$, or $x=\frac{720}{49}=14\frac{34}{49}=$ days for A.

49. The hour and minute hand of a clock are exactly together at 12 o'clock. When are they next together?

Let x be the time required in minutes; then will $\frac{x}{12}$ be the space passed over by the hour hand in that time, and which is

equal to $x-60$; that is, $\frac{x}{12}=x-60$, or $x=12x-720$, and $x=\frac{720}{11}$ $=65\frac{5}{11}=$1h. $5\frac{5}{11}$ min.

50. There is an island 73 miles in circumference, and three footmen all start together to travel the same way about it: A goes 5 miles a day, B 8, and C 10. When will they all come together again?

Put $p=73$ miles, the circumference of the island, $a=5$, $b=8$, $c=10$, and let x be the time required in days. And when the bodies have all come together again, let us suppose that C has overtaken B m times; and A, n times. Then $c-b$ is what C gains on B every hour, and therefore $c-b : 1 :: p : \frac{p}{c-b}=$ the time in which C first overtook B; consequently, $\frac{mp}{c-b}=$ the time in which C would overtake B, m times; hence $x=\frac{mp}{c-b}$. Again, $c-a$ is what C gains on A every hour, and therefore $c-a : 1 :: p : \frac{p}{c-a}=$ the time in which C first overtook A; consequently, $\frac{np}{c-a}=$ the time in which C would overtake A, n times; therefore $x=\frac{np}{c-a}$. Hence, $\frac{mp}{c-b}=\frac{np}{c-a}$, or $\frac{m}{c-b}=\frac{n}{c-a}$; and $m(c-a)=n(c-b)$, where m and n must be the least whole numbers that will satisfy the equation.

Now, in the question, $c-a=5$, and $c-b=2$; hence $5m=2n$, and the least integer values of m and n are evidently 2 and 5; that is, $m=2$, and $n=5$; whence $x=\frac{mp}{c-b}=\frac{2p}{2}=73$ days, the answer.

51. How much foreign brandy at 8*s*. per gallon, and British spirits at 3*s*. per gallon, must be mixed together, so that in selling the compound at 9*s*. per gallon, the distiller may clear 30 per ct.?

Let x be the number of gallons of brandy, and y the gallons of British spirits; then will $8x+3y$ be the prime cost, and $9x+9y$ the price sold for, so as to gain 30 per cent.

Hence $100 : 130 :: 8x+3y : 9x+9y$, or $130(8x+3y)=100(9x+9y)$; that is, $104x+39y=90x+90y$, or $14x=51y$, and $x : y :: 51 : 14$; whence it appears, that the brandy and spirits must be mixed in the proportion of 51 gallons to 14.

52. A person bought a certain number of sheep for \$94; having lost 7 of them, he sold $\frac{1}{4}$th of the remainder of them, at prime cost, for \$20. How many sheep had he at first?

Let x = the number of sheep he had at first; then $\frac{94}{x}$ $=\frac{\text{whole sum}}{\text{number of sheep}}=$ what each sheep cost. Now $x-7=$ number remaining when 7 were lost; $\therefore \frac{x-7}{4}=$ the number sold for \$20. But the number sold $\times$ price of each $=$ whole price of sheep sold. Hence, by substitution, $\frac{x-7}{4}\times\frac{94}{x}=20$, or $x=\frac{658}{14}$ $=47$.

53. A and B have the same income; A is extravagant, and contracts an annual debt amounting to $\frac{1}{7}$th of it; B lives upon $\frac{4}{5}$ths of it; at the end of 10 years B lends A money enough to pay off his debts, and has then \$160 to spare. What is their income?

Let x = their income; then $\frac{1}{7}$th of x, or $\frac{1}{7}x$, = A's annual debt, and $10\times\frac{1}{7}x$, or $\frac{10}{7}x=$ A's debt contracted in 10 years. As B lives upon $\frac{4}{5}$ths of his income, he saves annually $\frac{1}{5}$th of it; hence $\frac{1}{5}x=$ B's annual saving, and $10\times\frac{1}{5}x$, or $\frac{10}{5}x$, or $2x=$ B's savings in 10 years. But, by the question, B's savings = A's debt + 160; $\therefore$ by substitution, $2x=\frac{10}{7}x+160$, or $14x=10x+1120$, and $4x=1120$; or $x=\frac{1120}{4}=\$280$.

54. A person passed $\frac{1}{6}$th of his age in childhood, $\frac{1}{12}$th in youth, $\frac{1}{7}$th + 5 years in matrimony; he had then a son whom he survived 4 years, and who reached only half the age of his father. At what age did this person die?

Let x = age of the person at the time of his death. Then $\frac{1}{6}x$ = time spent in childhood; $\frac{1}{12}x=$ time in youth; $\frac{1}{7}x+5=$ time in matrimony; $\therefore \frac{1}{6}x+\frac{1}{12}x+\frac{1}{7}x+5=$ age of the person when the son was born, and $x-\frac{1}{6}x-\frac{1}{12}x-\frac{1}{7}x-5=$ interval between the birth of the son and the old man's death; $\therefore x-\frac{1}{6}x-\frac{1}{12}x-\frac{1}{7}x-5-4=$ the age of the son when he died. But, by the question, the son died at half the age of the father. Hence, $x-\frac{1}{6}x-\frac{1}{12}x-\frac{1}{7}x-9=\frac{1}{2}x$. Multiply by 12, then $12x-2x-x-\frac{12}{7}x-108=6x$, or $3x-\frac{12}{7}x=108$, and $21x-12x=756$; $\therefore 9x=756$; or $x=\frac{756}{9}=84$.

55. If three agents, A, B and C, can produce the effects a, b, c, in the times e, f, g, respectively, in what time would they jointly produce the effect d?

Let the time sought be denoted by x. Then $e:a::x::\frac{ax}{e}=$

part of the effect produced by A, $f:b::x:\frac{bx}{f}=$ part of the effect produced by B, $g:c::x:\frac{cx}{g}=$ part of the effect produced by C; whence $\frac{ax}{e}+\frac{bx}{f}+\frac{cx}{g}=d$, by the question, and $x=d\div\left\{\frac{a}{e}+\frac{b}{f}+\frac{c}{g}\right\}$, the time required.

56. A and B have certain sums of money; says A to B give me \$15 of your money, and I shall have 5 times as much as you will have left; says B to A, give me \$5 of your money, and I shall have exactly as much as you will have left. What sum of money had each?

Let $x=$ A's money, $y=$ B's, then $x+15=$ what A would have after receiving \$15 from B, $y-15=$ what B would have left. Again, $y+5=$ what B would have after receiving \$5 from A, $x-5=$ what A would have left.

Hence, by the question, $\left.\begin{array}{l}x+15=5\times(y-15)=5y-75,\\ \text{and } y+5=x-5.\end{array}\right\}$

By transposition, $\left.\begin{array}{rl}5y-x= & 90 \quad (A)\\ \text{and } y-x= & -10 \quad (B)\end{array}\right\}$

Subtract (B) from (A), $4y \quad =100$; $\therefore y=25$, B's money.

From equation (B), $x=y+10=25+10=35$, A's money.

57. Three men began talking about their money; the first said to the others, two-thirds of your money would make mine \$37; and the second said to the others, three-fifths of your money would make mine \$37; but, if the other two lent five-eighths of their money to the third, it would make his money amount to \$37. What sum did each possess?

Putting $a=\$37$, and z, y, x for the 1st, 2d and 3d person's dollars respectively, we shall have, by the question,

$$\begin{array}{l} z+\frac{2}{3}y+\frac{2}{3}x=a, \text{ or } 3z+2y+2x=3a,\\ y+\frac{3}{5}z+\frac{3}{5}x=a, \text{ or } 3z+5y+3x=5a,\\ x+\frac{5}{8}z+\frac{5}{8}y=a, \text{ or } 5z+5y+8x=8a.\end{array}$$

Taking the 1st equation from the 2d, we have $3y+x=2a$, or $x=2a-3y$; this substituted in the 1st and 3d, we have

$\left.\begin{array}{r}4y-3z=a\\ 19y-5z=8a\end{array}\right\}$; from 3 times the latter of these equations subtract 5 times the former, and there will be obtained $37y=19a$; hence $y=\frac{19a}{37}=\$19$. Then $x=2a-3y=\frac{17a}{37}=17$ dolls., and

$z=\frac{4y-a}{3}=\frac{13a}{37}=\$13.$

General Solution. 1st. Let there be given the two equations,
$\left\{\begin{array}{l} ax+by=c \\ a'x+b'y=c' \end{array}\right\}$ to find the values of x and y.

Multiplying the first by a', and the second by a, they become

$$aa'x+a'by=ca'$$
$$aa'x+ab'y=c'a;$$

and, subtracting the first from the second, we have

$$(ab'-a'b)y=c'a-ca', \text{ or } y=\frac{c'a-ca'}{ab'-a'b}.$$

In like manner, if the first equation be multiplied by b', and the second by b, and the latter product subtracted from the first, we have

$$(ab'-a'b)x=cb'-c'b, \text{ or } x=\frac{cb'-c'b}{ab'-a'b}.$$

In this case, where there are only two unknown quantities, it is evident what factors will render the coefficients of x and y, in the two equations, equal to one another, as above; but, when there are more unknown quantities than two, it is not so obvious what factors will answer; therefore, in what follows, I shall make use of a method for determining them, which is applicable to all cases, whatever may be the number of equations.

In the preceding example, if we multiply by the indeterminate factors m and n, the equations become $amx+bmy=cm$
$a'nx+b'ny=c'n$

and, subtracting the second equation from the first, we have

$$(am-a'n)x+(bm-b'n)y=cm-c'n.$$

Now, in order that y may disappear from this equation, $bm-b'n$ must be $=0$, then $x=\frac{cm-c'n}{am-a'n}$. But $bm-b'n$ will be equal 0, when $m=b'$ and $n=b$; and, these values being substituted for m and n in the expression for x, we have $x=\frac{cb'-c'b}{ab'-a'b}$.

Again, to make x disappear from the equation, $am-a'n$ must be $=0$, then $y=\frac{cm-c'n}{bm-b'n}$. But $am-a'n=0$, when $m=a'$ and $n=a$; therefore, substituting for m and n their values, we have $y=\frac{ca'-c'a}{ba'-b'a}$; or, changing the signs in the numerator and denominator, $y=\frac{ac'-a'c}{ab'-a'b}$.

When one of the unknown quantities has been determined by this method, the other may be found, without repeating the pro-

cess, by only changing the coefficients, or writing a in place of b, and a' in place of b', or the contrary. Thus, if in the value for x we write a for b, and a' for b', it becomes $\frac{ca'-c'a}{ba'-b'a}$, the value of y. And if in the value for y we write b for a, and b' for a', we have $\frac{cb'-c'b}{ab'-a'b}$, for the value of x. 2nd. Let there be three equations, $\left\{\begin{array}{l} ax+by+cz=d \\ a'x+b'y+c'z=d' \\ a''x+b''y+c''z=d'' \end{array}\right\}$ to find the values of x, y and z.

Multiplying by the indeterminate factors m, n, and p, we have

$$\begin{array}{l} amx+bmy+cmz=dm \\ a'nx+b'ny+c'nz=d'n \\ a''px+b''py+c''pz=d''p, \end{array}$$

and, subtracting the last from the sum of the other two, there remains $(am+a'n-a''p)x+(bm+b'n-b''p)y+(cm+c'n-c''p)z=dm+d'n-d''p$. Now, in order that y and z may disappear from this equation, and x only remain,

$$\begin{array}{l} bm+b'n-b''p=0, \text{ or } bm+b'n=b''p \\ cm+c'n-c''p=0, \text{ or } cm+c'n=c''p; \end{array}$$

then $x=\frac{dm+d'n-d''p}{am+a'n-a''p}$. From the equations $bm+b'n=b''p$ and $cm+c'n=c''p$, we have, by the first case, $m=\frac{p(c'b''-b'c'')}{bc'-b'c}$, and $n=\frac{p(bc''-b''c)}{bc'-b'c}$ and since p is arbitrary, we may take it equal to the denominator $bc'-b'c$, then $m=c'b''-b'c''$, and $n=bc''-b''c$. These values being substituted for m, n and p, in the expression for x, we have

$$x=\frac{d(c'b''-b'c'')+d'(bc''-b''c)-d''(bc'-b'c)}{a(c'b''-b'c'')+a'(bc''-b''c)-a''(bc'-b'c)}.$$

In like manner, if we make $am+a'n=a''p$ and $cm+c'n=c''p$, x and z will disappear from the equation, and we shall have $y=\frac{dm+d'n-d''p}{bm+b'n-b''p}$. And, proceeding as before, we find $m=c'a''-a'c''$, $n=ac''-ca''$, and $p=ac'-ca'$; therefore,

$$y=\frac{d(c'a''-a'c'')+d'(ac''-ca'')-d''(ac'-ca')}{a(c'a''-a'c'')+a''(ac''-ca'')-a''(ac'-ca')}.$$

Lastly, if we make $\left.\begin{array}{l} am+a'n=a''p \\ \text{and } bm+b'n=b''p \end{array}\right\}$, x and y will disappear from the equation, and we shall have $z=\frac{dm+d'n-d''p}{cm+c'n-c''p}$. And, substituting for m, n and p, their values found as above, namely,

$m=b'a''-a'b''$, $n=ab''-ba''$, and $p=ab'-ba'$, we have

$$z=\frac{d(b'a''-a'b'')+d'(ab''-ba'')-d''(ab'-ba')}{c(b'a''-a'b'')+c'(ab''-ba'')-c''(ab'-ba')}.$$

These values being multiplied out, and the terms so arranged as to be positive and negative alternately, we have

$$x=\frac{db'c''-dc'b''+cd'b''-bd'c''+bc'd''-cb'd''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}.$$

$$y=\frac{ad'c''-ac'd''+ca'd''-da'c''+dc'a''-cd'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}$$

$$z=\frac{ab'd''-ad'b''+da'b''-ba'd''+bd'a''-db'a''}{ac'b''-ab'c''+ca'b''-ba'c''+bc'a''-cb'a''},$$

the signs in the first and third expressions being changed, in order that the denominator of each may be the same.

When one of the unknown quantities is determined, the others may be found without going through the separate operations, by only changing the coefficients; thus, if in the first value of x, we substitute the respective coefficients of y in place of those of x, viz. b in place of a, b' in place of a', and b'' in place of a'', we shall have the value of y: and if we substitute the coefficients of z instead of those of x, viz. c in place of a, c' in place of a', and c'' in place of a'', we shall have the value of z.

The reason of this will be obvious, if we suppose the unknown quantities y and z to change places successively with x in the original equations. To apply these formulæ to the example in question, where

$$2x+2y+3z=3a$$
$$3x+5y+2z=5a$$
$$8x+5y+5z=8a$$

we have $a=2$, $a'=3$, $a''=8$, $b=2$, $b'=5$, $b''=5$, $c=3$, $c'=3$, $c''=5$, $d=3a$, $d'=5a$, and $d''=8a$.

And substituting these values in the above expressions,

$$x=\frac{3\cdot5\cdot5-3\cdot3\cdot5+3\cdot5\cdot5-2\cdot5\cdot5+2\cdot3\cdot8-3\cdot5\cdot8}{2\cdot5\cdot5-2\cdot3\cdot5+3\cdot3\cdot5-2\cdot3\cdot5+2\cdot3\cdot8-3\cdot5\cdot8}\times a=\frac{-17}{-37}\cdot a=\frac{17a}{37}$$

$$y=\frac{2\cdot5\cdot5-2\cdot3\cdot8+3\cdot3\cdot8-3\cdot3\cdot5+3\cdot3\cdot8-3\cdot5\cdot8}{2\cdot5\cdot5-2\cdot3\cdot5+3\cdot3\cdot5-2\cdot3\cdot5+2\cdot3\cdot8-3\cdot5\cdot8}\times a=\frac{-19}{-37}\cdot a=\frac{19a}{37}$$

$$z=\frac{2\cdot5\cdot8-2\cdot5\cdot5+3\cdot3\cdot5-2\cdot3\cdot8+2\cdot5\cdot8-3\cdot5\cdot8}{2\cdot5\cdot5-2\cdot3\cdot5+3\cdot3\cdot5-2\cdot3\cdot5+2\cdot3\cdot8-3\cdot5\cdot8}\times a=\frac{-13}{-37}\cdot a=\frac{13a}{37}$$

as before.

This method may be extended with the same facility to as many unknown quantities as we please; thus, if there be four equations

$$ax\ +by\ +cz\ +dw\ =e$$

$$a'x + b'y + c'z + d'w = e'$$
$$a''x + b''y + c''z + d''w = e''$$
$$a'''x + b'''y + c'''z + d'''w = e'''$$; then, multiplying the first by m, the second by n, the third by p, the fourth by q, and subtracting the last product from the sum of the other three, we find $(am + a'n + a''p - a'''q)x + (bm + b'n + b''p - b'''q)y + (cm + c'n + c''p - c'''q)z + (dm + d'n + d''p - d'''q)w = em + e'n + e''p - e'''q$. Therefore, to find x, we have $m + b'n + b''p - b'''q = 0$, $cm + c'n + c''p - c'''q = 0$ and: $dm + d'n + d''p - d'''q = 0$, and $x = \frac{em + e'n + e''p - e'''q}{am + a'n + a''p - a'''q}$.

The values of m, n and p must now be found from the formulæ in case 2nd, and substituted in the last expression, we shall then have the value of x; and from thence, by changing the coefficients, we shall obtain the values of y, z and w. The values of x, y, z, and w, are as below. The numerator of x is

$$\begin{aligned}
&eb'c''d''' - eb'd''c''' + ed'b''c''' - ed'c''b''' + ec'd''b''' - ec'b''d''' \\
&+ bd'c''e''' - bd'e''c''' + bc'e''d''' - bc'd''e''' + be'd''c''' - be'c''d''' \\
&+ ce'b''d''' - ce'd''b''' + cd'e''b''' - cd'b''e''' + cb'd''e''' - cb'e''d''' \\
&+ db'e''c''' - db'c''e''' + dc'b''e''' - dc'e''b''' + de'c''b''' - de'b''c'''
\end{aligned}$$

The numerator of y is

$$\begin{aligned}
&ae'c''d''' - ae'd''c''' + ad'e''c''' - ad'c''e''' + ac'e''d''' - ac'd''e''' \\
&+ ed'c''a''' - ed'a''c''' + ec'a''d''' - ec'd''a''' + ea'd''c''' - ea'c''d''' \\
&+ ca'e''d''' - ca'd''e''' + cd'a''e''' - cd'e''a''' + ce'd''a''' - ce'a''d''' \\
&+ de'a''c''' - de'c''a''' + dc'e''a''' - dc'a''e''' + da'c''e''' - da'e''c'''
\end{aligned}$$

The numerator of z is

$$\begin{aligned}
&ab'e''d''' - ab'd''e''' + ad'b''e''' - ad'e''b''' + ae'd''b''' - ae'b''d''' \\
&+ db'e''a''' - bd'a''e''' + be'a''d''' - be'd''a''' + ba'd''e''' - ba'e''d''' \\
&+ ea'b''d''' - ea'd''b''' + ed'a''b''' - ed'b''a''' + eb'd''a''' - eb'a''d''' \\
&+ db'a''e''' - db'e''a''' + de'b''a''' - de'a''b''' + da'e''b''' - da'b''e'''
\end{aligned}$$

The numerator of w is

$$\begin{aligned}
&ab'c''e''' - ab'e''c''' + ae'b''c''' - ae'c'b''' + ac'e''b''' - ac'b''e''' \\
&+ be'c''a''' - be'a''c''' + bc'a''e''' - bc'e''a''' + ba'e''c''' - ba'c''e''' \\
&+ ca'b''e''' - ca'e''b''' + ce'a''b''' - ce'b''a''' + cb'e''a''' - cb'a''e''' \\
&+ eb'a''c''' - eb'c''a''' + ec'b''a''' - ec'a''b''' + ea'c''b''' - ea'b''c'''
\end{aligned}$$

And the denominator of each is

$$\begin{aligned}
&ab'c''d''' + ab'd''c''' + ad'b''c''' + ad'c''b''' + ac'd''b''' - ac'b''d''' \\
&+ bd\,c''a''' - bd'a''c''' + bc'a''d''' - bc'd''a''' + ba'd''c''' - ba'c''d''' \\
&+ ca\,b''d''' - ca'd''b''' + cd'a''b''' - cd'b''a''' + cb''d''a''' - cb'a''d''' \\
&+ db'a''c''' - db'c''a''' + dc'b''a''' - dc'a''b''' + da'c''b''' - da'b''c'''
\end{aligned}$$

If there were 5 unknown quantities, the numerators and denominators of each would consist of 120 terms.

58. Given $\{x+\frac{1}{2}(y+z)=17,\ y+\frac{1}{3}(x+z)=17;$ and $z+\frac{1}{4}(x+y)=17\}$, or $\{x+\frac{1}{4}(y+z)=1000;\ y+\frac{1}{5}(x+z)=1000;$ and $z+\frac{1}{6}(x+z)=1000\}$; to find x, y and z, in both the above cases.

If $a=2$; $b=3$; $c=4$; Ans. $x=5$, $y=11$, $z=13$.

A general theorem for all questions of a similar nature:

$$\frac{(abc+b+c-2bc-a)n}{abc+2-a-b-c}=\frac{67n}{107}=\text{A's.}\ \frac{(abc+a+c-2ac-b)n}{abc+2-a-b-c}=\frac{77n}{107}$$

$=$B's. and $$\frac{(abc+a+b-2ab-c)n}{abc+2-a-b-c}=\frac{83n}{107}=\text{C's part}$$

In this question n is $=$ to one thousand.

EXPRESSION OF QUESTIONS.

As it is sometimes difficult for learners to know how to express the conditions of a question algebraically, the following remarks may be found useful:

If the question be concerning one unknown number, or quantity; it may be represented by x or y.

If the unknown quantity is to be multiplied by 5, that condition is expressed thus: $5x$, or $5y$.

If 4 be added to that product, and the sum is equal to 14, then, $5x+4=14$; but if 4 is to be subtracted, then $5x-4=14$.

If the unknown quantity is to be divided by 3, that condition may be expressed thus: $\frac{x}{3}$ or $\frac{y}{3}$; and if 7 be subtracted from that quotient, and the remainder is equal to 10, those conditions are expressed thus: $\frac{1}{3}x-7=10$.

If the third, fourth, and fifth of a number be added to itself, and the sum is equal to 35, that condition is expressed thus:

$$x+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=35.$$

If the question is concerning two numbers or quantities, they may be called x and y.

If it be required that the sum of the two numbers sought be 60, that condition is expressed thus: $x+y=60$.

If their difference must be 24, then $x-y=24$.

If their product is 156, then $xy=156$.

If their quotient must be 6, then $\frac{x}{y}=6$, or $x\div y=6$.

If their ratio is as 3 to 2, then $x:y::3:2$, and hence $2x=3y$.

If the sum of their squares is 100, then $x^2+y^2=100$.

If the difference of their squares is 27, then $x^2-y^2=27$.

If the product of their squares is 36, then $x^2y^2=36$.

If the quotient of their squares is 16,

Then $\frac{x^2}{y^2}=16$, or $x^2\div y^2=16$.

If their sum added to their difference be 10, and x be the greater number and y the less, then $2x=10$;

For $x+y+x-y=2x$.

If the difference of their sum and difference be 6, and x be the greater and y the less, then $2y=6$;

For $x+y-(x-y)=2y$.

If the product of their sum, multiplied by their difference, must be 64, then $(x+y)(x-y)=64$; whence $x^2-y^2=64$.

If the square of their sum be 36,

Then $(x+y)^2$, or $x^2+2xy+y^2=36$.

If the square of their difference be 16,

Then $(x-y)^2$, or $x^2-2xy+y^2=16$.

If the sum of the squares of their sum and of their difference be 100, then $2x^2+2y^2=100$.

For $x^2+2xy+y^2$ added to $x^2-2xy+y^2=2x^2+2y^2$.

If the difference of the squares of the sum and difference equal 64, then $4xy=64$. As may be seen by subtracting, $x^2-2xy+y^2$ from $x^2+2xy+y^2$.

If their sum, divided by their difference, be 3, then $\frac{x+y}{x-y}=3$.

If two-thirds of one, and four-sevenths of the other, make 60,

Then $\frac{2x}{3}+\frac{4y}{7}=60$.

If two-thirds of one, subtracted from four-sevenths of the other, leave 14, then $\frac{4x}{7}-\frac{2y}{3}=14$.

If their sum must be four times their difference,

Then $x+y=4(x-y)$, or $x+y=4x-4y$.

If the sum of their squares is five times the sum of the numbers, then $x^2+y^2=5(x+y)=5\times(x+y.)$

Or $x^2+y^2=5x+5y$.

If their product is six times their sum,

Then $xy=6(x+y)$, or $xy=6x+6y$.

If their product is nine times their quotient, then $xy=\frac{9x}{y}$.

If one number must be as much above 20 as the other wants of 20, then $x-20=20-y$.

If one number must be three times as much above 20 as the other wants of 20, then $x-20=3(20-y)=3\times(20-y)$.

Or $x-20=60-3y$.

If their difference and sum are to each other as the numbers two and three, then $x-y : x+y :: 2 : 3$; or,

Multiplying extremes and means, $3x-3y=2x+2y$.

If their sum and product are to each other as the numbers three and five, then $x+y : xy :: 3 : 5$; or,

Multiplying extrems and means, $5x+5y=3xy$.

If one number ought to be as many times contained in 20 as the other contains the number 4, then $\frac{20}{x}=\frac{y}{4}$; or $20 : x :: y : 4$; Whence $xy=80$.

If the number 20 must be a mean proportional between the two numbers sought, then $x : 20 :: 20 : y$; whence $xy=400$.

If the greater being divided by the less, and again the less by the greater, the first quotient must be to the second as 5 to 3;

Then $\frac{x}{y} : \frac{y}{x} :: 5 : 3$. Whence $\frac{3x}{y}=\frac{5y}{x}$.

If one number increased by 2, and multiplied by the other diminished by three, produce 40, then $(x+2)(y-3)=40$, or $xy-3x+2y-6=40$.

If three numbers, x, y, and z, must be in continued proportion, then $x : y :: y : z$. Whence, $xz=y^2$.

It is sometimes easier to employ fewer letters than there are unknown quantities. Thus, the solution becomes more easy and elegant. There are some examples of this kind of notation.

Conditions.	*Notation.*
The sum of the two numbers sought is 60	Let $x=$ one number, then will $60-x=$ the other.
Their difference is 24	x and $x+24$ (or $x-24$).
Their product is 146.	x and $\frac{146}{x}$.
Their quotient is 6.	x and $\frac{x}{6}$, or x and $6x$.

Their ratio is as 3 to 2. x and $\frac{2x}{3}$, or $3x$ and $2x$; because $\frac{3x}{2x}=\frac{3}{2}$, or $3x : 2x :: 3 : 2$.

The greater is 4 times the less .... x and $4x$.

These and the preceding are some of the relations which are easily expressed; many others occur which are less obvious, but as they cannot be described by particular rules, their expression is best explained by examples, and must be acouired by experience.

SOLUTION OF QUESTIONS.

To solve a simple equation containing but one unknown quantity.

RULE. Clear the equation of fractions by rule 3, and of radicles by rule 4; second, transpose the unknown terms or quantities to one side of the equation, and the known terms to the other, by rule 1.

Collect each side into one term, and the unknown quantities, with a known coefficient, will form one side of the equation, and a known quantity the other side. Divide each side by the coefficient of the unknown quantity, and the value of the unknown will be exhibited.

1. What number is that, from the treble of which if 18 be subtracted, the remainder is 6? Ans. 8.

Let $x=$ the number; $\therefore 3x-18=6$, or $3x=24$, and $x=8$.

2. What number is that, the double of which exceeds four-fifths of its half by 40? Ans. 25.

Let $x=$ the number; $\therefore 2x-\frac{4}{5}\cdot\frac{x}{2}=40$, and $10x-2x=200$, or $x=25$.

3. In fencing the side of a field, whose length was 450 yards, two workmen were employed; one of whom fenced 9 yards, and the other 6 per day. How many days did they work? Ans. 30.

Let x, $9x$ and $6x=$ the number of days and of yards fenced by each respectively; $\therefore (9x+6x)=15x=450$, and $x=30$, Ans.

4. A mercer bought 4 pieces of silk, which together measured 50 yards; the second was twice, the third three times, and the fourth four times as long as the first. What was the respective lengths of the pieces? Ans. 5, 10, 15, 20 yards.

Let x, $2x$, $3x$, $4x$, and $10x=50$, be the number of yards in the first, second, third, fourth, and the equation, respectively, and $x=5$.

5. A farmer sold 13 bushels of barley at a certain price; and afterwards 17 bushels at the same rate; and at the second time received 36 shillings more than at the first. What was the price of a bushel? Ans. 9 shillings.

Let $x=$ the price of a bushel, $13x$ and $17x$ the sum received for the first and second; $\therefore (17x-13x=4x)=36$, and $x=9$, Ans.

6. A person bought 198 gallons of beer, which exactly filled 4 casks; the first held twice as much as the second, the second twice as much as the third, and the third three times as much as the fourth. How many gallons did each hold?

Ans, 108, 54, 27, and 9 gallons.

Let x, $3x$, $6x$, and $12x=$ the number of gallons the fourth, third, second and first, respectively; $\therefore 22x=198$, and $x=9$, and 108, 54, 27 and 9 were the answers.

7. A silversmith has 3 pieces of metal, which together weigh 48 ounces. The second weighs 12 ounces more than the first, and the third 9 ounces more than the second. What were their respective weights? Ans. 5, 17, and 26 ounces.

Let x, $x+12$, $x+21=$ the number of ounces the first, second, and third weighed, respectively, and $3x+33=48$, or $x+11=16$, and $x=5$, and 17, 26 ounces, Ans.

8. A vinter fills a cask, containing 96 gallons, with a mixture of brandy, wine, and water. There are 20 gallons of water more than of brandy, and 17 more of wine than of water. How many are there of each?

Ans. 13 gallons of brandy, 33 of water, and 50 of wine.

Let x, $20+x$, $37+x$ be the number of gallons of brandy, of water, and of wine, respectively; $57+3x=96$, or $19+x=32$, and $x=13$, and the number of gallons of brandy, water, and wine, were, 13, 33, 50, respectively.

9. A gentleman buys 4 horses; for the second of which he gives \$12 more than for the first; for the third \$6 more than for the second; and for the fourth \$2 more than for the third. The sum paid for all was \$230. How much did each cost?

Ans. 45, 57, 63, and 65 dollars.

Let x, $x+12$, $x+18$, $x+20$ denote the price of the first, second, third, and fourth; $\therefore 4x+50=230$, or $4x=180$, and $x=45$.

10. A poor man had 6 children, the eldest of which could earn 7d. a week more than the second; the second 8d. more than the third; the third 6d. more than the fourth; the fourth 4d. more than the fifth; and the fifth 5d. more than the youngest. They altogether earned 10s. 10d. a week. How much could each earn a week? Ans. 38, 31, 23, 17, 13, and 8 pence per week.

Let x, $x+5$, $x+9$, $x+15$, $x+23$, and $x+30$ be the sum earned by the youngest, fifth, fourth, third, second, and eldest; $\therefore 6x+82=130$, or $x=8$.

11. An express set out to travel 240 miles in 4 days, but in consequence of the badness of the roads he found that he must go 5 miles the second day, 9 the third, and 14 the fourth day, less than the first. How many miles must he travel each day?

Ans. 67, 62, 58, and 53 miles.

Let x, $x-5$, $x-9$, $x-14=$ the number of miles travelled the first, second, third, and fourth day; $\therefore 4x-28=240$, or $x-7=60$, or $x=67$.

12. There are 5 towns, in the orders of the letters A, B, C, D, E. From A to E is 80 miles. The distance between B and C

is 10 miles more, between C and D is 15 miles less, and between D and E 17 miles more than the distance between A and B. What are the respective distances? Ans. from A to B 17; from B to C 27; from C to D 2; and from D to E 34 miles.

Let x, $x+10$, $x-15$, $x+17$ denote the distance from A to B, B to C, C to D, and D to E, and we have $4x+12=80$, or $x=17$.

13. A gentleman gave $27 to two poor persons; but he gave $5 more to one than to the other. What did he give to each?

Ans. 11, and 16 dollars.

Let x, and $x+5$ be the number of dollars given to one and the other, and $2x+5=27$, or $x=11$, and 16, are the answers.

14. What number is that, the treble of which is as much above 40, as its half is below 51? Ans. 26.

Let $2x=$ the number; $\therefore 6x-40=51-x$, and $7x=91$, or $x=13$, $2x=26$.

15. Two workmen received the same sum for their labor; but if one had received $15 more, and the other $9 less, then one would have had just three times as much as the other. What did they receive? Ans. 21 dollars each.

Let $x=$ the sum; $\therefore x+15=3(x-9)$, and $2x=42$, or $x=21$ dollars, Ans.

16. Two merchants entered into a speculation, by which one gained 54 dollars more than the other. The whole gain was 49 dollars less than three times the gain of the less. What were the gains? Ans. $103, and $157.

Let x, $x+54$ denote the less gain and the greater, and $2x+54=3x-49$.

17. The perimeter of a triangle is 75 feet, and the base is 11 feet longer than one of the sides, and 16 feet longer than the other. Required their respective lengths.

Let x, $x-11$, $x-16$ be the length of the base and the length of the sides, and $(x+x-11+x-16)=3x-27=75$; $\therefore x=34$, and the sides were 23 and 18.

18. A company settling their reckoning at a tavern, pay 8 shillings each; but observe, that if there had been 4 more, they should only have paid 7 shillings each. How many were there?

Ans. 28.

Let x and $8x$ be the number and the sum paid, and $8x=7(x+4)$; $\therefore x=28$.

19. Divide the number 46 into two such parts, that one of them being divided by 7 and the other by 3, the quotients may together be equal to 10. Ans. 18 and 28.

Let x, $46-x=$ one part and the other, and $\frac{x}{3}+\frac{46-x}{7}=10$, and $x=18$.

20. A certain sum is to be raised upon two estates, one of which pays 19 shillings less than the other; and if 5 shillings be added to treble the less payment, it will be equal to twice the greater. What are the sums paid? Ans. 33, and 52 shillings.

Let x, $x+19$ denote the payment of the less and greater in shillings, and $3x+5=2x+38$, and $x=33$, and $x+19=52$ shillings, the Ans.

21. Having bought a certain quantity of brandy at 19 shillings a gallon, and a quantity of rum exceeding that of the brandy by 9 gallons, at 15 shillings a gallon, I find that I paid one shilling more for the brandy than for the rum. How many gallons were there of each? Ans. 34 of brandy, and 43 of rum.

Let x and $x+9$ be the number of gallons of brandy and of rum, and $19x$ and $15(x+9)$ are the prices respectively; $\therefore 19x=15x+135+1$, and $4x=136$, or $x=34$ gallons of brandy, and 43 of rum, Ans.

22. Two persons, A and B, have each an annual income of 400 dollars. A spends every year 40 dollars more than B, and at the end of 4 years the amount of their savings is equal to one year's income of either. What does each spend annually?

Ans. \$370, and \$330, respectively.

Let x, and $x+40$ denote the sum spent by B and A, and $400-x$, $360-x$, denote the sum saved by B and A; whence, $4(760-2x)=100$, or $2x=660$ and $x=330=$B, and $370=$A.

23. A draper sold two pieces of cloth, by one of which he lost 6 dollars more than by the other; and his whole loss was 5 dollars less than treble the less loss. What were the losses sustained by each piece? Ans. \$11, and \$17.

Let x, $x+6$ denote the loss sustained by one and the other, and $2x+6=3x-5$, and $x=11$, and $x+6=17$, Ans.

24. A person engaged to reap a field of corn for 4 dollars an acre, but leaving 6 acres not reaped, he received 40 dollars. Of how many acres did the field consist? Ans. 16.

Let x denote the number of acres, and $4(x-6)=40$ dollars, or $4x-24=40$, or $4x=64$, and $x=16$ acres, Ans.

25. In a naval engagement, the number of ships taken was 7 more, and the number burnt 2 fewer, than the number sunk. Fifteen escaped, and the fleet consisted of 8 times the number sunk. Of how many did the fleet consist? Ans. 32.

Let x, $x+7$, $x-2$ denote the number sunk, taken and burnt; then $x+x+7+x-2+15=8x$, and $5x=20$, or $x=4$, and the fleet 32.

26. A farmer hires a farm for \$$175\frac{3}{4}$ a year; part of which he is not allowed to plough; he gives \$2 per acre for the arable, and for the rest, which was 5 acres less, he gives \$$1\frac{1}{4}$ per acre. How many acres were arable, and how many not?

Ans. 56 acres arable, 51 not.

Let x, and $x-5$ denote the number of acres of arable, and the rest; then $8x+5(x-5)=703$, or $13x=728$, and $x=56$ arable, and 51 the rest.

27. A cistern is filled in twenty minutes by three pipes, one of which conveys 10 gallons more, and the other 5 gallons less, than the third, per minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

Let x, $x+10$, and $x-5$ denote the number passing through the third, first and second, respectively; then $20(3x+5)=820$, or $x=12$. Ans. 22, 7, and 12 gallons.

28. A fortress is garrisoned by 2600 men; and there are 9 times as many infantry, and three times as many artillery, as cavalry. How many are there of each?

Let x, $3x$, $9x$ denote the number of cavalry, artillery, and infantry; then $13x=2600$, and $x=200$.

29. A and B began to play; A with exactly $\frac{4}{9}$ of the sum which B had. After winning 10 dollars, he found that they had each the same sum. What had each at first?

Let $9x$ and $4x$ denote the sums which B and A had at first; then $4x+10=9x-10$, or $5x=20$, $x=4$, and A had 16 and B 36.

30. A person has a certain number of horses at livery stables, and 3 times as many at grass. He keeps 15 in constant employment; and his whole number is 7 times the number in the stables. Required the whole number. Let x, $3x$ be the number at livery and at grass, $4x+15=7x$, and $x=5$, and 35, the Ans.

31. Two men at the distance of 150 miles set out to meet each other; one goes 3 miles in the time the other goes 7. What part of the distance does each travel?

Since they travel at the rates which are in the proportion of 3 to 7, let $3x$ and $7x$ be the number of miles each goes; then $(3x+7x=10x)=150$, or $x=15$, and they travel 45 and 105 miles.

32. A farm of 864 acres is divided between 3 persons. C has as many acres as A and B together; and the portions of A and B are in the proportion of $5:11$. How many acres has each?

Let $5x$, $11x$, $16x$ be the number A, B, and C each had; then $32x=864$, or $x=27$. Ans. A has 135, B 297, C 432.

Solutions by Arithmetic to Questions 43 and 46, page 96.

2 greyhound's leaps = 3 hare's leaps; hence 6 greyhound's leaps = 9 hare's leaps. But while the greyhound made 3 leaps, the hare made 4; ∴ the greyhound made 6 leaps in the same time which the hare made 8; consequently every 6 leaps the greyhound took he gained 1 hare's leap; 1 H.lp. : 6 G. lps. : : 50H.lps. : 300, Ans.

As $(30-12)=18:12::30:\frac{12\times30}{18}=\frac{360}{18}=20$ Ans.

33. A charitable person distributed £5 14*s*. amongst some poor women and children, giving to each woman 6 shillings, and to

each child two; the number of women was to the number of children as 4 : 7. How many were relieved?

Ans. 12 women, and 21 children.

Let $4x$ and $7x$ be the number of women and children, and $24x$ and $14x$ represent the number of shillings the women and children had; then $38x=114$, and $x=3$; $\therefore 4x=12$ and $7x=21$ children.

34. A and B begin trade, A with triple the stock of B. They each gain 50 dollars, which makes their stocks in the proportion of 7 to 3. What were their original stocks?

Ans. A's was 300 dollars, and B's 100.

Let x and $3x$ represent the stock of B and A; then $3x+50 : x+50 :: 7 : 3$, and $x=100$, and A's stock was 300 dollars, and B's 100.

35. There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, which being increased respectively by 6 and 5, are in the proportion of $\frac{3}{4}$ to $\frac{4}{5}$. Required the numbers. Ans. 30 and 40.

Let $3x$ and $4x$ be the numbers; then $3x+6 : 4x+5 :: \frac{3}{4} : \frac{4}{5} :: 4 : 5$, and $15x+30=16x+20$, and $x=10$, and the numbers 30 and 40.

36. A farmer has a stack of hay, from which he sells a quantity which is to the quantity remaining in the proportion of 4 to 5. He then uses 15 loads, and finds that he has a quantity left which is to the quantity sold as 1 to 2. How many loads did the stack at first contain? Ans. 45.

Let $9x$, $4x$, and $5x$ denote the number of loads it contained and sold and remaining; $\therefore 5x-15 : 4x :: 1 : 2$, or $3x=15$, and the number in the stack $\therefore$, was 45.

37. There are three pieces of cloth, whose lengths are in the proportion of 3, 5, and 7; and 6 yards being cut off from each, the whole quantity is diminished in the proportion of 20 to 17. Required the length of each piece at first.

Ans. 24, 40, and 56 yards.

Let $3x$, $5x$, and $7x$ be the lengths; and since the whole quantity is diminished in proportion to $20 : 17$, the whole quantity is to the part cut off $:: 20 : 3$, or $15x : 18 :: 20 : 3$, and then $x=8$.

38. The number of days that 4 workmen were employed were severally as the numbers 4, 5, 6, 7; their daily wages the same, viz. 3 shillings, and the sum received by the first and second was 36 shillings less than that received by the third and fourth. How much did each receive? Ans. 36, 45, 54, and 63 shillings.

Let $4x$, $5x$, $6x$, $7x$ denote the number of days employed; $\therefore 3(4x+5x)+36=3(6x+7x)$ or $9x+12=13x$, or $x=3$.

39. From two casks of equal size are drawn quantities, which are in the proportion of 6 to 7; and it appears that if 16 gallons

less had been drawn from that which is now the emptier, only half as much would have been drawn from it as from the other. How many gallons were drawn from each? Ans. 24 and 28.

Let $6x$ and $7x$ denote the number of gallons drawn off; $\therefore 7x-16=3x$, or $4x=16$, $x=4$, $6x=24$, $7x=28$.

40. On the inclosure of a parish, a proprietor had for his allotment two pieces of land, which were of the form of rectangular parallelograms. The longer sides of the parallelograms were in the ratio of 6 to 11, and the adjacent sides of the less as 3 to 2. The periphery of the less was 135 yards more than the longer side of the greater. Required the sides of the less, and the longer side of the greater. Ans. the sides of the less were 90 and 60; and the longer side of the greater was 165 yards.

Let $6x$ and $11x$ be the longer sides, and $4x=$ the shorter side of the less, and $2(6x+4x)=11x+135$, and $9x=135$; $\therefore x=15$.

41. Two persons, A and B, travelling, each with 80 dollars, meet with robbers, who take from A twice as much as from B and 5 dollars over, and leave A within 13 dollars half as much as B. How much is taken from each? Ans. 69, and 32 dollars.

Let x be the sum taken from B; then $2x+5=$ the sum taken from A; $\therefore 75-2x=$ sum left with A and $80-x=$ the sum left with B; $\therefore 2(75-2x+13)=80-x$, or $3x=16$; $\therefore x=32=$B's.

42. A person distributed forty shillings amongst 50 people; giving to some nine-pence each, and to the rest fifteen pence. How many were there of each? Ans. 45, and 5.

Let x and $50-x$ denote the numbers; then $9x+15(50-x)=40\times12$, or $3x+250-5x=160$, or $2x=90$, and $x=45$.

43. A person put out a certain sum to interest for $6\frac{1}{2}$ years, at 5 per cent. simple interest, and found that if he had put out the same sum for 12 years and 9 months at 4 per cent. he would have received 185 dollars more. What was the sum paid out?

Ans. 1000 dollars.

Let x denote the sum; $\therefore \frac{x}{20}=$ the interest for one year, and

$$\frac{x}{20}\times\frac{13}{2}+185=\frac{x}{25}\times12\tfrac{3}{4}=\frac{x}{25}\times\frac{51}{4},$$ and $130x+74000=204x$, or $74x=74000$, and $x=1000$, Ans.

44. A regiment of militia, containing 594 men, is to be raised from three towns, A, B, C. The contingents of A and B are in the proportion of three to five; and of B and C in the proportion of eight to seven. Required the numbers raised by each.

Ans. 144 by A, 240 by B, 210 by C.

Let $3x$ and $5x$ be the numbers raised by A and B; $\therefore 8:7::5x$; the contingent of $C=\frac{35}{8}x$; whence, $3x+5x+\frac{35}{8}x=594$, and $(64x+35x)=99x=8\times594$, and $x=48$.

45. Two persons, A and B, were partners. A's money remained in the firm 6 years, and his gain was one-fourth of his principal; and B's money, which was 50 less than A's, had been in the firm 9 years, when they dissolved partnership, and it appeared that if B had gained $6\frac{1}{4}$ dollars less, his gain and principal would have been to A's gain and principal as 4 to 5. What was the principal of each? Ans. 200 dollars, and 150 dollars.

Let $4x =$ A's principal, and $x =$ his gain; then $4x-50 =$ B's principal; $\therefore 6\times 4x : 9(4x-50) :: x :$ B's gain $= \frac{3}{4}(2x-25)$, and $5:4::5x:4x-50+\frac{3(2x-25)}{4}-\frac{25}{4}$, or $4x=4x-50+\frac{3(2x-20)}{4}-\frac{25}{4}$, and $6x=300$; $\therefore x=50$, and their principals were 200 dollars and 150 dollars.

46. The estate of a bankrupt, valued 21000 dollars, is to be divided amongst four creditors proportionably to what is due to them. The debts due to A and B are as $2:3$; B's claims and C's are in the proportion of $4:5$; and C's and D's in the proportion of $6:7$. What sum must each receive?

Let $2x$ and $3x$ denote the sums A and B received; then $4:5 :: 3x :$ sum C received $=\frac{15}{4}x$, and $6:7::\frac{15}{4}x$, sum D received, which $\therefore$ is $\frac{35}{8}x$, and therefore $2x+3x+\frac{15}{4}x+\frac{35}{8}x=21000$, or $(16x+24x+30x+35x=)105x=8\times 21000$; $\therefore x=1600$.

47. A merchant bought wheat at the rate of \$$3\frac{1}{2}$ for 5 bushels. He afterwards bought some inferior, which was in quantity to the former as 3 to 4, at the rate of \$$3\frac{3}{5}$ for 8 bushels, and sold the whole for 10 shillings a bushel, in consequence of which he lost \$$7\frac{4}{5}$ by the bargain. How much of each did he buy?

Let $4x$ and $3x$ denote the number of bushels of each sort, and $14\times 4x+9\times 3x=10+7x+156$, or $83x=70x+156$, or $x=12$.

48. Three persons, A, B and C, spent equal sums at a tavern. C having no money, the reckoning was paid by A and B. When C came to reimburse them, he paid 4 times as much to A as to B, and observed, that if B had paid 3 shillings more of his reckoning, their demands would have been equal. Required the sum each spent, and the respective part of C's reckoning that A and B paid.

Let $4x$ and x denote the part paid by A and B, and the reckoning $=5x$; therefore $x+3=4x-3$, and $x=2$; consequently, the reckoning was 10*s*. A paid 8*s*., and B 2*s*.

49. A certain sum is divided among three persons: A receives \$3000 more than the half, B \$1000 less than the third part, and C \$800 more than the fourth part of the whole. What is the sum divided, and what did each receive?

Let $12x$ be the sum divided; $6x-3000$, $4x-1000$, $3x+800$, represent the sums that A, B and C each received. Then, by the question, $13x-3200=12x$, and $x=3200$, and the whole \$38400. A receives \$16200, B \$11800, C \$10400.

50. A farmer had two flocks of sheep, one of which contained 40, and the other was sold for \$30; but one sheep of the latter was worth four of the other, and the value of the first flock was only \$4 more than the price of eight sheep of the second. How many sheep did the second flock contain, and what was the value of a sheep of each?

Let $x=$ the number, $\frac{30}{x}=$ the price of one, and $\frac{15}{2x}=$ the price of one of the first; $\therefore 8\{\frac{30}{x}\}+4=40\{\frac{15}{2x}\}$, or $\frac{60}{x}+1=\frac{75}{x}$; $\therefore 1=\frac{15}{x}$, and $x=15$, and the prices were \$2, and 50 cts.

51. A and B playing at billiards, A bet five shillings to 4 on every game, and found that, after a certain number of games, he had won 10 shillings. Had B won one game more, the number won by him would have been to the number won by A as 3 to 4. How many did each win?

Let $4x$ and $3x-1$ denote the number won by A and B; then $4\times4x=5(3x-1)+10$, or $16x=15x+5$; $\therefore x=5$, and A won 20, B 14.

52. A besieged garrison had such a quantity of bread as would, if distributed to each at 10 ounces a day, last 6 weeks; but, having lost 1200 men in a sally, the governor was enabled to increase the allowance to 12 ounces per day for 8 weeks. Required the number of men at first in the garrison.

Let x denote the number; then $6\times7\times10x=8\cdot7\cdot12\cdot(x-1200)$, or $5x=8x-9600$, and $3x=9600$; $\therefore x=3200$, Ans.

53. A composition of copper and tin, containing 100 cubic inches, weighed 505 ounces. How many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and a cubic inch of tin to weigh $4\frac{1}{4}$ ounces?

Let x, and $505-x$ be the number of ounces of copper and tin, and $\frac{4x}{21}+\frac{4(505-x)}{17}=100$, and $x=420=$ copper, 85 tin.

54. A and B are 131 miles distant from each other. A coach sets out from A at 6 o'clock in the morning, and travels at the rate of 4 miles an hour without intermission, in the direct road towards B. At 2 o'clock in the afternoon of the same day, a coach sets out from B to go to A, and goes at the rate of 5 miles an hour, constantly. Where will they meet?

Let x, and $x-8$ denote the number of hours the first and second travel before they meet, and $4x+5x-40=131$, and $x=\frac{171}{9}=19$; therefore, they meet 76 miles from A, and 55 miles from B.

55. Out of a certain sum a man paid his creditors 96*D.*; half of the remainder he lent his friend; he then spent one-fifth of what now remained, and after all these deductions had one-tenth of his money left. How much had he at first?

Let $2x$, $x-48$, and $\frac{4}{5}(x-48)$ denote the sum, what he lent, and what remained after spending; $\therefore$ $\frac{4}{5}(x-48)=\frac{1}{5}x$, and $4x-192=x$, and $\therefore$ $3x=192$, or $x=64$, and the sum was 128*D.* Ans.

56. At the review of an army, the troops were drawn up in a solid mass, 40 deep, when there were just one-fourth as many men in front as there were spectators. Had the depth, however, been increased by 5, and the spectators drawn up in the mass with the army, the number of men in front would have been 100 fewer than before. Of what number of men did the army consist?

Let x, $4x$, and $40x$ denote the number of men in front, spectators, and the army. Hence, $45(x-100)=44x$, and $x=4500$.

57. A and B, in order to keep up the price of copper to 86*D.* per ton, agree for a certain time to sell all the copper they raise, jointly, yet so that each shall be paid proportionally to the quantity he raises. Now, the whole quantity raised in the stipulated time was 235 tons; and A receives 4214*D.* more than B. Required the quantity raised by each.

Let x, $235-x$, be the quantity raised by A and B, and then $86x=86(235-x)+4214$, or $x=235-x+49$, and $2x=284$, $x=142$.

58. Bought two pieces of linen, one of which wanted twelve yards of being four times as long as the other. The longer cost 5 shillings, and the shorter 4 shillings a yard; 23 yards being cut off from the longer, and 5 from the shorter, and the remainders being sold for one shilling a yard more than they cost, I received 142*D.* How many yards of each were there?

Let x, and $4x-12$ be the length of the shorter and longer; then $5(x-5)+6(4x-35)=142$, or $(5x+24x)=29x=377$, and $x=13$.

59. A travels at the rate of 13 miles in 2 hours; 12 hours afterwards B passes through the same place, travelling the same road, at the rate of 26 miles in 3 hours. How long and how far must B travel before he overtakes A?

Let $6x$, and $6(x+2)$, denote the number of hours the second travels, and do. first; and $3:26::6x:$ the distance the second

travels after passing A, which $\therefore = 52x$. In the same manner, the distance which the first travels after passing B is $=39(x+2)$; consequently, $52x=39x+78$, and $x=6$; and he must $\therefore$ travel 36 hours and 312 miles, Ans.

60. As A and B were going to school, A shot an arrow in the direction in which they were going, which B took up and shot forward; and so on, alternately, till the arrow had passed exactly from one mile-stone to another; when it appeared that A had shot the arrow 8 times, and B 7 times. Some time afterwards, A and B were on the opposite banks of a river, the breadth of which they wished to ascertain. A first shot the arrow across the river, and it flew 13 yards beyond the bank on which B stood; B then took it up, and from the place where it had fallen, shot it back across the river; it now fell $9\frac{2}{7}$ yards beyond the bank upon which A stood. Required the breadth of the river.

Let x, $x+13$, $x+22\frac{2}{7}$, denote the breadth and the length of A's and B' shots, and $8(x+13)+7(x+22\frac{2}{7})=1760$, and $15x=1760-260=1500$, and $x=100$ yards, Ans.

61. Three merchants, A, B and C, enter into a speculation. B subscribes 10*D*. more than four-fifths of what A does; and C 30*D*. more than half of what B does. A's gain is two-fifths of his subscription, and B's is 148*D*. What are the respective sums subscribed, and whole gain?

Let $x =$ A's subscription, $4x+10 =$ B's, and $2x+35 =$ C's; and $2x =$ A's gain, and $5x : 2x :: (4x+10 : 148 ::)2x+5 : 74$, or $4x+10=370$, and $4x=360$, and $x=90$. Also, $5 : 2 :: 180+35$: C's gain, $\frac{2}{5}\cdot 215=86$; $\therefore$ the sums subscribed were 450, 370, 215, and the whole gain $= 180+148+86=414$.

62. There are two places, 154 miles distant, from which 2 persons set out at the same time to meet, one travelling at the rate of 3 miles in two hours, and the other at the rate of 5 miles in four hours. How long and how far did each travel before they met?

Let $4x =$ the number of hours; $\therefore 2 : 3 :: 4x :$ the number of miles the first travelled $= 6x$, and $5x =$ the second travelled; $\therefore$ $11x=154$, and $x=14$, the number of hours $= 56$.

63. A sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and eight hours afterwards B sets out from the same place, and travels the same road at the rate of five miles in three hours. How long and how far must A travel before he is overtaken by B?

Let $x =$ the number of hours; $\therefore 5 : 7 :: x :$ the number of miles A travels $= \frac{7}{5}x$; in the same way, $\frac{5(x-8)}{3} =$ the number B travels, and $\frac{5(x-8)}{3} = \frac{7}{5}x$, and $25x-25\times 8=21x$, and $4x= 25\times 8$, and $x=50$ hours, and 70 miles.

64. A man lent out a certain sum to interest at $8D.$ per cent. per annum. He suffered this to accumulate at simple interest for 12 years; and then putting out the principal and interest at the same rate, found that the present annual interest exceeded the former by $38\frac{2}{5}D.$ Required the sum put out each time.

Let x, $(\frac{8}{100}x=)\frac{2}{25}x$, and $\frac{24}{25}x$, and $\frac{49}{25}x$; $\frac{2}{25}\cdot\frac{49}{25}x$ = the sum, = the interest for one year, = the interest for 12 years, = the sum put out the second time, and = the annual interest, respectively; $\because 2\cdot\frac{49x}{25^2}=\frac{2x}{25}+\frac{192}{5}$, and $\frac{49x}{5\times25}=\frac{x}{5}+96$, and $\therefore \frac{24x}{5\times25}=96$, and $x=\$500$, and \$980 the second time.

65. A waterman finds by experience that he can, with the advantage of a common tide, row down a river from A to B, which is 18 miles, in an hour and a half, and that to return from B to A, against an equal tide, though he rows back along the shore, where the stream is only three-fifths as strong as in the middle, takes him just two hours and a quarter. It is required from hence to find at what rate per hour the tide runs in the middle, where it is strongest.

Let $5x$ = the rate required, $6 : 4 :: 18 :$ the distance he can row with the tide per hour $= 12$; $\therefore 12-5x$ = the distance without the tide, and $9 : 4 :: 18 :$ the distance he can row up the stream per hour against the tide $= 8$; $\therefore 8+3x$ = the distance per hour without the tide, and $8+3x=12-5x$, or $8x=4$, and $x=\frac{1}{2}$, and its rate $\therefore$ is $2\frac{1}{2}$ miles per hour.

66. The ingredients of a loaf of bread are rice, flour and water, and the weight of the whole is 15lbs. The weight of the rice augmented by 5lbs. is two-thirds of the weight of the flour, and the weight of the water is one-fifth of the weight of the flour and rice together. Required the weight of each.

Let $3x$, $2x-5$, and $x-1$ denote the weight of flour, rice and water respectively; then $6x-6=15$, and $6x=21$, and $x=3\frac{1}{2}$; $\therefore$ there were 2lbs. of rice, $10\frac{1}{2}$ of flour, and $2\frac{1}{2}$ of water.

67. In a battery two cannon were employed, the first of which had been fired 36 times before the second began to play; and afterwards was fired eight times whilst the second was fired seven. But the quantity of powder used for each shot of the first was less than what was used for the second in the proportion of $3 : 4$. How many times was the second fired before it had consumed as much powder as the first?

Let $7x$, and $8x+36$ denote the number of times the second and first were fired; and since the quantity of powder consumed is the same as $8x+36 : 7x :: 4 : 3$, or $2x+9 : 7x :: 1 : 3$; $\therefore 7x=6x+27$, and $x=27$, and $\therefore$ the number is 189 times.

68. Suppose two fingers of a watch, (a) and (b), were together on Sunday noon at 12 o'clock, and that the motion of each was such, that (a) moved round the horary circle in one hour, and (b) in $1\frac{1}{60}$ hour. When will they be together again for the first time?

Let $x =$ the number of hours; $\therefore$ the first makes x revolutions and $\frac{60}{61}x =$ the number of revolutions the second makes; $\therefore x = \frac{60}{61}x+1$, and $x=61$.

69. A draper bought a piece of cloth for \$69, from which he cut off 11 yards. He then met with another piece of equal goodness, for which he gave \$21, and found that if it had been 1 yard longer, its length would have been to the length of the remainder of the first as 2 to 3. How many yards were there in each piece, and what was the price of a yard?

Let $3x$, $\dfrac{23}{x}$, $2x-\dfrac{25}{3}$, and $\dfrac{63}{6x-25}$, denote the number of yards in the first, $=$ the price of a yard, $=$ the number in the second, $=$ the price of a yard, whence $\dfrac{23}{x}=\dfrac{63}{6x-25}$, and $23(6x-25) = 63x$; $\therefore 75x=23\times25$, and $x=\frac{23}{3}$.

70. A fruiterer sells for 19*s.* 6*d.* a certain number of oranges and apples, of which the latter exceed the former by 180. He sells the apples at the rate of five for 3*d.*, and fifteen oranges bring him in $1\frac{1}{2}d.$ more than 35 apples. How many are there of each sort, and what are the oranges worth apiece?

Let $5x =$ the number of apples; $\therefore 5x-180 =$ the number of oranges, and $5 : 3 :: 5x :$ the price of the apples $= 3x$; in the same way the price of 35 apples $= 21d.$, and $\frac{45}{2}\div15=\frac{3}{2} =$ the price of an orange; $\therefore 3x+\frac{3}{2}(5x-180) = 234$, or $x+\frac{5}{2}x-90 = 78$, or $\frac{7}{2}x=168$, and $x=48$.

71. Divide the number 198 into 5 such parts, that the first increased by one, the second increased by two, the third diminished by three, the fourth multiplied by four, and the fifth divided by 5, may be all equal.

Let x, $20x$, $4x+3$, $4x-2$, $4x-1$, be the fourth, fifth, 3d, 2d and first. Hence $33x = 198$, and $x = 6$, and the parts 23, 22, 27, 6, and 120.

72. A person has four casks, the second of which being filled from the first, leaves the first four-sevenths full. The third being filled from the second leaves it one-fourth full; and when the 3d is emptied into the fourth, it is found to fill one nine-sixteenths of it only. But the first will fill the third and fourth, and have fifteen quarts remaining. How many quarts does each hold?

Let $7x$, $3x$, $\frac{9}{4}x$, and 4, denote the first, second, 3d and fourth, respectively. Hence, $7x=\frac{9}{4}x+4x+15$, and $\frac{3}{4}x = 15$, and $x=20$, and the casks hold 140, 60, 45, and 80 gallons.

73. A packet sailing from Dover with a fair wind, arrives at Calais in two hours; and on its return, the wind being contrary, it proceeds six miles an hour slower than it went. Now, when it is half way over, the wind changing, it sails two miles an hour faster, and reaches Dover sooner than it would have done had the wind not changed, in the proportion of 6 : 7. Required the rates of sailing, and the distance between Dover and Calais.

Let x, $x-6$, $2x$, be the rate of sailing in going and in returning, and the whole distance. Hence $\frac{x}{x-6}$, and $\frac{x}{x-4}$ = the times of going half way before and after the change of the wind;

$\therefore \frac{1}{x-6}+\frac{1}{x-4} : \frac{1}{x-6} :: 6 : 7$, and $\frac{1}{x-4} : \frac{1}{x-6} :: 5 : 7$,

$\therefore x-6 : x-4 :: 5 : 7$, or $2 : x-4 : 2 : 7$, $x-4=7$, and $x=11$.

74. A and B set out from two places, C and D, at the same time, towards E; the road from C to E being through D. A travels 7 miles an hour, and at that rate of travelling would have overtaken B 5 miles before he got to E; but, after arriving at D, he travels $6\frac{2}{3}$ miles an hour, in consequence of which he overtakes B just as he enters E. Supposing B to travel 5 miles an hour, what are the distances between C, D and E?

Let $x=$CD; $\therefore \frac{1}{7}x =$ the time of travelling to CD, and $\frac{1}{7}x + \frac{3}{20}\text{DE}=\frac{\text{DE}}{5}$, or $\text{DE} = \frac{20x}{7}$. Now $\frac{\text{CE}-5}{7}=\frac{\text{DE}-5}{5}$, or $(x+\frac{20x}{7}-5)5=7(\frac{20x}{7}-5)$, $\therefore 10=\frac{40x}{7}-5x=\frac{5x}{7}$, and $x=14$, the distance from C to D, and the distance from D to E = 40, Ans.

75. A gentleman wishing his two daughters to receive equal portions when they became of age, bequeathed to the elder the accumulated interest of a certain sum of money, bought at the time of his death into the four per cent. stock at 88; and to the younger the accumulated interest of a sum less than the former by 3500*D.* bought at the same time into the 3 per cents. at 63. Supposing their ages at the time of their father's death to have been 17 and 14, what would be the sum bought into the stocks in each case, and what would be the fortune of each?

Let x, $\frac{x}{22}$, $\frac{2x}{11}$, $\frac{x-3500}{21}$, $\frac{x-3500}{3}$, and $\frac{x-3500}{3}=\frac{2x}{11}$, denote the sum bought into the 4 per cents., = the interest for one year, = the eldest's fortune, = interest for one year in the 3 per cts., = the youngest's fortune, and the equation, respectively; $\therefore 11x-11\times3500=6x$, and $5x=11\times3500$, and $x=7700D.$ = the sum bought into the 4 per cents., and 4200 = the sum bought into the 3 per cents., and \$1400 = the fortune.

76. Out of a common pack of cards, a certain number, including the ten of diamonds, was dealt equally among four persons, the dealer turning up the last card, which was the ten of spades, which he gave himself. Now, if twice the number of cards had been dealt to each, the ten of spades being turned up by the dealer, and the ten of diamonds being still dealt out, the chance of the dealer's having the ten of diamonds would be to the chance against him as $3 : 10$. Required the number of cards dealt to each the second time.

Let x be the number dealt to each; then $3x : x - 1 :: 10 : 3$, or $9x = 10x - 10$, or $x = 10$, Ans.

77. Two companies of soldiers, consisting of equal numbers, were sent out under A and B, from two hostile camps, to reconnoitre. Falling in with each other, a skirmish ensued, in which A lost 50 killed and prisoners, and B had 20 killed. A however, having been reinforced by a party equal to five-sevenths of the number which B had remaining; and B having been reinforced by a number greater by 46 than three-fifths of the number which A had remaining, they renewed the engagement, when A was forced to retire with the additional loss of 30 men. When the returns were made, B found he had again lost 20 men, but that he had then twice as many men remaining as A had. How many had each at first?

Let $x =$ the number;

$\because x - 50 + \frac{5}{7}(x - 20) - 30 = \frac{1}{2}\{x - 20 + 46 + \frac{3}{5}(x - 50) - 20\}$,

or $\frac{1}{7}(24x - 1320) = \frac{1}{5}(8x - 120)$, $\therefore x = \frac{5760}{64} = 90$.

78. A sportsman, who kept an account of the number of birds which he killed, found that each succeeding year he wanted 50, in order that the number killed might bear the proportion of $3 : 2$ to the number killed in the preceding year. In the fourth year he found that he had killed 170 fewer than three times the number killed in the first year. How many did he kill the first year?

Let $2x =$ the number killed the first year; $\therefore 2 : 3 :: 2x :$ the number killed in the second, $+ 50$; $\therefore 3x - 50 =$ the number killed in the second, and $\frac{3}{2}(3x - 50) - 50 = \frac{9x - 250}{2} =$ the number killed in the third; also, $\frac{3}{2}(\frac{9x - 250}{2} - 50 =$ the number killed in the fourth; $\therefore \frac{27x - 950}{4} = 6x - 170$, and $27x - 24x = 270$, or $x = \frac{270}{3} = 90$, Ans.

79. Several detachments of artillery divided a certain number of cannon balls. The first took 72, and one-ninth of the remainder; the next 144, and one-ninth of the remainder; the third 216,

and one-ninth of the remainder; the fourth 288, and one-ninth of those that were left; and so on: when it was found that the balls had been equally divided. Determine the number of detachments and balls.

Let x = the number of balls; $\therefore 72+\frac{x-72}{9}$ equal the number taken by the first detachment; $\therefore x-72-\frac{x-72}{9}=\frac{8}{9}(x-72)$ = the number remaining, and $144+\frac{1}{9}\{\frac{8}{9}(x-72)-144\}$ = the number taken by the second detachment; $\therefore 72+\frac{x-72}{9}=144+\frac{8}{81}(x-72)-16=128+\frac{8}{81}(x-72)$, and $\frac{x-72}{81}=56$, $\therefore x=72+56\times 81=4608$, and the number of the detachment $=\frac{4608}{72+504}=\frac{4608}{576}=8$.

80. Four men walking abroad, found a purse containing shillings only, out of which one of them took a number at a venture. Afterwards, comparing their numbers together, they found that if the first took 25 shillings from the second, it would make his number equal to what the second had left. If the second took thirty shillings from the third, his money would then be triple what the third had left. And if the third took 40 shillings from the fourth, his money would then be double of what the fourth had left. Lastly, the fourth taking 50 shillings from the first, he would then have three times as much as the first had left, and 5 shillings over. What had each?

Let x, $x+50$, $\frac{x+170}{3}$, $\frac{x+170}{6}+60$, and $\frac{x+170}{6}+60+50=3(x-50)+5$, denote what the first, second, third, fourth had, and the equation, respectively; then $x+170+660=18x-900+30$, and $x=\frac{1700}{17}=100$.

81. Fifteen current guineas should weigh 4 ounces; but a parcel of light gold being weighed and counted, was found to contain 9 more guineas than was supposed from the weight; and a part of the whole, exceeding the half by 10 guineas and a half, was found to be $1\frac{1}{3}$ oz. deficient in weight. What was the number of guineas?

Let x = the number; then $15:4::x:$ the proper weight, $=\frac{4}{15}x$, and in the same way, $\frac{36}{15}$ = the proper weight of 9 guineas, and $\frac{1}{15}(4x-36)$ = the apparent weight. Also, $x:\frac{4x-36}{15}::$

$\frac{x+21}{2}$: the apparent weight of $\frac{x+21}{2}$ guineas $=\frac{(4x-36)(x+21)}{30x}$ and $15 : 4 :: \frac{x+21}{2}$: their real weight $=\frac{2(x+21)}{15}$; $\therefore$ $\frac{(2x-18)(x+21)}{15x}+\frac{4}{3}=\frac{2(x+21)}{15}$; $\therefore$ $\frac{4}{3}=\frac{18}{15x}(x+21)=\frac{6}{5x}(x+21)$, and $10x=9x+189$, or $x=189$, Ans.

82. A merchant bought a quantity of wheat for \$200, half of which he reserved for his private use. He then sold five bushels more than $\frac{3}{4}$ of the remaining quantity at such a price as to gain \$40 per cent. But the price of wheat having advanced, he sold the remainder at such a price as to gain \$67 per cent. by what he sold. And had the whole been sold at this latter price, he would have gained \$160 per cent. How much did he buy, and how did he sell it?

Let $8x$ = the number of bushels he bought; $\therefore$ $4x$ = the quantity reserved, and $3x+5$ = the number first sold, and $x-5$ = the number second sold. Also $\frac{200}{8x}=\frac{25}{x}$ = the buying price of a bushel, and $(100 : 140 ::) 5 : 7 :: \frac{25}{x} : \frac{35}{x}$ = the first selling price of a bushel, and $(100 : 260 ::) 5 : 13 :: \frac{25}{x} : \frac{65}{x}$ = the second selling price; $\therefore$ $(3x+5x)\frac{35}{x}+(x-5)\frac{65}{x}-100=70-\frac{150}{x}$ = the gain; and as $100 : 67 :: 100 : 70-\frac{150}{x}$, $\therefore$ $x=\frac{150}{3}=50$, and the number of bushels $=400$. Also, the first selling price $=\frac{35}{50}=70$ cts., and the second $=\frac{65}{50}=1{,}30$ cts.

83. A brewer, from a certain quantity of ingredients which cost \$20, brews 500 gallons of ale, (on which there is a duty of 6*d.* a gallon,) and sells it at 2*s.* a gallon. Afterwards, from the same quantity of ingredients, he brews a certain number of gallons of strong beer, (on which he pays the ale duty,) and the remainder small beer, making together the same number of gallons as before; when, by mixing them together, and selling the mixture as ale, he finds his gains increased in the proportion of 10 : 7. Determine the number of gallons of strong beer, supposing the duty on small beer one-fourth of that on ale.

Let x = the number of gallons of strong beer, and as $12\frac{1}{2}$ = the duty on the ale, and 50 what he sells the ale for; $\therefore$ $17\frac{1}{2}$ = the gain; then $\frac{x}{40}$, $\frac{500-x}{160}$, $20+\frac{500+3x}{160}$, and $50-20-\frac{500+3x}{160}$ will represent the duty on the strong, and the duty paid on the small the sum expended in the second case, and = the second gain. Also, as

$17\frac{1}{2} : 30-\frac{500+3x}{160} :: 7 : 10$; $\therefore$ $\frac{5}{2} : 30-\frac{500+3x}{160} :: 1 : 10$, and

$$30-\frac{500+3x}{160}=25;\ \therefore 500+3x=800,\text{ and }x=\frac{300}{3}=100.$$

On the Solution of Simple Equations which involve more than one unknown Quantity.

70. If the equation involve several unknown quantities, and definite values of these are required, there must necessarily be as many independent equations as there are unknown quantities. In which case, the values will be found by exterminating all the unknown quantities except one; and this may be done by either of the three following methods:

1. By equalizing the coefficients of the same unknown quantity in the several equations.
2. By substitution.
3. By equating different values of the same unknown quantity.

1. Of exterminating an unknown quantity by the first method in equations where two unknown quantities are concerned.

If the coefficient of either unknown quantity in one equation be contained a certain number of times exactly in the coefficient of the same unknown quantity in the other, multiply the former equation by that number, then add it to, or subtract it from, the other equation, according as the signs are different or the same, and an equation arises, in which only one unknown quantity is found.

Thus, if $\left.\begin{array}{r}4x+y=34\\ \text{and } 4y+x=16\end{array}\right\}$ Here the coefficient of x in the second equation is contained 4 times exactly in the first; multiplying therefore the second equation by 4, and subtracting the first from it,

$$\begin{array}{r}4x+16y=64,\\ \text{and } \underline{4x+\ \ \ y=34;}\\ \therefore 15y=30,\text{ and } y=2.\end{array}$$

Having thus obtained a value of one of the unknown quantities, the other may be determined by substituting in either equation the value of the quantity found, and thus reducing the equation to one which contains only the other unknown quantity. Thus, from the second of the preceding equations, $x=16-4y=16-8=8$.

The values of x and y might be found in a similar manner, by multiplying the first equation by 4, and subtracting the second from it. But if neither of the coefficients be a measure of the coefficients of the same unknown quantity in the other equation, multiply the first equation by the coefficient of one of the unknown quantities in the second equation, and the second equation by the coefficient of the same unknown quantity in the first. If the signs of the unknown quantity be alike in both, subtract one equation

from the other; if unlike, add them together, and an equation arises in which only one unknown quantity is found.

Thus, if $\left.\begin{array}{r}2x+3y=23\\ \text{and } 5x-2y=10\end{array}\right\}$ In this case neither of the coefficients is a measure of the coefficient of the same unknown quantity in the other equation; and therefore, multiplying the first equation by 2, and the second by 3,

$$4x+6y=46,$$
$$\text{and } 15x-6y=30;$$

$\therefore$ by addition, $19x=76$, and $x=4$; whence, as before,
$3y=23-2x=23-8=15$, and $y=5$.

The values of x and y might also be obtained, by multiplying the first equation by 5, and the second by 2, and then subtracting the second from the first.

2. By substitution. Find the value of one of the unknown quantities, in terms of the other and known quantities, in the more simple of the two equations; and substitute this value instead of the quantity itself in the other equation; thus an equation is obtained in which there is only one unknown quantity.

Thus in the first of the preceding examples; from the second equation, $x=16-4y$; substituting therefore this value of x in the first equation, $4\cdot(16-4y)+y=34$, or $64-16y+y=34$; $\therefore$ by transposition, $(64-34=)30=15y$, and therefore $2=y$; whence, as before, $x=8$. Here a value of x might have been obtained from the second equation, and substituted for it in the first; whence an equation would have arisen, involving only y; the value of which being found, that of x also might be determined, as before, by substitution.

Or a value of y might be determined from either equation, and substituted in the other; from which would arise an equation involving only x, the value of which might be found; and therefore the value of y also might be obtained by substitution.

Again, in the second example; from the first equation is obtained $2x=23-3y$; and therefore $x=\frac{23-3y}{2}$;

substituting therefore this value in the second equation,

$5\cdot\frac{23-3y}{2}-2y=10$, or $115-15y-4y=20$; $\therefore$ by transposition, $115-20=15y+4y$, or $95=19y$; $\therefore 5=y$, and $x=\frac{23-3y}{2}=\frac{23-15}{2}=\frac{8}{2}=4$. Here also a value of x might be obtained from the second equation, and substituted in the first, which would give an equation involving only y; or a value of y might be obtained

from either equation, which substituted in the other would give an equation involving only x; the value of which might therefore be found, and consequently that of y might also be determined.

3. By equating different values of the same unknown quantity.

From each equation find the value of the same unknown quantity in terms of the other and known quantities; then, by equating the values so found, an equation arises containing only one unknown quantity. Thus, in the first of the preceding examples; from the first equation, $y=34-4x$, and from the second equation, $4y=16-x$; and therefore $y=\frac{16-x}{4}$; $\therefore \frac{16-x}{4}=34-4x$; consequently, $16-x=136-16x$; $\therefore$ by transposition, $16x-x=136-16$; or $15x=120$;

$\therefore x=8$, and $y=34-4x=34-32=2$, as before.

In this case, also, two values of x are deducible from the two equations, which would give an equation involving y only; and the value of y being determined, that of x might also be found.

Again, in the second of the preceding examples; from the first equation, $x=\frac{23-3y}{2}$, and from the second, $x=\frac{10+2y}{5}$; therefore, $\frac{10+2y}{5}=\frac{23-3y}{2}$, and $20+4y=115-15y$; by transposition, $4y+15y=115-20$, or $19y=95$; $\therefore y=5$, and $x=4$, as before. Here, again, two values of y might have been found, which would have given an equation involving only x; and from the solution of this new equation, a value of x, and therefore of y, might be found.

1. Given $\left.\begin{array}{l} 5x+4y=58 \\ \text{and } 3x+7y=67 \end{array}\right\}$ to find the values of x and y.

Multiplying the second equation by 5, and the first by 3,

$$\begin{array}{rr} & 15x+35y=335, \\ \text{and} & \underline{15x+12y=174}; \\ \therefore \text{by subtraction,} & 23y=161, \text{ and } y=7; \end{array}$$

whence, $5x=58-4y=58-28=30$, and therefore $x=6$.

If the second equation had been multiplied by 4, and subtracted from the first when multiplied by 7, an equation would have arisen, involving only x, the value of which might be determined, and thence, by substitution, the value of y.

Second Method.

From the second equation, $3x=67-7y$; $\therefore x=\frac{67-7y}{3}$. Substituting this value of x in the first equation, $5\cdot\frac{67-7y}{3}+4y=58$, and $335-35y+12y=174$; $\therefore$ by transposition, $335-174=$

$35y-12y$, or $161=23y$; $\therefore 7=y$; whence, as before, the value of x may be found. In the same manner, a value of x might be found from the first equation, which substituted in the second, would give an equation involving only y. Or a value of y might be obtained from either equation, and substituted for it in the other; whence an equation would arise involving only x, the value of which might be found, and therefore that of y also determined.

Third Method.

From the first equation, $5x=58-4y$; $\therefore x=\frac{58-4y}{5}$.

From the second, $x=\frac{67-7y}{3}$. $\therefore \frac{58-4y}{5}=\frac{67-7y}{3}$, and $174-12y=335-35y$;

by transposition, $35y-12y=335-174$, or $23y=161$; $\therefore y=7$; whence, as before, $x=6$. In this case, two values of y might be deduced from the two equations; and from equating these, there would arise an equation involving x only; whose value being found, that of y also might be determined by substitution.

2. Given $\left.\begin{array}{l} ax+by=m \\ cx+dy=n \end{array}\right\}$ to find the values of x and y.

Multiplying the first equation by c, and the second by a,

$$\begin{array}{l} acx+bcy=mc, \\ \underline{acx+ady=na;} \end{array}$$

$\therefore$ by subtraction, $(ad-bc)\,.\,y=na-mc$, and $y=\frac{na-mc}{ad-bc}$;

$$\text{whence } x=\frac{m}{a}-\frac{by}{a}=\frac{m}{a}-\frac{nab-mbc}{a^2d-abc},=\frac{mad-mbc}{a^2d-abc}-$$

$\frac{nab-mbc}{a^2d-abc},=\frac{mad-nab}{a^2d-abc}=\frac{md-nb}{ad-bc}$. Or the value of x might be determined from the second equation, $x=\frac{n}{c}-\frac{dy}{c}$.

If the first equation had been multiplied by d, and subtracted from the second multiplied by b, an equation would have arisen involving only x, the value of which might be determined; and this being substituted in either of the equations, the value of y might also be found.

Second Method.

From the first equation, $ax=m-by$; and $\therefore x=\frac{m-by}{a}$.

Substituting this value of x in the second equation,

$c\cdot\frac{m-by}{a}+dy=n$; $\therefore mc-bcy+ady=an$,

and $(ad-bc)\cdot y=an-mc$; $\therefore y=\frac{an-mc}{ad-bc}$; whence, the value of x may be determined, as before.

In the same manner, a value of x might be found from the second equation, which substituted in the first would give an equation involving only y, the value of which being found, that of x might also be determined. Or, a value of x might be obtained from either equation, which substituted in the other would give an equation involving only x, the value of which, and consequently that of y, might be found.

Third Method.

From the first equation, $x=\frac{m-by}{a}$, and from the second, $x=\frac{n-dy}{c}$; $\therefore \frac{m-by}{a}=\frac{n-dy}{c}$; and $mc-bcy=na-ady$; $\therefore$ by transposition, $ady-bcy=na-mc$;

$\therefore y=\frac{na-mc}{ad-bc}$; whence, as before, $x=\frac{md-nb}{ad-bc}$.

In this case, two values of y might be deduced from the two equations; and from equating these, there would arise another equation involving only x, the value of which being determined, that of y also might be found by substitution.

1. A draper bought two pieces of cloth for £12, 13*s*; one being 8*s*., and the other 9*s*. per yard. He sold them each at an advanced price of 2*s*. per yard, and gained by the whole £3. What were the lengths of the pieces?

Ans. 17 yards of the first, and 13 of the second.

Let x and y be the lengths, and $8x$ and $9y=$ what they cost, and by the question $\left\{\begin{array}{l}8x+9y=253\\2x+2y=\ 60\end{array}\right\}\begin{array}{l}x=17\\y=13.\end{array}$

2. A bill of £26 5*s*. was paid with half guineas and crowns, and twice the number of half guineas exceeded three times the number of crowns by 17. How many were there of each?

Let $2x$ and y denote the number of half guineas and of crowns; then $\left\{\begin{array}{l}21x+5y=525\\4x-3y=\ 17\end{array}\right\}$ or $\begin{array}{l}63x+15y=1575\\20x-15y=\ \ 85\end{array}\Bigg\}\begin{array}{l}x=20\\y=21.\end{array}$

3. A person expends half a crown in apples and pears, buying his apples at 4, and his pears at 5 a penny; and afterwards accommodated his neighbour with half his apples and one-third of his pears for 13 pence. How many did he buy of each?

Let $2x$ and $3y$ denote the number of apples and pears; then $\frac{1}{2}x=$ the price of apples, and $\frac{3}{5}y=$ the price of the pears, and by

the question, $\frac{x}{4}+\frac{y}{5}=13$, and $\frac{x}{2}+\frac{3y}{5}=30$, and $2x=72$, $3y=60$.

4. Two laborers, A and B, received £5 17*s*. for their wages; A having been employed 15, and B 14 days; and A received for four days 11*s*. more than B did for three days. What were their daily wages?

Let $\left.\begin{array}{r}x=\text{A's}\\ y=\text{B's}\end{array}\right\}$ daily wages. $\left\{\begin{array}{l}15x+14y=117\\ 4x-3y=11\end{array}\right\} \therefore \begin{array}{l}x=5\\ y=3.\end{array}$

5. A person had two casks, the larger of which he filled with ale, and the smaller with cider. Ale being half a crown, and cider 11*s*. per gallon, he paid £8 6*s*.; but had he filled the larger with cider, and the smaller with ale, he would have paid £11 5*s*. 6*d*. How many gallons did each hold?

Let x and $y=$ the number of gallons held by the larger and less: then $\left.\begin{array}{r}5x+22y=332\\ 22x+5y=451\end{array}\right\}$ $\left.\begin{array}{r}.\ 5(x+y=29)\\ \text{But } 15x+22y=332\end{array}\right\}$ and $\begin{array}{l}y=11\\ x=18.\end{array}$

Addition, $27x+27y=783$ Subtraction, $17y=187$

6. Two persons, A and B, played cards, each with a different sum. After a certain number of games, A had won half as much as he had at first, and found that if he had 15*s*. more, he would have had just three times as much as B. But B afterwards won 10*s*. back, and he had then twice as much as A. What had each at first?

Let $2x$ and $y=$ what A and B each had; then we have $3x+15=3(y-x)$; and $2(3x-10)=y-x+10$; here $x=7$ and $y=19$, and A 14, and B 19*s*.

7. A certain sum of money put out to interest, amounts in 8 months to \$$297\frac{3}{5}$; and in 15 months its amount is \$306, at simple interest. What is the sum, and the rate per cent.?

Let $x, y, \frac{xy}{100}, \frac{xy}{150}, \frac{xy}{80}$, and $(\frac{xy}{150}+x=297\frac{3}{5}$, and $\frac{xy}{80}+x=306)$ represent or $=$ the sum, $=$ the rate, $=$ the interest for one year, $=$ the interest for 8 months, $=$ the interest for 15 months, and $=$ the equations; whence $(297\frac{3}{5}-x)150=(306-x)80$, or

$4464-15x=2448-8x$; $\therefore x=\frac{2016}{7}=288$; $\therefore \frac{288y}{80}=18$, or $16y=80$.

8. A farmer being asked how many quarters of wheat he had sold in the market, answered if he had sold 8 quarters more, and got 7*s*. per quarter more than he did, he should have received £11 15*s*. more than he had; but if he had sold 7 quarters more at 8*s*. per quarter more, he should have had £11 17*s*. more. How many quarters did he sell, and what was the price?

Let x be the number of quarters, and $x=$ the price of one; then $\left\{\begin{array}{l}(x+8)(y+7)=xy+235 \\ (x+7)(y+8)=xy+237\end{array}\right\}$ or $\left\{\begin{array}{l}7x+8y=179 \\ 8x+7y=181\end{array}\right\}$ $\because \begin{array}{l}y=11 \\ x=13.\end{array}$

9. There is a number consisting of two digits, the second of which is greater than the first; and if the number be divided by the sum of its digits, the quotient is 4; but if the digits be inverted, and that number divided by a number greater by 2 than the difference of the digits, the quotient becomes 14. Required the number.

Let $10x+y$ be the numb.; then $\frac{10x+y}{x+y}=4$, $\frac{10y+x}{y-x+2}=14$; by the question, from the first, $10x+y=4x+4y$, or $2x=y$, and from the 2d, $10y+x=14y-14x+28$, or $15x-4y=28$, or $(15x-8x=)$ $7x=28$, or $x=4$, and $y=8$; and the number is 48, Ans.

10. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

Let $\frac{x}{y}$ be the fraction; $\therefore \frac{2x}{y+7}=\frac{2}{3}$ and $\left.\frac{x+2}{2y}=\frac{3}{5}\right\}$ $\therefore \begin{array}{l}x=4 \\ y=5.\end{array}$

11. A farmer parting with his stock, sells to one person 9 horses and 7 cows for \$300; and to another, at the same prices, 6 horses and 13 cows for the same sum. What was the price of each?

Let $x=$ the price of a horse, and $y=$ that of a cow; $\therefore 9x+7y=300=6x+13y$, and $x=2y$; then $12y+13y=300$, $y=\frac{300}{25}=12$, and $x=24$.

12. A farmer hires a farm for \$245 per ann., the arable land being valued at \$2 an acre, and the pasture at 28 shillings; now the number of acres of arable is to half the excess of the arable above the pasture as 28 : 9. How many acres were there of each?

Let $x=$ the number of acres of arable, and $y=$ that of the pasture; $\therefore x : \frac{1}{2}(x-y) :: 28 : 9$, and $2x+\frac{7}{5}y=245$; hence $x=98$, $y=35$.

13. A person owes a certain sum to two creditors. At one time he pays them \$53, giving to one four-elevenths of the sum which is due, and to the other \$3 more than one-sixth of his debt to him. At a second time he pays them \$42, giving to the first three-sevenths of what remains due to him, and to the other one-third of what is due to him. What were the debts?

Let $11x$ and $6y=$ the debts; $\therefore 4x+y+3=53$, and $3x+\frac{5}{3}y-1=42$; $\therefore 11x=121$, and $x=11$, and $6y=36$.

14. A and B playing at backgammon, A bet \$3 to \$2 on every

game, and after a certain number of games found that he had lost \$17. Now had A won 3 more from B, the number he would then have won, would have been to the number B would have won as 5 to 4. How many games did they play?

Let x and y denote the number A and B won; then $3y-2x=17$, and $x+3 : y-3 :: 5 : 4$, and $y=7$, and $x=2$.

15. A vinter has two casks of wine, from the greater of which he draws 15 gallons, and from the less 11; and finds the quantities remaining in the proportion of 8 to 3. After they become half empty, he puts 10 gallons of water into each, and finds that the quantities of liquor now in them are as 9 to 5. How many gallons will each hold? Let x and $y=$ the number of gallons each holds; then $x-15 : y-11 :: 8 : 3$, and $\frac{1}{2}x+10 : \frac{1}{2}y+10 :: 9 : 5$, by the question and from the first, $3x=8y-43$ and from the 2d equation, $5x=9y+80$; and we have $40y-27y=240+215$, or $y=\frac{455}{13}=35$, and $x=79$.

16. A person having laid out a rectangular bowling-green, observed that if each side had been 4 yards longer, the adjacent sides would have been in the ratio of 5 to 4; but if each had been 4 yards shorter, the ratio would have been 4 to 3. What are the lengths of the sides? Let x and $y=$ the sides; $\therefore$

$$\left.\begin{array}{l}\text{hence } x+4 : y+4 :: 5 : 4, \\ \text{and } x-4 : y-4 :: 4 : 3,\end{array}\right\} \begin{array}{l} x+4 : x-y :: 5 : 1, x=36 \\ x-y : x-4 :: 1 : 4, y=28. \end{array}$$

$$\text{ex æquali, } x+4 : x-4 :: 5 : 4, \text{ or } x : 4 :: 9 : 1.$$

17. At an election for two member of parliament, three men offer themselves as candidates, and all the electors give single votes. The numbers of voters for the two successful ones are in the ratio of 9 to 8; and if the first had had 7 more, his majority over the second would have been to the majority of the second over the third as 12 : to 7. Now if the first and third had formed a coalition, and had one more voter, they would each have succeeded by a majority of 7. How many voted for each?

Let $9x$, $8x$, and y be the numbers; then $8x-y : x+7 :: 7 : 12$, and $\frac{1}{2}(9x+y+1)=8x+7$; and from the first, $96x-12y=7x+49$; and from the 2d, $9x+y+1=16x+14$, and $x=\frac{205}{5}=41$; $\therefore y=7x+13=300$, and $8x=328$, $9x=369$.

18. To determine three numbers such that if 6 be added to the first and second, the sums will be in the proportion of 2 : 3; if 5 be added to the first and third, the sums will be in the proportion of 7 : 11; but if 36 be subtracted from the second and third, the remainders will be as 6 : 7.

Let $2x-6$, $3x-6$, and y be the numbers; then $2x-1 : y+5 :: 7 : 11$, and $3x-42 : y-36 :: 6 : 7$. Hence, $22x-11=7y+35$, and $7x+2y=26$, $x=\frac{90}{5}=18$, and $2y=7x-26=100$, or $y=50$.

19. Two shepherds, A and B, are intrusted with the charge of 2 flocks of sheep. A's consisting chiefly of ewes, many of which produced lambs, is at the end of the year increased by 80; but B finds his stock diminished by 20; when their numbers are in the proportion of 8 to 3. Now had A lost 20 of his sheep, and had B an increase of 90, the numbers would have been in the proportion of 7 to 10. What were the numbers?

Let x and y denote the numbers; $\therefore x+80 : y-20 :: 8 : 3$, and $x-20 : y+90 :: 7 : 10$. Hence $y=\frac{6490}{59}=110$, and $x=\frac{480}{3}=160$.

20. Two persons, A and B, can perform a piece of work in 16 days. They work together for 4 days, when A being called off, B is left to finish it, which he does in 36 days more. In what time would each do it separately?

Let x and y be the number of days; then by the question, $\frac{16}{x}+\frac{16}{y}=1$, and $\frac{4}{x}+\frac{4}{y}+\frac{36}{y}=1$, or $(\frac{4}{x}+\frac{40}{y}=1)-(\frac{4}{x}+\frac{4}{y}=\frac{1}{4})$, $=\{36\div y\}=\frac{3}{4}$; and $y=48$, and $x=24$.

21. There is a cistern, into which water is admitted by three cocks, two of which are of exactly the same dimensions. When they are all open, five-twelfths of the cistern is filled in four hours; and if one of the equal cocks be stopped, seven-ninths of the cistern is filled in ten hours and forty minutes. In how many hours would each cock fill the cistern?

Let x and y be the numbers; $\frac{8}{x}+\frac{4}{y}=\frac{5}{12}$ and $\frac{32}{3x}+\frac{32}{3y}=\frac{7}{9}$, $\therefore (\frac{8}{x}+\frac{8}{y}=\frac{7}{12})-(\frac{8}{x}+\frac{4}{y}=\frac{5}{12})=\frac{4}{y}=\frac{1}{6}$, and $y=24$, and $\frac{8}{x}=\frac{1}{4}$, and $x=32$.

22. Some hours after a courier had been sent from A to B, which are 147 miles distant, a second was sent, who wished to overtake him just as he entered B; in order to which he found he must perform the journey in 28 hours less than the first did. Now the time in which the first travels 17 miles added to the time in which the second travels 56 miles is 13 hours and 40 minutes. How many miles does each go per hour?

Let x and y be the number the first and second goes; then $\frac{17}{x}+\frac{56}{y}=\frac{41}{3}=13\frac{2}{3}$ hours, and $\frac{147}{x}-\frac{147}{y}=28$, by the question; then $\frac{8}{7}(\frac{147}{x}-\frac{147}{y}=28)-3(\frac{17}{x}+\frac{56}{y}=\frac{41}{3})=(\frac{168}{x}-\frac{168}{y}=32)$

$-(\frac{51}{x}+\frac{168}{y}=41)=\frac{219}{x}=73$, and $\frac{3}{x}=1$ and $x=3$, and $y=\frac{21}{3}=7$.

23. Two loaded waggons were weighed, and their weights were found to be in the ratio of 4 to 5. Parts of their loads, which were in the proportion of 6 to 7 being taken out, their weights were then found to be in the ratio of 2 to 3; and the sum of their weights was then 10 tons. What were their weights at first?

Let $4x$ and $5x=$ their weight, and $6y$ and $7y=$ the parts taken out; $\therefore$ $4x-6y : 5x-7y :: 2 : 3$. and $9x-13y=10$, $y=2$, $x=4$.

24. A gentleman gave away a certain sum in charity to 14 men and 15 women. Had the sum been less by 12 shillings, and only half the number of men relieved, the rest being divided amongst the women, each woman would have received two shillings more than each man did. But if there had been only 8 women, and the rest had been divided amongst the men, each man would have received twice as much as each woman. How much money was given away?

Let x and y be the number of shillings one man and one woman received; $\therefore$ $14x+15y=$ the whole sum, and $\frac{7x+15y-12}{15}=x+2$, and $\frac{14x+7y}{14}=2y$. Hence $x=21$, and $y=14$.

25. When wheat was 5 shillings a bushel, and rye 3 shillings, a man wanted to fill his sack with a mixture of rye and wheat for the money he had in his purse. If he bought 7 bushels of rye, and laid out the rest of his money in wheat, he would want 2 bushels to fill his sack; but if he bought 6 bushels of wheat, and filled his sack with rye, he would have 6 shillings left. How must he lay out his money, and fill his sack?

Let x and y denote the number of bushels of wheat and rye he must buy; $5x+3y=$ his money; and then by the question, $\frac{5x+3y-21}{5}+7=x+y-2$, $5x+3y-21+35=5x+5y-10$ or $24=2y$ and $30+3(x+y-6)=5x+3y=6$; hence, $y=12$, and $x=9$.

26. A draper bought two pieces of cloth of different kinds for \$744; there were 6 yards of the coarser more than there were of the finer; and had the coarser cost 2 dollars a yard more than it did, 6 yards of the coarser would have cost just as much as 5 yards of the finer. He afterwards bought 4 yards of the finer, and 12 of the coarser at the same prices per yard, and found their value less than that of the former pieces in the ratio of 20 : 31. How many yards did he buy the first time, and what did he give per yard for each?

Let $x=$ the number of yards of the finer, and $y=$ the number of the coarser; $\therefore y=$ the price of a yard of the coarser, and the price of a yard of the finer $=\frac{6(y+2)}{5}$; $\therefore \frac{6x(y+2)}{5}+y(x+6)=744$; and $\frac{24(y+2)}{5}+12y : 744 :: 20 : 31$, or $\frac{24(y+2)}{5}+12y : 24 :: 20 : 1$, or $\frac{2y+4}{5}+y=40$, or $y=\frac{196}{7}=28$, and $x=\frac{576}{64}=9$.

27. A mercer bought two pieces of silk of different lengths for £50; the price of two yards of the shorter was 6*s*. 8*d*. more than the price of 3 yards of the longer; and each piece cost the same sum. He cut off two yards from each, and sold the rest for £53 12*s*. Now if he had sold the whole at that rate, he would have gained £5 by each piece. How many yards did each piece contain?

Let $x=$ the number contained in the shorter, and y that of the larger; then $\frac{25}{x}$ and $\frac{25}{y}=$ the price of a yard of each, and $\frac{50}{x}-\frac{75}{y}=\frac{1}{3}$; also, $\frac{30}{x}(x-2)+\frac{30}{y}(y-2)=53\frac{3}{5}$; $\therefore 60-\frac{60}{x}-\frac{60}{y}=53\frac{3}{5}$; whence $\{(\frac{10}{x}+\frac{10}{y}=\frac{16}{15})-(\frac{10}{x}-\frac{15}{y}=\frac{1}{15})\}=\frac{25}{y}=1$, and $y=25$, and $x=15$.

28. A sets out express from C towards D, and three hours afterwards B sets out from D towards C, travelling 2 miles an hour more than A. When they meet, it appears that the distances they have travelled are in the proportion of 13 to 15; but had A travelled five hours less, and B gone 2 miles an hour more, they would have been in the proportion of 2 : 5. How many miles did each go per hour, and how many hours did they travel before they met?

Let $x=$ the number of miles A went per hour; $\therefore x+2=$ the number B went; let $y=$ the number of hours B travelled; $\therefore y+3=$ the number A travelled; and $x(y+3) : y(x+2) :: 13 : 15$; $\therefore 15xy+45x=13xy+26y$, or $2xy=26y-45x$. Again, $x(y-2) : y(x+4) :: 2 : 5$; $\therefore 5xy-10x=2xy+8y$, or $3xy=10x+8y$; whence, $20x+16y=78y-135x$, or $155x=62y$, and $5x=2y$; $\therefore 3xy=4y+8x=12y$, and $x=4$, $y=10$.

29. A and B engaged to reap a field of corn in 12 days. The times in which they could severally reap an acre are as 2 : 3. After some time, finding themselves unable to finish it in the stipulated time, they called in C to help them; whose rate of working was such, that if he had wrought with them from the beginning, it would have been finished in 9 days. Also the times in which he could severally have reaped the field with A alone, and with B alone, are in the proportion of 7 to 8. When was C called in?

Let $3x=$ A's daily work; $\therefore 2x=$ B's; let $z=$ C's, and $y=$ number of days C worked; $\therefore (3x+2x)12+yz=$ the whole quantity of work done; and $9(3x+2x+z)=$ the same; $\therefore 12(3x+2x)+yz=9(3x+2x+z)$, and $15x+yz=9z$. Now $3x+z : 2x+z :: 8 : 7$, and $5x=z$; $\therefore 3z+yz=9z$, or $y+3=9$; $\therefore y=6$, and C was called in after 6 days.

30. The letters l, m, n representing any given numbers, as for instance, 16, 17 and 18, it is required to find a, b, c.

$$\text{Suppose}\quad \begin{matrix}(1.)\\(2.)\\(3.)\end{matrix}\left\{\begin{matrix}a^2+bc=l=16\\ b^2+ac=m=17\\ c^2+ab=n=18\end{matrix}\right\}\quad \text{What are the numbers } a, b, c\,?$$

Before I proceed to the solution, I shall give Dr. Wallis's account of the problem in his own words, from which I have been induced to think that it may have been intended by the first proposer, Dr. Pell, as a trial of the skill of the author of the arithmetic of infinities, of whose algebra it occupies four whole chapters, viz. 60th, 61st, 62d, and 63d, extending over three and thirty folio pages.

"For a further explanation of Dr. Pell's method, I here subjoin another question, not undetermined as the former; but (in itself determined) proposed to myself, long since, by Col. Silas Titus, (then of his majesty's bed-chamber) a very ingenious person, and well skilled in affairs, civil and military, and very well accomplished in mathematics and other learning.

The process of which (because I understood from the colonel, it was a question proposed by Dr. Pell) I drew up in general terms, (after Dr. Pell's method, with which the colonel was well acquainted,) in this form; as I find it yet amongst my loose papers."

The 62d chapter concludes with the following paragraph:

"And thus I have pursued the problem, through all its cases and varieties, to an absolute and universal solution and determination of them. Which I have done more at large and distinctly, to shew a method how the like may be done in other problems in like manner proposed."

I think it must now appear highly probable that this was a sort of challenge from Dr. Pell to Dr. Wallis, and that the latter considered the problem worthy his best attention, and the solution as fit to be a pattern to succeeding analysts: that he retained this opinion in 1693 is a strong presumtion that his cotemporaries had given him no reason for changing it, yet this solution has been pronounced "remarkably operose and inelegant," and so it certainly is, but for patient labor, exactness and completeness almost beyond example or imitation.

By transposing a^2 in the first, and b^2 in the second equation, we obtain (4.) $bc=l-a^2$,

and (5.) $ac=m-b^2$, then by the multiplication of equals, there results (6.) $abc^2=m(l-a^2)-(l-a^2)b^2$,
and by transposing ab in the third equation, (7.) $n-ab=c^2$, whence by multiplying the two left and the two right-hand members of (6.) and (7.), and omitting the common factor c^2, we shall obtain the following quadratic in b, viz. $nab-a^2b^2=m(l-a^2)-(l-a^2)b^2$, which arranged according to the powers of b becomes (8.) $(l-2a^2)b^2+nab=m(l-a^2)$ Another equation in b and a without c will enable us to solve the problem, by finding an equation in a, and the given quantities; this we proceed to find as follows: We have seen that $bc=l-a^2$, and that $m-b^2=ac$, now multiply their left, and also their right-hand members together, and omit the common factor c, there results, for a second equation in a and b, the following cubic, viz.

(9.) $mb-b^3=(l-a^2)a$.

But by (8.) $m(l-a^2)=(l-2a^2)b^2+nab$, therefore, by multiplying their left, and also their right-hand members, and omitting the common factor $(l-a^2)b$ there will result $m^2-mb^2=a(l-2a^2)b+na^2$, and by transposition, we obtain

(10.) $m^2-na^2=mb^2+a(l-2a^2)b$.

But by (8.) $(l-2a^2)b^2+nab=m(l-a^2)$ therefore multiplying as before, and omitting the common factor b,
$(m^2-na^2)(l-2a^2)b+na(m^2-na^2)=m^2(l-a^2)b+m(l-a^2)(l-2a^2)a$;
by transposition, $\{(m^2-na^2)(l-2a^2)-m^2(l-a^2)\}b=$
$\{m(l-a^2)(l-2a^2)-n(m^2-na^2)\}a$.
Whence, by division, we obtain the following value of b, viz.

$$(11.)\ b=\frac{a\{m(l-a^2)(l-2a^2)-n(m^2-na^2)\}}{(l-2a^2)(m^2-na^2)-m^2(l-a^2)}$$
$$=\frac{2ma^4+(n^2-3lm)a^2+m(ll-mn)}{a(2na^2-ln-m^2)}.$$

Again, by multiplying (10.) by $l-2a^2$, and (8.) by m, we obtain
$m(l-2a^2)b^2+a(l-2a^2)^2b=(l-2a^2)(m^2-na^2)$
and $m(l-2a^2)b^2+mnab=m^2(l-a^2)$, whence, by subtraction,
$\{(l-2a^2)^2-mn\}ab=(l-2a^2)(m^2-na^2)-m^2(l-a^2)$.
Whence, by division, b is found, viz. (12.) $b=$
$$\frac{(l-2a^2)(m^2-na^2)-m^2(l-a^2)}{a\{(l-2a^2)^2-mn\}}=\frac{a(2na^2-ln-m^2)}{4a^4-4la^2+ll-mn}.$$

Now by equating the values of b in (11.) and (12.) we obtain the following final equation in a, viz.

(13.) $8a^8-20la^6+2(9ll-mn)a^4+(5lmn-7lll-m^3-n^2)a^2+(ll-mn)^2=0$; if this equation be multiplied by 2, and then e be substituted for $2aa$, there will result

(14.) $e^3 5le^6+(9ll-mn)e^4+(5l-mn-7l^2-m^3-n^2)e^2+2(ll-mn)^2=0$,

which coincides with that found by Dr. Wallis, Algebra, pp. 230, edit. 1685, also pp. 250, lat. edit. 1693.

Now then e^2 being found from (14.) $a=\sqrt{(\frac{1}{2}e^2)}$; then b is found by (11.) or (12.), and lastly $c=\frac{l-a^2}{e}$.

When 16, 17, 18 are substituted for l, m, n in the final equation, it becomes $e^8-80e^6+1998e^4-14937e^2+5000=0$, and hence, Wallis finds the approximate value of e^2 to be 12.7564,4179,4460,744; or 0·350,987,046; or 34·832,280,28; or 32·060,290,88; and the corresponding values of a, b, c are

$a = 2{\cdot}525,513,986,744,158$	$a = 0{\cdot}418,919,470$
$b = 2{\cdot}969,152,768,619,848$	$b = 3{\cdot}912,226,866$
$c = 3{\cdot}240,580,681,617,074$	$c = 4{\cdot}044,884,670$
$a = +\ 4{\cdot}173,264,926$	$a = +\ 4{\cdot}003,766,407$
$b = +\ 4{\cdot}287,022,553$	$b = -\ 0{\cdot}007,099,744$
$c = -\ 0{\cdot}330,331,815$	$c = +\ 4{\cdot}245,989,3$

And he observes that if the signs of every one of any set be changed, they will solve the problem, because the signs of the planes aa, bb, cc, ab, ac, cb are not changed.

As it was not generally known, when the Doctor solved this problem, that b and c might be expressed by rational functions of a, he determines b from a quadratic equation in a, and l, m, n, as also from a cubic, and is at pains to show that only one root of the quadratic will serve, and that the cubic furnishes none serviceable but this same; c he finds from the equation $c=\frac{l-a^2}{b}$. The quadratic the Doctor found is less simple than mine; a ascending in it to the 4th power: the quadratic and cubic resulting for the extermination of b after c has been expelled are the same in Kirkby, as in this of mine; but his solution is complicated with the unnecessary factors a^2 and $l-a^2$ as that of Wallis is by the square of this last; and I think they both seem to have imagined these factors somehow connected with the general problem, though it seems evident, from the solution here given, that they are not in the least. I have only to add that Dr. Wallis's solution is, in my opinion, as curious a morsel as can easily be found, as from its copiousness and minuteness, it enables us to ascertain very exactly the analytical skill of one who made so great a figure in the world of science, during the latter half of the seventeenth century, being along with Lord Brounker, the unfoiled champion of England in the mathematical contests with M. M. Frenicle, Pascal, and Fermat, and in the doctrine of series and quadrature of curves, the precursor of Newton.

Given $x^4-80x^3+1998x^2-14937x=-5000$ to find x.

—80	1998	—14937	—5000(12.75644179448074
+12	— 816	14184	—9036 Answer.
—68	1182	—753 Div.	4036 See rule page 197.
12	— 672	6120	3737.3441
—56	510	5367 Tr.	298.6559
12	— 528	— 27.937	264.59730625
—44	— 18	5339.063	34.05859475
12	—21.91	42.931	31.723000237296
—32	—39.91	5296.132	2.335593512704
.7	—21.42	—4.185875	2.114644002206
—31.3	—61.33	5291.946125	220949510498
.7	—20.93	— 4.258625	211462866009
—30.6	—82.26	5287.687500	9486644489
.7	—1.4575	— .520793784	5286568076
—29.9	—83.7175	5287.166706216	4200076413
.7	— 1.4550	—.521837352	3700597549
—29.2	—85.1725	5286.644868864	499478864
.05	— 1.4525	— 34863350	475791108
—29.15	—86.6250	5286.61000551,4	23687756
5	— 173964	— 3486798	21146271
—29.10	—86.798964	5286.57513753	2541485
5	— 173928	— 348731	2114627
—29.05	—86.972892	5289.5716502,2	426858
5	— 173892	— 34874	422925
—29.00	—87.146784	5286.5681628	3933
.006	— 1159	— 872	3701
—28.994	—87.15837,4	5286.568075,6	232
6	— 115	— 872	211
—28.988	—87.1699,6	5286.567988	21
.6	116	— 61	21
—28.982	—87.1816	5286.56792,7	
6	— 12	— 6	
—28.976	—87.182,8	5286.56787	
	—87.184		

1. Given $x^2+1728\frac{1}{3}x=123578$, to find the value of x. Here the value of $x=68.7652857$.
2. Given $x^2-\frac{137}{364}x=\frac{2557}{59825}$, to find the value of x.*
3. Given $x^2+\frac{7}{24}x=\frac{1695}{12716}$, to find the value of x.

* Ans. $x=.44464894$. Ans. $x=.29411765$.

On Quadratic Equations.

65. Quadratic Equations are divided into pure and adfected. Pure quadratic equations are those which contain only the square of the unknown quantity; such as $x^2=36$; $x^2+5=54$; $ax^2-b=c$; &c. Adfected quadratic equations are those which involve both the square and simple power of the unknown quantity, such as $x^2+4x=45$; $3x^2-2x=21$; $ax^2+2bx=c+d$; &c. &c.

On the Solution of Pure Quadratic Equations.

66. The Rule for the solution of pure quadratic equations is this: "Transpose the terms of the equation in such a manner, that those which contain x^2 may be on one side of the equation, and the known quantities on the other; divide (if necessary) by the coefficient of x^2; then extract the square root of each side of the equation, and it will give the value of x."

Let $x^2+5=54$. By transposition, $x^2=54-5=49$.

Extract the square root of both sides of the equation, then

$$x=\sqrt{49}=7.$$

Let $3x^2-4=71$. By transposition, $3x^2=71+4=75$. Divide by 3, $x^2=\frac{75}{3}=25$. Extract the square root, $x=\sqrt{25}=5$.

On the Solution of Adfected Quadratic Equations.

67. The most general form under which an adfected quadratic equation can be exhibited is $ax^2+bx=c$; where a, b, c, may be any quantities whatever, positive or negative, integral or fractional. Divide each side of this equation by a, then $x^2+\frac{b}{a}x=\frac{c}{a}$. Let $\frac{b}{a}=p$, $\frac{c}{a}=q$; then this equation is reduced to the form $x^2+px=q$, where p and q may be any quantities whatever, positive or negative, integral or fractional.

68. From the two-fold form under which adfected quadratic equations may be expressed, there arise two Rules for their solution.

69. Let $\pm ax^2 \pm bx \pm c = \pm d$ be an adfected quadratic equation, then, by transposing and dividing by $\pm a$, it becomes $x^2 \pm \frac{b}{a}x=\frac{\pm d \mp c}{\pm a}$; or, by putting p for $\frac{b}{a}$, and q for $\frac{d\mp c}{\pm a}$, it is $x^2 \pm px = \pm q$.

Add now the square of $\frac{1}{2}p$ to each side of this equation, and there results $x^2 \pm px + \frac{1}{4}p^2 = \pm q + \frac{1}{4}p^2$, where it is readily perceived that the first side is a complete square, viz. $(x \pm \frac{1}{2}p)^2$; consequently, if

the square root of each side be extracted, we obtain $x \pm \frac{1}{2}p = \pm \sqrt{(\pm q + \frac{1}{4}p^2)}$, the double sign $\pm$ being placed before the radical, because the square root of a quantity may be either $+$ or $-$; hence it appears that

$$x = +\sqrt{(\pm q + \tfrac{1}{4}p^2)} \mp \tfrac{1}{2}p, \text{ or } -\sqrt{(\pm q + \tfrac{1}{4}p^2)} \mp \tfrac{1}{2}p.$$

70. The above general values of x evidently include every possible case, from which separate formulæ for each distinct case are readily obtained, and are as follow:

In equations of the form,

Case 1. $x^2 + px = q$, $x = \begin{cases} +\sqrt{(q+\frac{1}{4}p^2)} - \frac{1}{2}p, \\ \text{or } -\sqrt{(q+\frac{1}{4}p^2)} - \frac{1}{2}p, \end{cases}$

Case 2. $x^2 - px = q$, $x = \begin{cases} +\sqrt{(q+\frac{1}{4}p^2)} + \frac{1}{2}p, \\ \text{or } -\sqrt{(q+\frac{1}{4}p^2)} + \frac{1}{2}p, \end{cases}$

Case 3. $x^2 + px = -q$, $x = \begin{cases} +\sqrt{(-q+\frac{1}{4}p^2)} - \frac{1}{2}p, \\ \text{or } -\sqrt{(-q+\frac{1}{4}p^2)} - \frac{1}{2}p, \end{cases}$

Case 4. $x^2 - px = -q$, $x = \begin{cases} +\sqrt{(-q+\frac{1}{4}p^2)} + \frac{1}{2}p, \\ \text{or } -\sqrt{(-q+\frac{1}{4}p^2)} + \frac{1}{2}p. \end{cases}$

In the third and fourth forms, when $(\frac{p}{2})^2$, or $\frac{p^2}{4}$ is less than $-q$, the root will be impossible, (the square root of a negative quantity being impossible, viz. $\sqrt{(-q + \frac{1}{4}p)}$) because in the solution of problems, if the expression for the root happens to be the square root of a negative quantity, it will show that the conditions, in such equations, are inconsistent. Every quadratic equaton has 2 equal or 2 unequal roots, either real or imaginary. Sometimes one root is *affirmative* and the other *negative*, and sometimes they are both *affirmative*, or both *negative*.

Rule I. Hence it appears, that if to each side of the equation be added the square of half the coefficient of the second term, there will arise, on the left-hand side of the equation, a quantity which is the square of $x + \frac{1}{2}p$; and by extracting the square root of each side of the resulting equation, we obtain a simple equation, from which the value of x may be determined.

Rule. II. *Hindoo Method of Solving Quadratics.*

71. Let the equation $ax^2 \pm bx = c$ be multiplied by $4a$, then $4a^2x^2 \pm 4abx = 4ac$, and if b^2 be added to each side, the equation becomes $4a^2x^2 \pm 4abx + b^2 = 4ac + b^2$; now the first side is evidently a square, $= (2ax \pm b)^2$, whence

$$2ax \pm b = \pm\sqrt{(4ac + b^2)}, \therefore x = \frac{\pm\sqrt{(4ac + b^2)} \mp b}{2a}.$$

From which we infer, that if each side of the equation be multiplied by four times the coefficient of x^2, and to each side there be added the square of the coefficient of x, the quantity on the left-hand side of the equation will be the square of $2ax + b$. Extract

the square root of each side of the equation, and there arises a simple equation, from which the value of x may be determined.

If $a=1$, the equation is reduced to the form $x^2+px=q$; in this case, therefore, the Rule may be applied, by multiplying each side of the equation by 4, and adding the square of the coefficient of x.

Rule Third, from Emerson's Algebra, (1764.)

When you have large numbers to deal with, it is better to proceed thus. Clear the equation, and if $a^2+ba=d$, then $a=\frac{d}{b+a}$, the form. To find the first quotient figure, take $\frac{d}{b}$, when b is far greater than a; or take $\sqrt{d}$, when a is far greater than b; or take $\frac{d}{2b}$ when a and b are nearly equal; thus it will easily be found by a few trials. Or, in general, take the first figure such, that when it is multiplied by the sum of itself and b, it will produce the first figure or figures of d, or the next less: this is all the difficulty. Then multiply and subtract as usual; the remainder is the resolvend.

Then, to continue the division, you must find a new divisor, for each quotient figure, thus. Add the last quotient figure to the last divisor, (duly observing their places,) for a new divisor; see how oft this is contained in the resolvend, set the answer in the quotient, and also add it to the divisor; then multiply the whole divisor by that quotient figure, and subtract the product for a new resolvend. But when any of the signs are negative, the proper quantities are to be subtracted, instead of being added. This work is always to be repeated for each quotient figure.

When any quotient figure is so great that the product exceeds the resolvend, place a less figure in the quotient.

When you have got more than half your intended number of figures in the quotient, you may continue the division, without adding the new quotient figures to the divisor.

Observe, each quotient figure is to be added twice to the divisor, once before multiplication, and once after; just as in extracting the square root, and for the same reason. For this method extracts the square root when $b=0$.

When one root is had, the other is found by adding this to the coefficient b; for the sum, changing its sign, is the other root.

This rule is the foundation of the method for extracting the roots of adfected equations.

Scholium. If $x^4+bx^2=d$. Put $a=x^2$, then $a^2+ba=d$; and find a as above. Then $x=\sqrt{a}$, by extracting the root. And the same for higher equations.

Let $a^2+32a=4644$, to find a.

then $a=\frac{4644}{32+a}$. Suppose $\frac{4600}{32}=100$ too great for a. $\sqrt{4644}=60$, which is also too great for a. Take $a=\frac{4600}{64}=7$, too great. Take $a=50$.

32
+50
82) 4644 (50
+ 54 410·
136) 544 (4
544 54 = a.

2. Let $a^2+35a=28349994$

$a=\frac{28349994}{35+a}$

Here $a=\sqrt{28}$, &c. $=5000$
+35
5000
5035 } 28349994(5307 = a
5300 } 25175...
10335 } 31749
307 } 31005
10642) 74494
74494

3. Suppose $a^2-5307a=-184520$, to find a.

then $a=\frac{-184520}{-5307+a}=\frac{184520}{5307-a}$.

Here $a=\frac{184}{6}=30$ nearly.

5307)184520(35 = a.
−30 15831·
5277) 26210
−35 26210
5242) ··

4. Let $x^2+463x=26698$, or $x=\frac{26698}{463+a}$.

c r P N
$r+C$ $=463+50=513$) 26698 (51.855342 $=x$
$r+s=+51$ 2565
$r+P+s=564$ 1048 $=N'$
$s+t=+1.8$ 564
$r+P+2s+t=565.8$ 484.0000 $=N''$
$t+u=+.85$ 452.64
$r+P+2s+2t+u=566.65$ 31.3600 $=N'''$
5 28.3325
5|6|6.|70 3.0275
2.8335
1940
1700
240
226
14−11=3 remain

4. Given $364x^2 - 127x = 15.49699958$, to find x, by Emerson's method.

—127	+15.4969(—.0956747
—32.76	+14.3784
—159.76 D	1.11859958
—32.76	.971700
—192.52	.14689958
—1.820	.13749036
—194.340 D	940922
—1.820	786737
—196.160	154185
—2548	137691
—196.4148 D	16496
2548	15736
—196.669|6	758
—146	580
—196.684|2 D	178
—146	177
—196.69|9	1
3	
—19|6.|7|0|2	

839 Root.

5. Given $102x^2 - 101x = -16.27843657$, to find the value of x.

	—101	—16.27843657
	+ 71.4	—20.72
Divisor	—29.6	+ 4.45
	+ 71.4	3.9968
	41.8	.444763
	8.16	411838
Divisor	49.96	3292543
	8.16	2979950
	58.12	312593
	714	298280
Divisor	58.834	14313
	714	11932
	59.548	2381
	51	1790
Divisor	59.599	591
	5	537
Divisor	59.656	54
Divisor	59.|6|6|1	54

Ans. (.787552399

See page 220.

Rule 4. When an equation assumes the form of $x^3 + cx = ab$, a solution may be effected by quadratics as often as $b^2 + c = a$; for multiplying both sides by x, there is produced $x^4 + cx^2 = abx$, $= a \times bx$; and by adding b^2x^2 to these equals, $x^4 + (b^2 + c)x^2 = b^2x^2 + abx$; therefore, by page 141, art. 70, I have $x^4 + (b^2 + c)x^2 + \frac{1}{4}(b^2 + c)^2 = b^2x^2 + abx + \frac{1}{4}(b^2 + c)^2$, that is, $x^4 + ax^2 + \frac{1}{4}a^2 = (bx)^2 + a(bx) + \frac{1}{4}a^2$; and by extracting the root, $x^2 + \frac{1}{2}a = bx + \frac{1}{2}a$, or $x^2 = bx$, and $x = b$.

Given $x^3 + 3x = 140$, to find x. Here 140 is a composite number, and if it be resolved into the factors 28×5, then $a = 28$, $b = 5$, and $c = 3$, $b^2 + c$ being equal to a, therefore $x = 5$.

Rule 5. When an equation assumes the form of $x^{4m} - 2x^{3m} + x^m = a$, a solution may be effected by quadratics, in this manner: add x^{2m} to both sides of the equation, and transpose x^m; then $x^{4m} - 2x^{3m} + x^{2m} = x^{2m} - x^m + a$; that is, $(x^{2m} - x^m)^2 - (x^{2m} - x^m) = a$; by art. 70, case 1, I have $(x^{2m} - x^m) - \frac{1}{2} = \sqrt{(a + \frac{1}{4})}$, or $x^{2m} - x^m = \sqrt{(a + \frac{1}{4})} + \frac{1}{2}$, a quadratic equation.

Rule 6. When the sum of two quantities, and the sum of their first or second powers, reciprocally divided by their first powers,

are given, substitute, as in the last case, $m+n$ for the greater, and $m-n$ for the less. Now the same method applies, when the difference of two quantities, and the difference of their second powers, reciprocally divided by their first, are given.

Rule 7. When the product of two quantities, and the sum of their squares respectively, constitute the unknown part of the equations, add twice the equation containing the product to the equation containing the squares; and subtract twice the equation containing the product from the equation containing the squares; so shall the square root of the first result give the sum of the quantities sought, and the square root of the second result their difference. Now any examples in the preceding pages may be solved, by substituting $m+n$ for the greater, and $m-n$ for the less of the unknown values; for if double the equivalent of the product of the unknown quantities, in terms of m and n, be added to the equivalent of the sum of the squares, in the same terms, the value of $4m^2$ will be obtained, and if subtracted, the value of $4n^2$; whence m and n are easily determinable. See Index.

Rule 8. When two unknown quantities, x and y, are proposed, such that the two equations containing them cannot, by any method yet given, be freed from the product and the square of one or both the quantities sought, or from the two squares, assume $vx=y$, and substitute this value of y for y; then determine the value of v, and thence that of x. Lastly, substitute known values in the equation $vx=y$, and thus make y known.

Rule 9. When the value of the product of two quantities and of their sum or difference are given, if the sides of the equation containing the sum or the difference be squared, and if four times the equation containing the product be ADDED to, or SUBTRACTED from the result, (that is, added when the difference is squared, and the contrary for the sum); there will be obtained an equation in which, the square root being taken on both sides, the value of the *difference* of the unknown quantities will appear, where their sum was given; and of their *sum*, where their difference was known.

Rule 10. When the sum or difference of two quantities, and the sum or difference of their squares or cubes, are given, substitute $(m+n)$ for the greater and $(m-n)$ for the less. Then involve $(m+n)$ and $(m-n)$ respectively to the second or third power, as the case may be; and by addition or subtraction form an equation in terms of m and n. Now since the sum or *difference* of x and y is given, the value of m or n is known; substitute therefore the known value in the formed equation, and determine the value which is unknown. But m and n being known, the problem or question is solved.

72. *A new method of resolving Equations of all orders, however complicated.*

A universal method, which may be applied to equations of all dimensions, was first given by W. Emerson, in his Algebra in 1764, first edition, with a new demonstration, thus: let $x^2+ax=n$ be the general form of a quadratic equation, let r denote the first figure of the root, and y the remaining part of the root; that is, if the root be integral, then r denotes the left integer, with as many ciphers as figures in the root. If the root consist of an integer and decimals, then r will denote the integer, and y the decimal part; then we shall have $x=r+y$, or $(y+r)$, and then, by substituting r for x in the equation, we have $r^2+ar=n$. Hence, $\frac{n}{r+a}$ will give r, or the first figure of the root. To find the remaining figures, substitute $r+y$ for x in the given equation; 2dly, Then $\left.\begin{array}{l} y^2+2ry+r^2 \\ \quad ay+ar \end{array}\right\}=n$

$\overline{y^2+a'y+p=n}$, or $y^2+a'y=n-p=n'$. From this equation, which is to the first y, can be found $y=s+z$, or $(z+3)$; then by substitution, 2dly,

$\left.\begin{array}{l} z^2+2^s+s^2 \\ \quad s^2+as \end{array}\right\}=n.$

$\overline{z^2+a'z+p}=n$, or $z^2+a''z=n'-p=n''$. Also, $\frac{n'}{s+a}=s'$

From this equation z can be determined. Let $z=t+q$; then $\frac{n'}{t+a}=t$. In the same manner, we find $\frac{m'''}{u+a''}=u$, &c.

Now, the value a', a'', a''', &c. can be easily determined, and hence we have the following expressions for s, t, u, &c. or the remaining figures of the root, thus

$$\frac{n}{r+a}=r;\ \frac{n'}{s+a'},=\frac{n'}{s+2r+a}=s;\ \frac{n''}{t+a''}=\frac{n''}{t+2s+2r+a}=t;$$

$$\text{and } \frac{n'''}{u+a'''}=\frac{n''}{u+2t+2s+2r+a}=u, \text{ \&c.}$$

I shall close the subject of quadratic equations by one observations. Consequently this method appears to be a generalization of the extraction of the square root, if we suppose $a=0$, then $x^2=n$, or $x=\sqrt{n}$, and the formulas become

$$\frac{n}{r}=r,\ \frac{n'}{s+2r}=s,\ \frac{n''}{t+2s+2r}=t, \text{ and } \frac{n'''}{u+2t+2s+2r}=u,$$

This is a demonstration of the rule usually given for the extraction of the square root of numbers.

Given $x^4+x^3-5x=25$, to find the value of x true to about 10 figures, and also $x^3-x^2-2x=-1$, to find x.

				—1	
1	0	—5	25(2.236067977	—2 —1(1·80193	
2	6	12	14	+0 —2	794
3	6	7	11	—2 1	
2	10	32	9.0736	+1 .992	
5	16	39	1.9264	2.8 ... 2.24	8
2	14	6.368	1.59570141	1.24	4124401
7	30	45.368	33069859	64	3875599
2	1.84	6.744	32697893	4.401 4401	3719490
9.2	31.84	52.112	371966	4.124401	155109
2	1.88	1.078047	328311	1	124106
9.4	33.72	53.190047	43654	4.40|39....3964	31003
2	1.92	1.086921	38305	4.132767	28959
9.6	35.64	54.276968	4350	4|.4|0573 ...132	2044
2	2949	219521	4925	4.1368|63	1655
9.83	35.9349	54.49648(9	425	31	389
3	2958	21988	383	4|.1|3|7(0	372
9.86	36.2307	54.716|37	42		17
3	2967	2|20	38		16
9.89	36.5274	54.718|57			
3	595	2|2			
|9.9|26	36.586|9	5|4|.7|2|1			
	60|				
	36.64|6				
	36|70|			

See Rules that follow page 195.

All the possible forms of Cubic Equations exhibited at one view.

1. Given $x^3+ax^2=r$, or $x^3+x^2=500$.
2. Given $x^3-ax^2=r$, or $x^3-3x^2=5$.
3. Given $x^3-ax^2=-r$, or $x^3-48x^2=-200$.
4. Given $x^3+ax=r$, or $x^3+9x=6$.
5. Given $x^3-ax=r$, or $x^3-27x=36$.
6. Given $x^3-ax=-r$, or $x^3-12x=-12$.
7. Given $x^3+ax^2+bx=r$, or $x^3+5x^2+29x=1829$.
8. Given $x^3+ax^2-bx=r$, or $x^3+2x^2-3x=9$.
9. Given $x^3-ax^2+bx=r$, or $x^3-39,6x^2+585,6x=2937,6$.
10. Given $x^3-ax^2-bx=*r$, or $x^3-120x^2-300x=8487$.
11. Given $x^3-ax^2-bx=-r$, or $x^3-x^2-2x=-1$.
12. Given $x^3-ax^2+bx=-r$, or $x^3-5x^2+2x=-12$.
13. Given $x^3+ax^2-bx=-r$, or $x^3+2x^2-23x=-70$.

It may be observed, that all equations may be solved as quadratics, by completing the squares, in which there are two terms involving the unknown quantity or any function of it, and the index of one is double that of the other. Thus,

$$x^6 + px^3 = q,\ x^{2n} - px^n = q,\ x^{\frac{n}{2}} + x^{\frac{n}{4}} = a,\ a^2x^2 + ax = b,$$
$$x^{3n} + ax^{\frac{3n}{2}} = b,\ p^2x^{4n} - px^{2n} = d,\ \overline{x^2 + px + q}|^2$$
$$+ (x^2 + px + q) = r,\ x^2 \cdot (x^2 + ax)^2 + bx \cdot (x^2 + ax) = d,$$

are of the same form as quadratics, and the value of the unknown quantity may be determined in the same manner. Many equations also, in which more than one unknown quantity are involved, may in a similar manner be reduced to lower dimensions by completing the square, as $x^2y^2 + pxy = q$, $(x^3+y^3)^2+p\cdot(x^3+y^3)=r$.

Instances of this kind occur in the following.

Thus, if there be taken any general equation of the abovementioned form, $x^{2m}+ax^m=b$, we shall have, by first finding the square root of x^{2m} in terms of the rest, according to the common rule, and then taking the mth rule of the result.

$$x^m=-\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}a^2+b)},\ \text{and}\ x=\{-\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}a^2+b)}\}^{\frac{1}{m}}.$$

And if the equation, which is proposed to be resolved, be of the following form, $x^m-ax^{\frac{m}{2}}=b$, we shall have, according to the same principle,

$$x^{\frac{m}{2}}=\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}a^2+b)},\ \text{and}\ x=\{\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}b^2-b)}\}^{\frac{2}{m}}.$$

1. Given $x^2+4x=140$, to find the value of x.

Here $x^2 + 4x = 140$, by the question, whence $x=-2\pm\sqrt{(4+140)}$, by the rule, or, which is the same, $x=-2\pm\sqrt{144}$, Therefore, $x=-2+12=10$, or $-2-12=-14$; from which solution it appears that one of the values of x is positive and the other negative.

2. Given $x^2-12x+30=3$, to find the value of x.

Here $x^2-12x=3-30=-27$, by transposition, whence $x=6\pm\sqrt{(36-27)}$, by the rule, or, which is the same thing, $x=6\pm\sqrt{9}$, therefore, $x=6+3=9$, or $\pm6-3=3$, where it appears that x has two positive values.

3. Given $2x^2+8x-20=70$, to find the value of x.

Here $2x^2+8x=70+20=90$, by transposition, ad $x^2+4x=45$, by dividing each side by 2, whence $x=-2\pm\sqrt{(4+45)}$; by the rule, or, which is the same thing, $x=-2\pm\sqrt{49}$ therefoe $x=-2+7=5$, or $=-2-7=-9$, where one of the values of x is positive, and the other negative.

4. Given $3x^2-3x+6=5\tfrac{1}{3}$, to find the value of x.

Here $3x^2-3x=5\frac{1}{3}-6=-\frac{2}{3}$, by transposition, and $x^2-x=-\frac{2}{9}$, by dividing each side by 3, whence $x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}-\frac{2}{9})}$, by the rule, or, by subtracting $\frac{2}{9}$ from $\frac{1}{4}$, $x=\frac{1}{2}\pm\sqrt{\frac{1}{36}}$; therefore $x=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$, or $=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$, in which case x has two positive values.

5. Given $\frac{1}{2}x^2-\frac{1}{3}x+20\frac{1}{2}=42\frac{2}{3}$, to find the value of x.

Here $\frac{1}{2}x^2-\frac{1}{3}x=42\frac{2}{3}-20\frac{1}{2}=22\frac{1}{6}$ by transposition, and $x^2-\frac{2}{3}x=44\frac{1}{3}$, by dividing by $\frac{1}{2}$, or multiplying by 2, whence we have $x=\frac{1}{3}\pm\sqrt{(\frac{1}{9}+44\frac{1}{3})}$, by case 2. or, by adding $\frac{1}{9}$ and $44\frac{1}{3}$ together, $x=\frac{1}{3}\pm\sqrt{\frac{400}{9}}$; therefore $x=\frac{1}{3}+6\frac{2}{3}=7$, or $=\frac{1}{3}-6\frac{2}{3}=-6\frac{1}{3}$, where one value of x is positive, and the other negative.

6. Given $ax^2+bx=c$, to find the value of x.

Here $x^2+\frac{b}{a}x=\frac{c}{a}$ by dividing each side by a. Whence, by the rule, $x=-\frac{b}{2a}\pm\sqrt{\left\{\frac{b^2}{4a^2}+\frac{c}{a}\right\}}$, or, multiplying c and a by $4a$, $x=-\frac{b}{2a}\pm\sqrt{\frac{b^2+4ac}{4a^2}}$, and, by $x=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{(b^2+4ac)}$.

7. Given $ax^2-bx+c=d$, to find the value of x.

Here $ax^2-bx=d-c$, by transposition, and $x^2-\frac{b}{a}x=\frac{d-c}{a}$, by dividing each side by a, whence $x=\frac{b}{2a}\pm\sqrt{\left\{\frac{d-c}{a}+\frac{b^2}{4a^2}\right\}}$, by the rule, or multiplying $d-c$ and a by $4a$, $x=\frac{b}{2a}\pm\frac{1}{2a}\sqrt{\{4a(d-c)+b^2\}}$.

8. Given $x^4+ax^2=b$, to find the value of x.

Here $x^4+ax^2=b$, by the question, or $x^2=-\frac{1}{2}a\pm\sqrt{\left\{\frac{a^2}{4}+b\right\}}$ $=-(\frac{1}{2}a\pm\frac{1}{2}\sqrt{(a^2+4b)}$, by the rule, whence $x=\pm\sqrt{\{-\frac{1}{2}a\pm\frac{1}{2}\sqrt{(4b+a^2)}\}}$, by extraction of roots.

9. Given $\frac{1}{2}x^6-\frac{1}{4}x^3=-\frac{1}{32}$, to find the value of x.

Here $\frac{1}{2}x^6-\frac{1}{4}x^3=-\frac{1}{32}$, by the question, and $x^6-\frac{1}{2}x^3=-\frac{1}{16}$, by multiplying by 2, whence $x^3=\frac{1}{4}\pm\sqrt{(\frac{1}{16}-\frac{1}{16})}=\frac{1}{4}$, by the rule, and consequently, $x=\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{2}{8}}=\frac{1}{2}\sqrt[3]{2}$.

10. Given $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$, to find the value of x.

Here $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$, by the question, and $x^{\frac{2}{3}}+\frac{3}{2}x^{\frac{1}{3}}=1$, by dividing each side by 2, whence $x^{\frac{1}{3}}=-\frac{3}{4}\pm\sqrt{(\frac{9}{16}+1)}=-\frac{3}{4}\pm\frac{5}{4}=\frac{1}{2}$, or -2, therefore $x=(\frac{1}{2})^3=\frac{1}{8}$, or, $(-2)^3=-8$.

11. Given $x^4-12x^3+44x^2-48x=9009(a)$, to find the value of x.

This equation may be expressed as follows: $(x^2-6x)^2+8(x^2-6x)=a$, whence $x^2-6x=-4\pm\sqrt{(16+a)}$, by the common rule; and, by a second operation, $x=3\pm\sqrt{\{9-4\pm\sqrt{(16+a)}\}}$; therefore, by restoring the value of a, we have

$$x=3\pm\sqrt{(5\pm\sqrt{9025})}$$

Or, by extraction of roots, $x=13$, the answer.

12. Given $x^2-8x+10=19$, to find the value of x.

By transposition, $x^2-8x=9$, by completing the square, and extracting the root by case 2d, we have $x=5+4=9$, Ans.

13. Given $x^2-x-40=170$, to find the value of x.

By transposition, $x^2-x=210$, whence $x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+210)}=\frac{1}{2}\pm\sqrt{\frac{841}{4}}$; therefore, $x=\frac{1}{2}\pm\frac{29}{2}=15$, or -14.

14. Given $3x^2+2x-9=76$, to find the value of x.

By transposition, $3x^2+2x=85$, or $x^2+\frac{2}{3}x=\frac{85}{3}$; where $x=\frac{-1}{3}\pm\sqrt{(\frac{1}{9}+\frac{85}{3})}=\frac{-1}{3}\pm\sqrt{\frac{256}{9}}$; therefore $x=\frac{-1}{3}\pm\frac{16}{3}=5$, or $-\frac{17}{3}$.

15. Given $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8}=8$, to find the value of x.

By transposition, $\frac{1}{2}x^2-\frac{1}{3}x=\frac{5}{8}$, or $x^2-\frac{2}{3}x=\frac{10}{8}$, where $x=\frac{1}{3}\pm\sqrt{(\frac{1}{9}+\frac{10}{8})}=\frac{1}{3}\pm\sqrt{\frac{49}{36}}$; therefore, $x=\frac{1}{3}\pm\frac{7}{6}=1\frac{1}{2}$, or $-\frac{5}{6}$.

16. Given $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$, to find the value of x.

Multiply by 2; $x-\frac{2}{3}\sqrt{x}=44\frac{1}{3}$; where $\sqrt{x}=\frac{1}{3}\pm\sqrt{(\frac{1}{9}+\frac{133}{3})}=\frac{1}{3}\pm\sqrt{(\frac{400}{9})}$; therefore $\sqrt{x}=(\frac{1}{3}\pm\frac{20}{3})=7$, or $\frac{-19}{6}$, consequently, $x=7^2=49$, or $(\frac{-19}{6})^2=\frac{361}{36}$.

17. Given $x+\sqrt{(5x+10)}=8$, to find the value of x.

By transposing $\sqrt{(5x+10)}=8-x$, by squaring $5x+10=64-16x+x^2$, where $x^2-21x=-54$; therefore $x=\frac{21}{2}\pm\sqrt{(\frac{441}{4}-54)}=\frac{21}{2}\pm\sqrt{\frac{225}{4}}$; that is, $x=\frac{21}{2}\pm\frac{15}{2}=18$,*, or 3, the answer.

18. Given $(10+x)^{\frac{1}{2}}-(10+x)^{\frac{1}{4}}=2$, to find the value of x.

Here, since the first index is double the second, the equation is a quadratic; therefore, by the rule, $(10+x)^{\frac{1}{4}}=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+2)}=\frac{1}{2}\pm\sqrt{\frac{9}{4}}=2$, whence $(10+x)^{\frac{1}{4}}=2$; $(10+x)^{\frac{1}{2}}=4$, $10+x=16$, and $x=6$.

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* This value of x does not answer the condition of the question; because, from the transposed equation, x must be less than 8.

19. Given $2x^4-x^2+96=99$, to find the value of x.

By transposition, $2x^4-x^2=3$, or $x^4-\frac{1}{2}x^2=\frac{3}{2}$, whence $x^2=\frac{1}{4}\pm\sqrt{(\frac{1}{16}+\frac{3}{2})}=\frac{1}{4}\pm\sqrt{\frac{25}{16}}$; therefore $x^2=\frac{6}{4}$, and $x=\sqrt{\frac{6}{4}}=\frac{1}{2}\sqrt{6}$.

20. Given $x^6+20x^3-10=59$, to find the value of x.

By transposition, $x^6+20x^3=69$; whence $x^3=-10\pm\sqrt{(100+69)}=-10\pm13=3$, or -23; and $x=\sqrt[3]{3}$, or $-\sqrt[3]{23}$.

21. Given $3x^{2n}-2x^n+3=11$, to find the value of x.

By transposition, $3x^{2n}-2x^n=8$, or $x^{2n}-\frac{2}{3}x^n=\frac{8}{3}$, whence $x^n=\frac{1}{3}\pm\sqrt{(\frac{1}{9}+\frac{8}{3})}=\frac{1}{3}\pm\sqrt{\frac{25}{9}}$; therefore $x^n=\frac{6}{3}=2$, and $x=2^{\frac{1}{n}}$.

21. Given $5\sqrt[4]{x}-3\sqrt{x}=1\frac{1}{3}$, to find the value of x.

Here $3\sqrt{x}-5\sqrt[4]{x}=-1\frac{1}{3}$; then, by division, $x^{\frac{1}{2}}-\frac{5}{3}x^{\frac{1}{4}}=\frac{-4}{9}$ where $x^{\frac{1}{4}}=\frac{5}{6}\pm\sqrt{(\frac{25}{36}-\frac{4}{9})}=\frac{5}{6}\pm\sqrt{\frac{9}{36}}$, whence $x^{\frac{1}{4}}=\frac{5}{6}\pm\frac{3}{6}=\frac{4}{3}$, or $\frac{1}{3}$ consequently $x=(\frac{4}{3})^4=3\frac{13}{81}$, or $\frac{1}{81}$.

22. Given $\frac{2}{3}x\sqrt{(3+2x^2)}=\frac{1}{2}+\frac{2}{3}x^2$, to find the value of x.

Multiply by $\frac{3}{2}$, and we have $x\sqrt{(3+2x^2)}=\frac{3}{4}+x^2$, squaring, $3x^2+2x^4=\frac{9}{16}+\frac{6}{4}x^2+x^4$, whence $x^4+\frac{3}{2}x^2=\frac{9}{16}$; therefore, by the rule, $x^2=\frac{-3}{4}\pm\sqrt{(\frac{9}{16}+\frac{9}{16})}=\frac{3}{4}\pm\sqrt{\frac{18}{16}}$ or $x^2=\frac{-3}{4}\pm\frac{3}{4}\sqrt{2}$, or $x=\frac{1}{2}\sqrt{(-3+3\sqrt{2})}$.

23. Given $x\sqrt{(\frac{6}{x}-x)}=\frac{1+x^2}{\sqrt{x}}$, to find the value of x.

Multiply by $\sqrt{x}$, $x\sqrt{(6-x^2)}=1+x^2$; or, by squaring, $6x^2+x^4=1+2x^2+x^4$, whence $2x^4-4x^2=-1$, or $x^4-2x^2=-\frac{1}{2}$; therefore, $x^2=1\pm\sqrt{(1-\frac{1}{2})}=1\pm\frac{1}{2}\sqrt{2}$, consequently $x=\sqrt{(1\pm\frac{1}{2}\sqrt{2})}$.

24. Given $\frac{1}{x}\sqrt{(1-x^3)}=x^2$, to find the value of x.

Multiplying by x, we have $\sqrt{(1-x^3)}=x^3$, or $1-x^3=x^6$, or $x^6+x^3=1$; whence also we have, $x^3=-\frac{1}{2}\pm\sqrt{(\frac{1}{4}+1)}=\frac{-1}{2}\pm\frac{1}{2}\sqrt{5}$, or $x=(-\frac{1}{2}\pm\frac{1}{2}\sqrt{5})^{\frac{1}{3}}$.

25. Given $x\sqrt{(\frac{a}{x}-1)}=\sqrt{(x^2-b^2)}$, to find the value of x.

By squaring $ax-x^2=x^2-b^2$, or $2x^2-ax=b^2$; that is, $x^2-\frac{1}{2}ax=\frac{1}{2}b^2$, or $x=\frac{1}{4}a\pm\sqrt{(\frac{1}{16}a^2+\frac{1}{2}b^2)}=\frac{1}{4}a\pm\frac{1}{4}\sqrt{(8b^2+a^2)}$.

26. Given $\sqrt{(1+x-x^2)}-2(1+x-x^2)=\frac{1}{9}$, to find the value of x.

Here $(1+x-x^2)-\frac{1}{2}(1+x-x^2)^{\frac{1}{2}}=\frac{-1}{18}$; therefore $(1+x-x^2)^{\frac{1}{2}}=\frac{1}{4}\pm\sqrt{(\frac{1}{16}-\frac{1}{18})}=\frac{1}{4}\pm\sqrt{\frac{2}{16\cdot18}}=\frac{1}{4}\pm\sqrt{\frac{1}{144}}=\frac{1}{4}\pm\frac{1}{12}=\frac{1}{3}$, or $\frac{1}{6}$;

consequently $1+x-x^2=(\frac{1}{3})^2=\frac{1}{9}$, or $(\frac{1}{6})^2=\frac{1}{36}$. From the 1st we have $x^2-x=\frac{8}{9}$, which gives $x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+\frac{8}{9})}=\frac{1}{2}\pm\frac{1}{6}\sqrt{41}$. From the 2d, we have $x^2-x=\frac{35}{36}$, which gives $x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+\frac{35}{36})}=\frac{1}{2}\pm\frac{1}{6}\sqrt{41}$.

27. Given $\sqrt{(x-\frac{1}{x})}+\sqrt{(1-\frac{1}{x}}=x$, to find the value of x.

Multiply by $\sqrt{(x-\frac{a}{x})}-\sqrt{(1-\frac{1}{x})}$, and we have $x-1=x\sqrt{(x-\frac{1}{x})}$; divide by x, $1-\frac{1}{x}=\sqrt{(x-\frac{1}{x})}-\sqrt{(1-\frac{1}{x})}$, and from the 1st, $x=\sqrt{(x-\frac{1}{x})}+\sqrt{(1-\frac{1}{x})}$; therefore $1+x-\frac{1}{x}=2\sqrt{(x-\frac{1}{x})}$, or which is the same, $(x-\frac{1}{x})-2\sqrt{(x-\frac{1}{x})}+1=0$; or, by extracting $\sqrt{(x-\frac{1}{x})}-1=0$, or $x-\frac{1}{x}=1$, or $x^2-x=1$; consequently $x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+1)}=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.

28. Given $x^{4n}-2x^{3n}+x^n6$, to find the value of x.

Then this expression is evidently $=$ to $(x^{2n}-x^n)^2+(x^{2n})=6$. Put $y=x^{2n}-x^n$; then $y^2-y=6$, and $y^2-y+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$, by completing the square; $x-\frac{1}{2}=\frac{5}{2}=2\frac{1}{2}$, by extracting the root. Whence $y=3$; therefore $x^{2n}-x^n=3$; and, completing the square, $x^{2n}-x^n+\frac{1}{4}=3\frac{1}{4}=\frac{13}{4}$, or $x^n-\frac{1}{2}=\frac{1}{2}\sqrt{13}$, by extracting the root; therefore $x^n=\frac{1}{2}+\frac{1}{2}\sqrt{14}$, and $x=\sqrt[n]{(\frac{1}{2}+\frac{1}{2}\sqrt{13})}$, the Ans.

29. Given $x^4-2x^3+x=a$, to find the value of x.

Here this equation may be expressed thus: $(x^2-x^2)^2-(x^2-x)=x$; then $(x^2-x)^2-(x^2-)+\frac{1}{4}=a+\frac{1}{4}$, by completing the square; $x^2-x-\frac{1}{2}=\pm\sqrt{(a+\frac{1}{4})}$, by extracting the root, and $x^2-x=\frac{1}{2}\pm\sqrt{(a+\frac{1}{4})}$. Again, $x^2-x+\frac{1}{4}=\frac{3}{4}\pm\sqrt{(a+\frac{1}{4})}$, by completing the square; $x-\frac{1}{2}=\sqrt{(\frac{3}{4}\pm\sqrt{(a+\frac{1}{4})}}$, by extracting the root; therefore $x=\frac{1}{2}\pm\sqrt{(\frac{3}{4}\pm\sqrt{(a+\frac{1}{4})}}$, the Ans.

When there are more equations and unknown quantities than one, a single equation, involving only one of the unknown quantities, may sometimes be obtained by the rules before laid down for the solution of simple equations; and, in this case, one of the unknown quantities being determined, the others may be found by substituting its value in the remaining equations.

1. Given $\left\{\begin{array}{r} x^2+y^2=65 \\ xy=28 \end{array}\right\}$ to find the values of x and y.

Here, from the second equation, we have $y=\frac{28}{x}$; and, by sub-

stituting this in the first, $x^2+\frac{784}{x^2}=65$, or $x^4-65x^2=-784$. Whence, by the common rule before given, we have $x=\pm\sqrt{\{\frac{65}{2}\pm\sqrt{(\frac{4225}{4}-784)}\}}$; or, by reducing the parts under the last radical, and extracting the root, $x=\pm\sqrt{(\frac{65}{2}\pm\frac{33}{2})}=7$, or -4, and consequently, $y=\frac{28}{7}$, or $-\frac{28}{4}=4$, or -7.

Or the solution, in cases of this kind, may often be more readily obtained by some of the artifices frequently made use of upon these occasions, which can only be learned from experience; thus, taking, as before, (1.) $x^2+y^2=65$, (2.) $xy=28$, we shall have, as in the former method, by multiplying by 2, $2xy=56$, and by adding this equation to the first, and subtracting it from the same, $x^2+2xy+y^2=121$, and $x^2-2xy+y^2=9$. Whence, by extracting the square roots of each of these last equations, there will arise $x+y=\pm 11$, and $x-y\pm=3$, and consequently, by adding and subtracting these, we shall have $2x=\pm 14$, or $x=7$, or -7, and $y=4$, or -4. It will also sometimes facilitate the operation, by substituting for one of the unknown quantities the product of the other, and a third unknown quantity; which method may be applied with advantage whenever the sum of the dimensions of the unknown quantities is the same in every term of the equation.

2. Given $\left\{\begin{matrix} x^2+xy=56 \\ xy+2y^2=60 \end{matrix}\right\}$ to find the values of x and y.

Here, agreeably to the above observation, let $x=vy$, then $v^2y^2+vx^2=56$, and $vy^2+2y^2=60$, whence, from the first of these equations, $y^2=\frac{56}{v^2+v}$, and from the second, $y^2=\frac{60}{v+2}$. Therefore, by equating the right hand member of these two expressions, we shall have $\frac{60}{v+2}=\frac{56}{v^2+v}$, or $60v^2+60v=56v+112$. And, by transposing $56v$, and dividing the result by 60, $v^2+\frac{1}{15}v=\frac{28}{15}$ Hence, by the common rule for quadratics, we have $v=-\frac{1}{30}\pm\sqrt{(\frac{1}{900}+\frac{28}{15})}=-\frac{1}{30}+\frac{41}{30}=\frac{4}{3}$. And, consequently, by the former part of the process, $y^2=\frac{60}{v+2}=\frac{60}{1\frac{1}{3}+2}=18$, or $y=\sqrt{(18)}=3\sqrt{2}$, and $x=vy=\frac{4}{3}\times 3\sqrt{2}=4\sqrt{2}$. The work may also be sometimes shortened, by substituting for the unknown quantities the sum and difference of two other quantities; which method may be used when the unknown quantities, in each equation, are similarly involved.

3. Given $\left\{\begin{matrix} \frac{x^2}{y}+\frac{y^2}{x}=18 \\ x+y=12 \end{matrix}\right\}$ to find the values of x and y.

Here, according to the above observation, let there be assumed $x=z+v$, and $y=z-v$. Then, by adding these two equations together, we shall have $x+y=2z=12$, or $z=6$; also, since $x=6+v$, $y=6-v$, and by the first equation $x^3+y^3=18xy$, we shall obtain, by substitution, $(6+v)^3+(6-v)^3=18(6+v)(6-v)$, or, by involving the two parts of the first member, and multiplying those of the second, $432+36v^2=648-18v^2$, whence, by transposition, $54v^2=216$; and, by division, $v^2=\frac{216}{54}=4$, or $v=\pm 2$; and therefore, by the first assumption, and the first part of the process, we have $x=z+v=6\pm 2=8$, or 4, and $y=z-v=6\mp 2=4$, or 8.

4. A and B gained \$18 by trade. A's money was in trade 12 months, and he received for his principal and gain \$26. Also B's money, which was \$30, was in trade sixteen months, What money did A put into trade?

Let $x=$ the number of dollars he put in; $\therefore 26-x=$ the number he gained, and $12x+16\times 30 : 12x :: 18 . 26-x$, or, $18x=1040-14x-x^2$, or $x^2+32x=1040$; and, by case 1, $x=20$, or -52; consequently A put in \$20, Ans.

5. A sold a quantity of brandy for \$39, and gained as much per cent. as the brandy cost him. What was the price of the brandy?

Let $x=$ the price of the brandy; then $100 : x :: x : =$ the gain, $\frac{x^2}{100}$, and $\therefore \frac{x^2}{100}=39-x$, or $x^2=3900-100x$, or $x^2+100x=3900$; and, by case 1, $x=30$, or -130; consequently, the price was \$30, Ans.

6. A sold a piece of cloth for \$24, and gained as much per cent. as it cost him. What was the price of the cloth?

Let $x=$ cost price of the cloth in dollars; then $24-x=$ the whole gain. Hence $100+x : x :: 24 : 24-x$; $(100+x)\cdot(24-x)=24x$; $\therefore 2400-100x-x^2=0$; $\therefore x^2+100x=2400$, by case 1, we have $x=\$20$, the price of the cloth.

7. A horse dealer bought a horse for a certain number of dollars, and sold it again for \$119, by which means his profit was as much per cent. as the horse cost him. Required the cost price of the horse? Ans. \$70.

$100 : x :: x : 119-x$, or $x^2+100x=11900$.

8. A person bought a horse which he afterwards sold for \$24, and by so doing lost as much per cent. as the horse cost him. What sum did he cost? Ans. \$60.

Let x denote the number of dollars the horse cost; then, as the owner lost x per cent., the loss upon x dollars will be found thus: $100 : x :: x : \frac{x^2}{100}$; since $\therefore$ he lost $\frac{x^2}{100}$, and the horse cost him

x dollars, it is obvious he must have sold it for $x-\frac{x^2}{100}$; therefore $x-\frac{x^2}{100}=24$, or $x^2-100x=-2400$; by case 4 we have $x^2-100x+2500=2500-2400$, or $x-50=\sqrt{100}=\pm10$, and $\therefore x=50\pm10=60$, or 40. Hence it appears that the price of the horse might be either 60 or 40 dollars, for both these values of x equally satisfy the conditions of the question.

9. A sets out from C towards D, and travels 8 miles a day. Af- he had gone 27 miles, B set out from D towards C, and goes every day $\frac{1}{20}$th of the whole journey; and, after he had travelled as many days as he goes miles in one day, he met A. Required the distance of the places C and D.

Let x be the distance in miles; then B travelled $\frac{x}{20}$ miles a day, and $\frac{x^2}{400}$ miles in all. Also, A travelled $27+\frac{8}{20}x$, or $27+\frac{2x}{5}$ miles in all. By the question, $\frac{x^2}{400}+27+\frac{2x}{5}=x$, or $x^2+10800+160x=400x$; by transposition, $x^2-240x=-10800$; by case 4, $x^2-240x+(120)^2=14400-10800=3600$; root is $x-120=\sqrt{3600}=\pm60$; therefore $x=120\pm60=180$ or 60, both which values answer the conditions of the question. The distance $\therefore$ of C from D is 180 or 60 miles.

10. An oblong pond was surrounded by a terrace walk 4 yards broad; the pond measured 15200 square yards, and the walk 2104 square yards. Required the length and breadth of the pond.

Let $x=$ the length of the pond; then $\frac{15200}{x}=$ the breadth. $\therefore (x+4)\times(\frac{15200}{x}+8)=(15200+2104)=17304$; that is, $8x+\frac{121600}{x}+15264=17304$, or $8x^2-2040x=+121600$. Divide by 8, $x^2-255x=-15200$; by case 4, $x=127\cdot5\pm32\cdot5=160$, or 95, both of which values answer the conditions of the question. The length of the pond, therefore, is 160 or 95 yards, and the breadth $\frac{15200}{160}=95$, or $\frac{15200}{95}=160$ yards.

11. A and B gained \$140 by trade. A's money was three months in trade, and his gain was \$60 less than his stock; and B's money, which was \$50 more than A's, was in trade 5 months. What was A's stock?

Put $140=g$, $60=a$, $50=b$, and $x=$A's stock; then will $x-a$

be A's gain; also, $x+b$ will be B's stock, and $g-(x-a)=(g+a)-x$ B's gain; but the gains must be proportional to the stocks multiplied by the respective times; therefore $3x:5(x+b)::x-a:g+a-x$, or $3x(g+a-x)=(5x+4b)(x-a)$; $\therefore 8x^2-8ax-3gx+5bx=5ab$, by multiplying and transposing; or $8x^2-650x=15000$, in numbers by case 2d, $x^2-\frac{325}{4}x+(\frac{325}{8})^2=1875+(\frac{325}{8})^2=\frac{225625}{64}$; by completing the square, its root is $x-\frac{325}{8}=\sqrt{(\frac{225625}{64})}=\frac{475}{8}$, and $x=\frac{325}{8}+\frac{475}{8}=\frac{800}{8}=100$.

12. A tailor bought a piece of cloth for \$147, from which he cut off 12 yards for his own use, and sold the remainder for \$$\frac{481}{8}$ and gained \$$\frac{1}{4}$ per yard. How many yards were there, and what did it cost him per yard?

Let x, $\frac{147}{x}$, and $\frac{481}{4(x-12)}$, $\therefore \frac{147}{x}=\frac{481}{4(x-12)}-\frac{1}{4}$ denote the number of yards, the buying price per yard, the selling price, and the equation, respectively, and $x^2+95x+(\frac{95}{2})^2=7056+\frac{9025}{4}=\frac{37249}{4}$, and $x+\frac{95}{2}=\frac{193}{2}$, or $x=49$, Ans.

13. A farmer received \$288 for a certain quantity of wheat, and an equal sum at a price less by \$3 per bushel for a quantity of barley, which exceeded the quantity of wheat by 16 bushels. How many bushels were there? Ans. 32 bushels of wheat, and 48 of barley.

Let x, $x+16$, $\frac{288}{x}$, and $\frac{288}{x+16}$ denote the number and price of a bushel of wheat and barley, respectively, by the question, $\frac{288}{x}=\frac{288}{x+16}+3$, or $\frac{96}{x}=\frac{96}{x+16}+1$, $96(x+16)=96x+x^2+16x$.

General solution by which all such questions may be solved; let $x=$ the number of bushels of wheat; $\therefore x+a=$ the number of barley. Also $\frac{b}{x}=$ the price of a bushel of wheat, and $\frac{b}{x+a}=$ the price of a bushel of barley; $\therefore \frac{b}{x}=\frac{b}{x+a}+c$; $\therefore bx+ab=bx+cx^2+acx$, or $cx^2+acx=ab$. By case first, $x^2+ax=\frac{ab}{c}$, or

$$x^2+ax+\frac{a^2}{4}=\frac{ab}{c}+\frac{a^2}{4}=\frac{4ab+ca^2}{4c};\quad x+\frac{a}{2}=\sqrt{(\frac{4ab+cx^2}{4c})},\ \text{and}$$

$$x=\sqrt{(\frac{4ab+ca^2}{4c})}-\frac{a}{2}.$$

If $a=16$, $b=288$, and $c=3$, we have $x=\frac{80}{2}=40-8=32$; there were 31 bushels of wheat, and 48 of barley.

14. A and B were dispatched at the same time to a place 90 miles distant, the former of whom riding one mile an hour more than the other, arrived at the end of his journey an hour before

him; at what rate did each travel per hour? Ans. A went 10, and B 9 miles per hour.

Let $x=$ the number of miles A went per hour; $\therefore x-1=$ the number B went, and $\frac{90}{x}=\frac{90}{x-1}-1$, or, $90(x-1)=90x-x^2+x$, or $x^2-x=90$; by case 2d, $x=10$, the miles.

15. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he should have been 6 hours longer in performing the same journey. How many miles did he go per hour? Ans. 7 miles.

Let $x=$ the number; $\therefore \frac{105}{x}=\frac{105}{x-2}-6$, or $x^2-2x=35$; by case 2d, $x=\pm\sqrt{(35+1)}+1=7$, and $\frac{105}{7}=\frac{105}{5}-6=15$ hours.

16. A and B distributed 1200 dollars each among a certain number of persons; A relieves 40 persons more than B, and B gives 5 dollars a piece to each person more than A. How many persons were relieved by A and B respectively?

Let $x=$ the number of persons B relieves, then $x+40$ will $=$ the number A relieves. But $\frac{1200}{x+40}+5=\frac{1200}{x}\times$ by (x^2+40x), or $1200x+5x^2+200x=1200x+48000$; transpose and divide by 5, I have $x^2+40x=9600$; by case 1, $x=\pm100-20=80$, or 120.

17. A person bought a certain number of sheep for 1140 dollars. Having lost 8 of them, and sold the remainder at 8 dollars a head profit, he is no loser by the bargain. How many sheep did he buy? Ans. 38.

Let x denote the number of sheep; then each sheep will cost $\frac{1140}{x}$. But $\frac{1140}{x-8}=\frac{1140}{x}+8\times(x^2-8x)$, or $x^2-8x=1140$; by case 2, $x=4+\sqrt{1156}=34+4=38$, Ans.

18. A and B set off at the same time to a place at the distance of 300 miles. A travels at the rate of one mile an hour faster than B, and arrives at his journey's end 10 hours before him; at what rate did each person travel per hour?

Let $x=$ the rate per hour B travels; then $x+1=$ the rate A travels. But $\frac{300}{x+1}=\frac{300}{x}-10\times(x^2+x)$, or $300x=300x-10x^2-10x+300$, or $x^2+x=30$, and by case 1, $x=\frac{11}{2}-\frac{1}{2}=\frac{10}{2}=5$, rate B travels per hour, and $5+1=6=$ A's rate per hour.

19. A person bought some sheep for 72 dollars, and found that if he had bought 6 more for the same money, he would have paid 1 dollar less for each; how many did he buy, and what was the price of each? Ans. 18 sheep, and the price 4 dollars.

Let $x, \frac{72}{x}, \frac{72}{x+6}$, and $\therefore \frac{72}{x+6}+1=\frac{72}{x}$, denote the number of sheep bought, the price of one in dollars, the price of one, if he had bought 6 more, and the equation respectively; then $x^2+6x=432$, and by case 1, $x=\pm21-3=18$.

20. A and B set off at the same time to a place at the distance of 150 miles. A travels 3 miles an hour faster than B, and arrives at his journey's end 8 hours and 20 minutes before him; at what rate did each person travel per hour?

Let $x=$ the rate per hour at which B travels, then $x+3=$ the rate per hour at which A travels, and $\frac{150}{x}=$ the number of hours for which B travels, and $\frac{150}{x+3}=$ the number of hours for which A travels. But A is 8 hours 20 minutes ($8\frac{1}{3}$ hours) sooner at his journey's end than B. Hence $\frac{150}{x+3}+8\frac{1}{3}=\frac{150}{x}$, or $\frac{150}{x+3}+\frac{25}{3}=\frac{150}{x}$, and $\therefore$ the equation is $x^2+3x=54$. By case 1st, $x+\frac{3}{2}=\frac{15}{2}$, and $x=\frac{15-3}{2}=6$ miles per hour for B, and $x+3=9$, for A per hour.

21. A person bought a certain number of sheep for 2400 dollars. If there had been 8 more, each sheep would have cost him 10 dollars less. How many sheep were there? Ans. 40.

By the question, we have $\frac{2400}{x}=\frac{2400}{x-8}+10$, and $\therefore x^2+8x=1920$. By case 1st, $x=44-4=40$, Ans.

22. Given $16x^2-225x=225$, to find x.

Rule II. Multiplying by $(4\times16)=64$; $1024x^2-14400x=14400$; completing the square,

$$1024x^2-14400x+(-225)^2=14400+50625=65025.$$

Extracting the roots, $32x-225=\sqrt{65025}=255$. By transposition, $32x=255+225=480$. Dividing by 32, $x=\frac{480}{32}=15$, Ans.

Questions producing Quadratic Equations.

The methods of expressing the conditions of questions of this kind, and the consequent reduction of them, till they are brought to a quadratic equation, involving only one unknown quantity and its square, are the same as those already given for simple equations.

1. To find two numbers such that their difference shall be 8, and their product 240.

Let $x=$ the least number, then will $x+8=$ the greater, and

$x(x+8)=x^2+8x=240$, by the question; whence $x=-4+\sqrt{(16+240)}=-4+\sqrt{256}$ by the common rule, before given; therefore $x=16-4=12$. the less number, and $x+8=12+8=20$, the greater.

2. It is required to divide the number 60 into two such parts that their product shall be 864.

Let $x=$ the greater part, then will $60-x=$ the less, and $x(60-x)=60x-x^2=874$, by the question, or, by changing the signs on both sides of the equation, $x^2-60x=-864$; $x=30\pm\sqrt{(900-864)}=30\pm\sqrt{36}=30\pm6$, by the rule, and consequently, $x=30+6=36$, or $30-6=24$, the two parts sought.

3. It is required to find two numbers such, that their sums shall be $10(a)$, and the sum of their squares $58(b.)$

Let $x=$ the greater of the two numbers, then will $a-x=$ the less, and $x^2+(a-x)^2=2ax+a^2=b$, by the question, or $2x^2-2ax=b-a^2$, by transposition, and $x^2-ax=\frac{b-a^2}{2}$, by division; whence $x=\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}+\frac{b-a^2}{2})}=\frac{a}{2}\pm\frac{1}{2}\sqrt{(2b-a^2)}$ by the rule, and if 10 be put for a, and 58 for b, we shall have $x=\frac{10}{2}+\frac{1}{2}\sqrt{(116-100)}=7$, the greater number. And $10-x=\frac{10}{2}-\frac{1}{2}\sqrt{(116-100)}=3$, the less.

4. Having sold a piece of cloth for \$34, I gained as much per cent. as it cost me; what was the price of the cloth?

Let $x=$ pounds the cloth cost, then will $24-x=$ the whole gain; but $100:x::24-x$, by the question, or $x^2=100(24-x)=2400-100x$, that is, $x^2+100x=2400$; whence $x=-50+\sqrt{(5200+2400)}=-50+70=20$ by the rule, and consequently, \$20= price of the cloth.

5. A person bought a number of oxen for \$80, and if he had bought four more for the same money, he would have paid \$1 less for each; how much did he buy?

Let x represent the number of oxen, then will $\frac{80}{x}$ be the price of each, and $\frac{80}{x+4}=$ price of each, if $x+4$ cost \$80. But $\frac{80}{x}=\frac{80}{x+4}+1$, by the question, or $80=\frac{80x}{x+4}+x$, by multiplication. And $8x+320=80x+x^2+4x$, by the same, or, by leaving out $80x$ on each side, $x^2+4x=320$, whence $x=-\sqrt{(4+320)}=-2+18$, by the rule, and consequently, $x=16$, the number of oxen.

In this and in many other cases, especially in the solution of philosophical questions, we deduce, from the algebraical process, answers which do not correspond with the conditions. The reason seems to be, that the algebraical expression is more general

than the common language, and the equation which is a proper representation of the condition, will also express other conditions, and answer other suppositions. In the foregoing instance, x may either represent a positive or a negative quantity, and cannot in the operation represent a positive quantity alone, (Art. 70,) and the equation $\frac{80}{x-4}=\frac{80}{x}-1$, where x is negative, or represents the denomination of stock, will be a proper expression for the solution of the following problem: A person sells a certain number of oxen for 80 dollars, and, had he sold 4 fewer for the same sum, he would have received 1 dollar apiece more for them; required the number sold. Ans. 20.

6. It is required to find two numbers, such that their sum, product, and difference of their squares, shall be all equal to each other.

Let $x=$ the greater number, and $y=$ the less.

Then $\left\{\begin{array}{l} x+y=xy \\ x+y=x^2-y^2 \end{array}\right\}$ by the question.

Hence $1=\frac{x^2-y^2}{y+y}=x-y$, or $x=y+1$, by 2d equation. And $(y+1)+y=y(y+1)$, by 1st equation, that is, $2y+1=y^2+y$; $y^2+y=1$. Whence, $y=\frac{1}{2}+\sqrt{(\frac{1}{4}+1)}=\frac{1}{2}+\frac{1}{2}\sqrt{5}$, by rule 1. Therefore, $y=\frac{1}{2}+\frac{1}{2}\sqrt{5}=1.6180\ldots$ and $x=y+1=\frac{3}{2}+\frac{1}{2}\sqrt{5}=2.6180\ldots$ where ... denotes that the decimal does not end.

7. It is required to find four numbers in arithmetical progression, such that the product of the two extremes shall be 45, and the product of the means 77.

Let $x=$ the least extreme, and $y=$ common difference; then x, $x+y$, $x+2y$, and $x+3y$, will be the four numbers;

Hence $\left\{\begin{array}{l} x(x+3y)=x^2+3xy=45 \\ (x+y)\ (x+2y=x^2+3xy+2y^2=77 \end{array}\right\}$ by the question, and $2y^2=77-45=32$, by subtraction, or $y^2=\frac{32}{2}=16$ by division, and $y=\sqrt{16}=4$; therefore, $x^2+3xy=x^2+12x=45$, by the 1st equation, and consequently, $x=-6+\sqrt{(36+45)}=-6+9=3$, by case 1. Whence the numbers are 3, 7, 11, and 15.

8. It is required to find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84.

Let x, y, and z, be the three numbers, then $xz=y^2$, by the nature of proportion,

And $\left\{\begin{array}{l} x+y+z=14 \\ x^2+y^2+z^2=84 \end{array}\right\}$ by the question;

Hence, $x+z=14-y$, by the second equation, and $x^2+2xz+z^2=196-28y+y^2$, by squaring both sides, or $x^2+z^2+2y^2=196-$

$28y+y^2$ by putting $2y^2$ for its equal $2xz$, that is, $x^2+y^2+z^2=196-28y$ by subtraction, or $196-28y=84$ by equality; hence $y=\frac{196-84}{28}=4$, by transposition and division. Again, $xz=y^2=16$, or $x=\frac{16}{z}$, by the 1st equation, and $x+y+z=\frac{16}{z}+4+z=14$, by the 2d equation, or $16+4z+z^2=14z$, or $z^2-10z=-16$; whence, $z=5\pm\sqrt{(25-16)}=5\pm3=8$, or 2 by the rule; therefore, the 3 numbers are 2, 4, and 8.

9. It is required to find two numbers, such that their sum shall be 13 (a), and the sum of their fourth powers 4721 (b).

Let $x=$ the difference of the two numbers sought; then will $\frac{1}{2}a+\frac{1}{2}x$, or $\frac{a+x}{2}=$ the greater number, and $\frac{1}{2}a+\frac{1}{2}x$, or $\frac{a+x}{2}=$ the less; but $\frac{(a+x)^4}{16}+\frac{(a-x)^4}{16}=b$, by the question, or $(a+x)^4+(a-x^4)=16b$, by multiplication, or $2a^4-12a^2x^2+2x^4=16b$, by involution and addition, and $x^4+6a^2x^2=8b-a^4$, by transposition and division; whence $x^2=-3a^2+\sqrt{(9c^4+8b-a^4)}=3a^2+\sqrt{\{8(a^4+b)\}}$, by the rule, and $x=\sqrt{(-3a^2+2\sqrt{\{2[a^4+b)]\}}}$, by extracting the root; where, by substituting 13 for a, and 4721 for b, we shall have $x=3$; therefore $\frac{13+x}{2}=\frac{16}{2}=8$, the greater number, and $\frac{13-x}{2}=\frac{10}{2}=5$, the less number. The sum of which is 13, and $8^4+5^4=4721$.

10. Given the sum of two numbers equal s, and their product $=p$, to find the sum of their squares, cubes, biquadrates, &c.

Let x and y denote the two numbers; then (1.) $x+y=s$, (2.) $xy=p$. From the square of the first of these equations take twice the second, and we shall have (3.) $x^2+y^2=s^2-2p=$ sum of the squares. Multiply this by the 1st equation, and the product will be $x^3+xy^2+x^2y+y^3=s^3-2sp$. From which subtract the product of the first and second equations, and there will remain (4.) $x^3+y^3=s^3-3sp=$ sum of the cubes. Multiply this likewise by the 1st, and the product will be $x^4+xy^3+x^3y+y^4=s^4-3s^2p$; from which subtract the product of the second and third equations, and there will remain (5.) $x^4+y^4=s^4-4s^2p+2p^2=$ sum of the biquadrates. And, again, multiplying this by the 1st equation and subtracting from the result the product of the second and fourth, we shall have (6.) $x^5+y^5=s^5-5s^3p+5sp^2=$ the sum of the fifth powers, and so on; the expression for the sum of any powers in

general being $x^m + y^m = s^m - ms^{m-2}p + \frac{m(m-3)}{2}s^{m-4}p^2 - \frac{m(m-4)(m-5)}{2\cdot3}s^{m-6}p^3 + \frac{m(m-5)(m-6)(m-7)}{2\cdot3\cdot4}s^{m-8}p^4 -$, &c. Where it is evident that the series will terminate when the index of s becomes $= 0$.

11. It is required to divide the number 40 into two such parts, that the sum of their squares shall be 818.

Let x and y represent the two parts; then we have $x+y=40$, and $x^2+y^2=818$. From the 1st, $x=40-y$, or $x^2=1600-80y+y^2$. By substitution, $1600-80y+2y^2=818$, or $2y^2-80y=-782$, or $y^2-40y=-391$; whence, $y=20\pm\sqrt{(400-391)}=20\pm3=23$, or 17, but $x=40-y=17$, or 23; therefore 17 and 23 are the parts sought.

12. To find a number such, that if you subtract it from 10, and then multiply the remainder by the number itself, their product shall be 21.

Let x represent the number sought; then by the question $(10-x)x=21$, or $x^2-10x=-21$. Whence $x=5\pm(25-21)=5\pm2=7$ or 3.

13. It is required to divide the number 24 into two such parts, that their product shall be equal to 35 times their difference.

Let x and y be the two parts; then by the question we have $\left.\begin{array}{l} x+y=24 \\ 35(x-y)=xy \end{array}\right\}$ Here $x=24-y$, which substituted in the 2d eq. gives $35(24-2y)=y(24-y)$, or $840-70y=24y-y^2$, or $y^2-94y=-840$; whence $y=47\pm\sqrt{(47^2-840)}=47\pm37=10$; consequently $x=24-y=24-10=14$.

14. It is required to divide a line of 20 inches in length, into two such parts that the rectangle of the whole and one of the parts shall be equal to the square of the other.

Let one part $=x$; then the other will be $20-x$. And by the question $20(20-x)=x^2$, or $x^2+20x=400$; therefore $x=-10\pm\sqrt{(100+400)}=-10+10\sqrt{5}$, and the other part $=20-x=30-10\sqrt{5}$.

15. It is required to divide the number 60 into two such parts that their product shall be in the sum of their squares in the ratio of 2 to 5.

Let x and y represent the two parts, then $x+y=60$, and $xy : x^2+y^2 :: 2 : 5$, or $x+y=60$, and $\frac{5xy}{2}=x^2+y^2$. Squaring the 1st, $x^2+2xy+y^2=3600$. Subtract 2d, $x^2-2\frac{1}{2}xy+y^2=0$, we have $4\frac{1}{2}xy=3600$, or $xy=\frac{7200}{9}=800$. From this we have $x=\frac{800}{y}$;

which substituted in the 1st gives $\frac{800}{y}+y=60$, or $y^2-60y=-800$. Whence $y=30\pm\sqrt{(900-800)}=30\pm10=40$, or 20; consequently $x=60-40=20$, or $60-20=40$.

In all quadratics of this kind, in which x may be changed for y, and y for x, in the original equations, without altering their form, the two values of one of the quantity, may be taken for the value of the two quantities sought.

16. It is required to divide the number 146 into two such parts, that the difference of their square roots shall be 6.

Here, in order to avoid radicals, let us assume x^2 and y^2 for the two parts; then by the question $x^2+y^2=146$, and $x-y=6$, which may now be solved the same as Ex. 1. Another method is as follows: By squaring the second, $x^2-2xy+y^2=36$; subtract it from twice the first, $2x^2+2x^2=292$, and we have $x^2+2xy+y^2=256$. Hence by extracting, $x+y=16$; but $x-y=6$. Whence by addition $2x=22$, or $x=11$, and $x^2=121$, and by subtraction, $2y=10$, or $y=5$, and $y^2=25$.

17. What two numbers are those whose sum is 20 and their product 36?

Let x and y represent the two numbers, then by the question, $\left.\begin{array}{l} x+y=20 \\ \text{and } xy=36 \end{array}\right\}$ Squaring the first, $x^2+2xy+y^2=400$; subtract 4 times the 2d, $4xy=144$, and we have $x^2-2xy+y^2=256$; whence $x-y=16$; and since also $x+y=20$; by addition we have also $2x=36$, or $x=18$, and by subtraction, $2y=4$, or $y=2$.

But the more direct solution is as follows: From the second equation $x=\frac{36}{y}$, which substituted in the first, gives $\frac{36}{y}+y=20$, or $y^2-20y=-26$; whence $y=10\pm\sqrt{(100-36)}=10\pm8=18$ or 2; therefore 18 and 2 are the parts sought.

18. The sum of two numbers is $1\frac{1}{3}$, and the sum of their reciprocals $3\frac{1}{5}$; required the numbers.

Let x and y be the two numbers, and consequently $\frac{1}{x}$ and $\frac{1}{y}$ their reciprocals; then, $x+y=\frac{4}{3}$ and $\frac{1}{x}+\frac{1}{y}=\frac{16}{5}$; where the 2d becomes $x+y=\frac{16xy}{5}$; whence $\frac{16xy}{5}=\frac{4}{3}$, or $xy=\frac{20}{48}=\frac{5}{12}$; $\therefore$ $x=\frac{5}{12y}$; which substituted in the first gives $\frac{5}{12y}+y=\frac{4}{3}$, or $5+12y^2=16y$, or $y^2-\frac{4}{3}y=\frac{-5}{12}$; whence $y=\frac{2}{3}\pm\sqrt{(\frac{4}{9}-\frac{5}{12})}=\frac{2}{3}+\frac{1}{6}=\frac{1}{2}$, or $\frac{5}{6}$, therefore $\frac{1}{2}$ and $\frac{5}{6}$ are the quantities sought.

19. The difference of two numbers is 15, and half their product is equal to the cube of the less number; required the numbers.

Let x and y represent the two numbers; then by the question $x-y=15$, and $\frac{xy}{2}=y^3$. The second equation gives $xy=2y^3$, or $x=2y^2$; whence by substitution in the first we have $2y^2-y=15$, or $y^2-\frac{1}{2}y=\frac{15}{2}$; and hence $y=\frac{1}{4}\pm\sqrt{(\frac{1}{16}+\frac{15}{2})}=\frac{1}{4}\pm\frac{11}{4}=3$. Consequently $x=15+y=15+3=18$.

Here the two values of y are not those of x and y, because y is made to represent the less number, and cannot, therefore, be changed for x without altering the conditions of the question.

20. The difference of two numbers is 5, and the difference of their cubes 1685; required the numbers.

Let x and y be the two numbers; then by the question $x-y=5$, and $x^3-y^3=1685$. By the 1st, $x=5+y$, or $x^3=125+75y+15y^2+y^3$. Consequently $125+75y+15y^2+y^3-y^3=1685$; that is, dividing by 15, $y^2+5y=104$; whence $y=\frac{-5}{2}\pm\sqrt{(\frac{25}{4}+104)}$, or $y=\frac{-5}{2}+\frac{21}{2}=8$; therefore $x=5+y=5+8=13$. Consequently 8 and 13 are the numbers required.

21. A person bought cloth for \$675, which he sold again at \$48 per piece, and gained by the bargain as much as one piece cost him; required the number of pieces.

Let x be the number of pieces, and y the dollars that each piece cost; then by the question $xy=675$, and $48x=675+y$. From the 1st, $y=\frac{675}{x}$; whence by substitution we have $48x=675+\frac{675}{x}$, or $48x^2-675x=675$, or $x^2-\frac{225}{16}x=\frac{225}{16}$; whence $x=\frac{225}{32}\pm\sqrt{(\frac{225^2}{32}+\frac{225}{16})}=\frac{225}{32}+\frac{255}{32}=15$, and $y=\frac{675}{x}=\frac{675}{15}=45$, the dollars each cost.

22. What two numbers are those, whose sum, multiplied by the greater, is equal to 77, and whose difference, multiplied by the less, is equal to 12.

Let x and y represent the two numbers; then by the question, $x(x-y)=77$, or $x^2+xy=77$, and $y(x-y)=12$, or $xy-y^2=12$. By subtraction we have $x^2+y^2=65$; whence $y=\sqrt{(65-x^2)}$; which by substitution in the first gives $x^2+x\sqrt{(65-x^2)}=77$, or $\sqrt{(65-x^2)}=\frac{77-x^2}{x}$, and by squaring, $65-x^2=\frac{77^2-154x^2+x^4}{x^2}$

or $2x^4-219x^2=-5929$, or $x^4-\frac{219}{2}x^2=-2964\frac{1}{2}$. Whence $x^2=\frac{219}{4}\pm\sqrt{(\frac{219^2}{16}-2964\frac{1}{2})}$, or $x^2=\frac{219}{4}\pm\frac{23}{4}=\frac{242}{4}=60\frac{1}{2}$; or $\frac{196}{4}=$

49. Consequently $x=\sqrt{60\frac{1}{2}}$, or $\sqrt{49}=7$; and adopting the latter, $y=\sqrt{(65-x^2)}=\sqrt{(65-49)}=4$, that is, 7 and 4 are the two numbers required.

23. A grazier bought as many sheep as cost him \$1200, and after reserving 15 out of the number, sold the remainder for \$1080, and gained \$2 a head by them; how many sheep did he buy?

Let x represent the number of sheep, and y the dollars each cost, and consequently $y+2$, what they sold for; then by the question we have $\left.\begin{array}{l} xy=1200 \\ \text{and } (x-15).(y+2)=1080 \end{array}\right\}$; the latter by multiplying, gives $xy+2y-45y-30=1080$, or since $xy=1200$, we have $2x-15y=-90$. Now $x=\dfrac{1200}{x}$; and $15y=\dfrac{18000}{x}$, which substituted in the latter, gives $2x-\dfrac{18000}{x}=-90$, or $2x^2-18000=-90x$, or $x^2+45x=9000$; whence $x=\dfrac{-45}{2}\pm\sqrt{(\dfrac{45^2}{4}+9000)}=\dfrac{-45}{2}\pm\frac{395}{2}=175$, the number of sheep, and $\frac{1200}{175}=16$ dollars, the price each cost.

24. It is required to find two numbers, such that their product shall be equal to the difference of their squares, and the sum of their squares equal to their cubes.

Let $x=$ less number, and xy the greater; then $xy\times x=x^2y^2-x^2$, and $x^2y^2+x^2=x^3y^3=x^3$, by the question; and, by division, $y=y^2-1$, and $y^2+1=xy^3-x$. From the first, $y^2-y=1$, and, by completing the squares, $y^2-y+\frac{1}{4}=1+\frac{1}{4}=\frac{5}{4}$; therefore $y-\frac{1}{2}=\frac{1}{2}\sqrt{5}$, by extracting the root, or $y=\frac{1}{2}+\frac{1}{2}\sqrt{5}$. Again, $x=\dfrac{y^2+1}{y^3-1}=\dfrac{y+2}{2y}$ (because $y^2=y+1$) $=\frac{1}{2}+\dfrac{1}{y}=\frac{1}{2}+\dfrac{2}{1+\sqrt{5}}=\frac{1}{2}\sqrt{5}$, and $xy=\frac{1}{2}\sqrt{5}\times(\frac{1}{2}+\frac{1}{2}\sqrt{5})=\frac{1}{4}(5+\sqrt{5})$, the answer.

25. The difference of two numbers is 8, and the difference of their fourth powers is 14560; required the numbers.

Let $(m+n)$ and $(m-n)$ represent the two numbers,

Then by the question $(m+n)-(m-n)=2n=8$

And $(m+n)^4-(m-n)^4=14560.$

Now $(m+n)^4=m^4+4m^3n+6m^3n^2+4mn^3+n^4$

$(m-n)^4=m^4-4m^3n+6m^3n^2-4mn^3+n^4$. Whence by subtraction, $8m^2n+8mn^3=14560$, or by division, and substituting $n=4$, we have $m^3+16m=455$. Multiply by m, $m^4+16m^2=455m=65\times7m$. Add $49m^2$ to both sides, and it becomes $m^4+65m^2=49m^2+65\times7m$. Therefore, by completing the square, $m^4+65m^2+\dfrac{65^2}{4}=(7m)^2+65(7m)+\dfrac{65^2}{4}$. Whence $m^2+\frac{65}{2}=7m+\frac{65}{2}$, or

$m^2=7m$, or $m=7$. Consequently $m+n=7+4=11$, one number, and $m-n=7-3=4$, the other.

26. A company at a tavern had \$175 to pay for their reckoning; but before the bill was settled, two of them went away, in consequence of which those who remained had \$ 10 apiece more to pay than before; how many were there in company?

Let x be the whole number of persons, and y the number of dollars each would have had to pay; then after two were gone, the number was only $(x-2)$, and their reckoning $x+10$. Now by the question, $xy=175$, and $(x-2)(y+10)=175$; from the latter $xy+10x-2y-20=175$, or since $xy=175$, we have $10x-2y=20$, or $5x-y=10$; but $x=\frac{175}{x}$; therefore $5x-\frac{175}{x}=10$, or $5x^2-10x=175$, or $x^2-2x=35$. Whence $x=1+\sqrt{(1+35)}=7$, the number sought.

27. A person ordered \$144 to be distributed among some poor people; but before the money was divided, there came in, unexpectedly, two claimants more, by which means the former received a dollar apiece less than they would otherwise have done; what was their number at first?

Let x be the number of persons at first, and y the dollars each would have received; then $x+2$ was the number at last, and $y-1$ what each actually received; hence the following equations: $xy=144$, and $(x+2)(y-1)=144$. From the latter $xy+2y-x-2=144$, or since $xy=144$, $2y-x=2$; from the first $y=\frac{144}{x}$; therefore $\frac{288}{x}-x=2$; or $x^2+2x=288$; consequently $x=-1\pm\sqrt{(288+1)}=-1+17=16$, Ans.

28. It is required to find four numbers in geometrical progression such, that their sum shall be 15, and the sum of their squares 85.

Let $\frac{x^2}{y}$, x, y, and $\frac{y^2}{x}$ represent any four numbers in geometrical progression; then we have

$$\left.\begin{array}{l}\frac{x^2}{y}+x+y+\frac{y^2}{x}=15=a \\ \frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=85=b\end{array}\right\},$$

Make $x+y=s$, and $xy=r$; then will $x^2+y^2=s^2-2r$, $\frac{x^2}{y}+\frac{y^2}{x}=a-s$; $\frac{x^4}{y^2}+\frac{y^4}{x^2}=(a-s)^2-2r$.

Here equations first and third are drawn from this considera-

tion, that the sum of the squares of any two quantities is equal to the squares of their sum minus the double rectangle of them. Whence by adding the first and third, we have $\frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=b$, or $s^2+(a-s)^2-4r=b$; and from the second we have $x^2+y^2=xy(a-s)$, or $x^3+y^3=r(a-s)$. Also since $x^3+y^3=(x+y)^3-3xy(x+y)$ or $=s^3-3rs$, we have $r(a-s)=s^3-3rs$, or $r=\frac{s^3}{2s+a}$, which value substituted for r in the above, gives $s^2+(a-s)^2-\frac{4s^3}{2s+a}=b$, or $2s^2-2as+a^2-\frac{4s^3}{2s+a}=b$; whence $4s^3-4as^2+2a^2s+2as^2-2a^2s+a^3-4s^3=2sb+ab$, or by reduction and transposition, $2as^2+2bs=a^3-ab$, or $s^2+\frac{b}{a}s=\frac{1}{2}a^2-\frac{1}{2}b$, consequently $s=-\frac{b}{2a}\pm\sqrt{(\frac{b^2}{4a^2}+\frac{1}{2}a^2-\frac{1}{2}b)}$; where, by substituting $a=15$, and $b=85$, we obtain $s=6$; but $r=\frac{s^3}{2s+a}=\frac{216}{27}=8$.

Here then $x+y=6$, and $xy=8$; from which are determined $x=2$, and $y=4$; and therefore the numbers sought will be 1, 2, 4, and 8.

29. The sum of two numbers is 11, and the sum of their fifth powers is 17831; required the numbers.

Let $m+n$ be one of the numbers, and $m-n$ the other; then we shall have

$$(m+n)+(m-n)=11, \text{ or } 2m=11$$
$$(m+n)^5+(m-n)^5=17831=a.$$

Now $(m+n)^5=m^5+5m^4n+10m^3n^2+10m^2n^3+5mn^4+n^5$, and $(m-n)^5=m^5-5m^4n+10m^3n^2-10m^2n^3+5mn^4-n^5$. Consequently, by addition we have $10mn^4+20m^3n^2+2m^5=a$, or $n^4+2m^2n^2=\frac{a-2m^5}{10m}$. Or substituting for m its value, found as above, in the second term, we have $n^4+\frac{121}{2}n^2=\frac{a-2m^5}{10m}$. Whence $n^2=\frac{-121}{4}\pm\sqrt{(\frac{14641}{16}+\frac{a-2m^5}{10})}$. Where, by substituting for a and m, we derive $n^2=\frac{9}{4}$, or $n=\frac{3}{2}$; consequently, $m+n=5\frac{1}{2}+1\frac{1}{2}=7$, and $5\frac{1}{2}-1\frac{1}{2}=4$, that is, the two numbers are 4 and 7.

30. It is required to find four numbers in arithmetical progression, such, that their common difference shall be 4, and their continued product 176985. *See page 180*

Let $x-3y$, $x-y$, $x+y$, and $x+3y$ be the four numbers; then, by the question, $(x-3y)(x-y)(x+y)(x+3y)=176985=a$, or $(x^2-9y^2)(x^2-y^2)=a$, or $x^4-10x^2y^2+9y^4=a$; or, since by the question $(x+3y)-(x+y)=2y=4$, or $y=2$, this becomes $x^4-40x^2=176841$; consequently, $x^2=20\pm\sqrt{(176841+400)}=441$; hence,

$x=\sqrt{441}=21$, and $x-3y=15$, $x-y=19$, $x+y=23$, and $x+3y=27$, the numbers sought.

31. Two detachments of foot being ordered to a station at the distance of 39 miles from their present quarters, begin their march at the same time; but one party, by travelling $\frac{1}{4}$ of a mile an hour faster than the other, arrive there an hour sooner; required their rates of marching.

Let $x=$ the number of hours' march of the first detachment, and y the miles per hour; then, $x+1$ will be the hours of the second, and $y-\frac{1}{4}$ the miles per hour. Whence, by the question, we have $xy=39$, and $(x+1)(y-\frac{1}{4})=39$, or $xy-\frac{1}{4}x+y-\frac{1}{4}=39$; or, since $xy=39$, we have $-\frac{1}{4}x+y-\frac{1}{4}=0$, or $4y-x=1$. Again, $x=\frac{39}{y}$; whence, $4y-\frac{39}{y}=1$, or $4y^2-39=y$, or $y^2-\frac{1}{4}y=\frac{39}{4}$; therefore, $y=\frac{1}{8}\pm\sqrt{(\frac{1}{64}+\frac{39}{4})}=\frac{26}{8}=3\frac{1}{4}$. Consequently, $3\frac{1}{4}$ and 3 miles per hour are their rates of marching.

Otherwise, let $x=$ the number of miles travelled per hour by the one, and $x+\frac{1}{4}$ those travelled by the other; then $\frac{39}{x}$ and $\frac{39}{x+\frac{1}{4}}$ are the respective times they are in travelling 39 miles. Whence, by the question, $\frac{39}{x}-\frac{39}{x+\frac{1}{4}}=1$; and, multiplying by x and $x+\frac{1}{4}$, $\frac{39}{4}=x^2+\frac{1}{4}x$; therefore, by completing the square, $x^2+\frac{1}{4}x+\frac{1}{64}=\frac{39}{4}+\frac{1}{64}=\frac{625}{64}$, and extracting the root, $x+\frac{1}{8}=\frac{25}{8}$; whence $x=\frac{25}{8}-\frac{1}{8}=\frac{24}{8}=3$, and $x+\frac{1}{4}=3\frac{1}{4}$; therefore the answers are 3 and $3\frac{1}{4}$.

32. It is required to find two numbers, such that the square of the first plus their product shall be 140, and the square of the second minus their product 78.

Let x and y represent the two numbers; then, by the question, $\left.\begin{array}{l}x^2+xy=140\\ y^2-xy=\ 78\end{array}\right\}$ By addition, $x^2+y^2=218$, or $y=\sqrt{(218-x^2)}$. But, by the first equa., $y=\frac{140-x^2}{x}$; whence $\frac{140-x^2}{x}=\sqrt{(218-x^2)}$; or by squaring and reducing $19600-280x^2+x^4=218x^2-x^4$, or $2x^4-498x^2=-19600$; whence, by dividing by 2, we have $x^4-249x^2=-9800$; that is, $x^2=\frac{249}{2}\pm\sqrt{(\frac{249^2}{4}-9800)}$, or $x^2=\frac{249}{2}\pm\frac{151}{2}$, viz. $x^2=200$, or 49; therefore by adopting the latter we shall have $x=7$, and $y=\frac{140-49}{7}=13$; the numbs. sought.

33. It is required to find two numbers, such that their difference shall be $13\frac{55}{216}$, and the difference of their cube roots $1\frac{1}{6}$.

Let x^3= greater, and y^3= less; $x^3-y^3=13\frac{55}{216}$, and $x-y=1\frac{1}{6}$, by the question. Dividing the former equations by the latter, $x^2+xy+y^2=\frac{409}{36}$, and $x^2-2xy+y^2=\frac{49}{36}$, squaring the 2d. Hence.

$3xy=\frac{360}{36}=10$, by subtracting. And, from the second equation, $x=1\frac{1}{6}+y$; which value being substituted for x in the above equation, and we have $3y\times(1\frac{1}{6}+y)=10$, or $x^2+\frac{7}{6}y=\frac{10}{3}$, whence by completing the square, and extracting the root $y=-\frac{7}{12}\pm\surd(\frac{49}{144}+\frac{10}{3})$, or $y=-\frac{7}{12}+\frac{23}{12}=1\frac{1}{3}$, hence $x=1\frac{1}{6}+1\frac{1}{3}=2\frac{1}{2}$, and consequently $x^3=15\frac{5}{8}$, and $y^3=2\frac{10}{27}$.

34. It is required to find three numbers in arithmetical progression, such that the sum of their squares, shall be 93; and if the first be multiplied by 3, the second by 4, and the third by 5, the sum of their products shall be 66.

Let $x-y$, x, and $x+y$ = the numbers; then, $(x-y)^2+x^2+(x+y)^2=93$, or $3x^2+2y^2=93$, and $(x-y)\times x\times(x+y)=66$ or $12x+2y=66$, by the question. From the 2d equation $y=33-6x$; hence $y^2=1089-396x+36x^2$, which value being substituted for y in the 1st, we shall have $3x^2+2178-792x+72x^2=93$, or $25x^2-264x=-2085$; whence $x=\frac{132}{25}\pm\surd(\frac{17424}{625}-\frac{695}{25})$ or $x=\frac{132}{25}\pm\frac{7}{25}$, hence $x=\frac{139}{25}$, or $x=5$, and by taking the latter value for x, we have $y=33-6x=3$; and, consequently, the numbers are 2, 5, and 8.

35. The sum of three numbers in harmonical proportion is 191, and the product of the first and third is 4032; required the numbers.

Let x, y, and z= the numbers; then $x-y:y-z::x:z$; whence $xz-zy=xy-zx$, or $(x+z)y=2xz$. But $x+z=191-y$, and $xz=4032$, by the question; therefore $(191-y)y=2\times4032$, or $y^2-191y=-8064$; hence $y=63$, by completing the square, &c. Now $x+z=191-y=191-63=128$, and $xz=4032$.

From the square of the 1st, $x^2+2xz+z^2=46384$,
Take 4 times the second, $4xz\quad=16128$,

And we shall have $x^2-2xz+z^2=\quad 256$.

Or, $x-z=16$, and $x+z=128$; whence, by addition and subtraction, $x=72$, and $z=56$. And, consequently, 72, 63, and 96 are the numbers.

36. It is required to find four numbers in arithmetical progression, such that if they are increased by 2, 4, 8, and 15 respectively, the sums shall be in geometrical progression.

Let x denote the least number, and y the common difference. Then the four numbers will be expressed by x, $x+y$, $x+2y$, and $x+3y$; and the four specified sums by $x+2$, $x+y+4$, $x+2y+8$, and $x+3y+15$. Whence, by the nature of geometrical proportionals, we have $(x+2)\times(x+2y+8)=(x+y+4)^2$, and $(x+2)\times$

$(x^2+9y+15)=(x+y+4)\times(x+2y+8)$: that is, $x^2+4y=2x$, and $2y^2+10y+2=5x$, by multiplication and transposition. Hence, $5y^2+20y=4y^2+20y+4$; therefore $y^2=4$, or $y=2$; whence $x=6$; and, consequently, the numbers are 6, 8, 10, and 12.

37. It is required to find two numbers, such, that if their difference be multiplied into their sum, the product will be 5; but if the difference of their squares be multiplied into the sum of their squares, the product will be 65.

Let x and $y=$ the numbers; then $(x+y)\times(x-y)=x^2-y^2=5$, and $(x^2+y^2)\times(x^2-y^2)=65$, by the question. By dividing the latter equation by the former, we have $x^2+y^2=15$; whence by addition and subtraction, $x=3$, and $y=2$.

38. It is required to divide the number 10 into two such parts that if the square root of the greater part be taken from the greater part, the remainder shall be equal to the square root of the less part added to the less part.

Let $x^2=$ greater, and $y^2=$less: then $x^2-x=y^2+y$, and $x^2+y^2=10$, by the question. From the first, by adding $\frac{1}{4}$ to both sides, we have $x^2-x+\frac{1}{4}=y^2+y+\frac{1}{4}$, and by extracting the square root, $x-\frac{1}{2}=y+\frac{1}{2}$, or $x=y+1$; which value being substituted for x in the second equation, we have $y^2+2y+1+y^2=10$, or $y^2+y=\frac{9}{2}$; hence $y=-\frac{1}{2}\pm\frac{1}{2}\sqrt{19}$, and $x=y+1=\frac{1}{2}\pm\sqrt{19}$. And, consequently, $x^2=5+\frac{1}{2}\sqrt{19}$, and $y^2=5-\frac{1}{2}\sqrt{19}$.

39. It is required to find two numbers, such that if their product be added to their sum, it shall make 61; and if their sum be taken from the sum of their squares, it shall leave 88.

Let $x=\frac{1}{2}$ sum, and $y=\frac{1}{2}$ difference; then $x+y=$greater, and $x-y=$ less. Therefore $x^2-y^2+2x=61$, and $2x^2+2y^2-2x=88$, by the question; or dividing the second equation by 2, we have $x^2+y^2-x=44$; hence, by adding this to the first, we have $2x^2+x=105$, or $x^2+\frac{1}{2}x=\frac{105}{2}$. Whence $x=-\frac{1}{4}\pm\sqrt{(\frac{1}{16}+\frac{105}{2})}=\frac{1}{4}\pm\sqrt{\frac{841}{16}}=-\frac{1}{4}\pm\frac{29}{4}=7$; hence $y=\sqrt{(44-x^2+x)}=\sqrt{(44-49+7)}=\sqrt{2}$. Therefore $7+\sqrt{2}$, and $7-\sqrt{2}$ are the numbers.

40. It is required to find two numbers, such that their difference multiplied by the difference of their squares, shall be 576; and their sum multiplied by the sum of their squares, shall be 2336.

Let $x=$ greater, and $y=$ less; then $(x-y)\times(y^2-y^2)=576$, and $(x+y)\times(x^2+y^2)=2336$, by the question; that is, $x^3-x^2y-xy^2+y^3=576$, and $x^3+x^2y+xy^2+y^3=2336$; hence, $2x^2y+2xy^2=1760$, by subtraction, and by adding this to the second equation we have $x^3+3x^2y+3xy^2+y^3=4096$, or $x+y=\sqrt[3]{4096}=16$, by extracting the cube root; but $2x^2y+2xy^2=2xy(x+y)=1760$; then $32xy=1760$, by substitution, or $xy=\frac{1760}{32}=55$. Whence, the sum and product being given, (ex. 17,) we have $x=21$, and $y=5$.

41. It is required to find three numbers in continual proportion, whose sum shall be 20, and the sum of their squares 140.

Let x, y, and z = the numbers; then, $x+y+z=20$, $xz=y^2$: and $x^2+y^2+z^2=140$. From the 1st. equation $x+z=20-y$, or $x^2+2xz+z^2=400-40y+y^2$; hence $x^2+z^2+2y^2=400-40y+y^2$, by substitution, or $x^2+z^2+y^2=400-40y$; therefore $140=400-40y$, $40y=400-140=260$, or $y=6\frac{1}{2}$; and, consequently,

$$x=\frac{20-y+\sqrt{(400-40y-3y^2)}}{2}=6\tfrac{3}{4}+\sqrt{3\tfrac{5}{16}}, \text{ and}$$

$$z=\frac{20-y-\sqrt{(400-40y-3y^2)}}{2}=6\tfrac{3}{4}-\sqrt{3\tfrac{5}{16}}.$$

42. It is required to find two numbers whose product shall be 320, and the difference of their cubes to the cube of their difference, as 61 is to unity.

Let x= greater, and y= less; then will $xy=320$, and $x^3-y^3:(x-y)^3::61:1$, by the question; hence, dividing by $x-y$, we have $x^2+xy+y^2:(x-y)^2::61:1$; from whence, by subtracting the consequents from their antecedents, we have $3xy:(x-y)^2::60:1$, or $xy:(x-y)^2::20:1$; hence, $xy=20(x-y)^2$, or $20(x-y)^2=320$ by substitution, or $(x-y)^2=16$; and consequently, $x-y=4$. Now $x=y+4$, which value being substituted for x in the first equation, we have $y^2+4y=320$, and, consequently, $y=-2\pm\sqrt{324}=-2\pm18=16$, or -20, and by taking the positive value of y, we have $x=\frac{320}{16}=20$.

NOTE. This can be done otherwise by putting $x=\frac{1}{2}$ difference. Then $x+y$= greater and $x-y$= less, from whence the answer can be easily found.

43. The sum of 700 dollars was divided among four persons, A, B, C, and D, whose shares were in geometrical progression; and the difference between the greatest and least, was to the difference between the two means, as 37 to 12. What were all the several shares?

Let x= the share of A, or the first term of the progression, and let the common ratio$=\frac{1}{y}$, or as 1 to y; then, $x+xy+xy^2+xy^3=700$, and $xy^3-x:xy^2-xy::m:n$, by the question. From which proportion we have $y^3-1=(y-1)\times\frac{my}{n}$, or $y^2+y+1=\frac{my}{n}$ (by dividing the whole by $y-1$). Hence y is found $=\frac{m-n+\sqrt{(m^2-2mn-3n^2)}}{2n}=\frac{25+7}{24}=\frac{4}{3}$; $\therefore x(=\frac{700}{1+y+y^2+y^3}$ from the first equation) is given $=\frac{27\times700}{27+36+48+64}=108$.

Therefore the four shares are 108, 144, 192, and 256 dollars.

On the Solution of Quadratic Equations by the Tables of Sines and Tangents.

When the numerical parts of a quadratic equaiion are either large numbers or complicated fractions, its solution may often be more readily obtained, by means of one of the following trigonometrical formulæ, than by the common method, which, in cases of this kind, becomes very laborious.

$$1.\ x^2+ax=b,\quad \text{Put } \frac{2r}{a}\sqrt{b} = \text{tan. } z\text{; then } y = \left\{\begin{array}{l} +\frac{1}{r}\sqrt{b}\times\text{tan. }\tfrac{1}{2}z, \\ -\frac{1}{r}\sqrt{b}\times\text{cot. }\tfrac{1}{2}z. \end{array}\right.$$

$$2.\ x^2-ax=b,\quad \text{Put } \frac{2r}{a}\sqrt{b} = \text{tan. } z\text{; then } x = \left\{\begin{array}{l} -\frac{1}{r}\sqrt{b}\times\text{tan. }\tfrac{1}{2}z, \\ +\frac{1}{r}\sqrt{b}\times\text{cot. }\tfrac{1}{2}z. \end{array}\right.$$

$$3.\ x^2-ax=b,\quad \text{Put } \frac{2r}{a}\sqrt{b} = \text{sin. } z\text{; then } x = \left\{\begin{array}{l} +\frac{1}{r}\sqrt{b}\times\text{tan. }\tfrac{1}{2}z. \\ +\frac{1}{r}\sqrt{b}\times\text{cot. }\tfrac{1}{2}z. \end{array}\right.$$

In the last of which cases it is to be observed, that if $\frac{2r}{a}\sqrt{b}$ be greater than the radius r, or $4b$ greater than a^2, the two roots, or values of x, will be impossible.

1. Given $x^2+\frac{7}{44}x=\frac{1695}{12716}$, to find the roots of the equation, or the two values of x.

Here tan. $z = \frac{88r}{7}\sqrt{\frac{1695}{12716}}$

Log. 1695 . .	3.2291697
Log. 12716 . .	4.1043505
	2)$\bar{1}$.1248192
Square root .	$\bar{1}$.5624096
Log. 88 . . .	1.9444827
Colong. 7 . .	9.1549020
Log. tan. z . .	10.6617943

Or $z=77°\ 42'\ 32''$, and $\frac{1}{2}z=38°\ 51'\ 16''$, whence

Log. tan. $\frac{1}{2}z$ . .	9.9061115
Former root . .	$\bar{1}$.5624096
Log. $\frac{1}{r}$ . . .	$\overline{10}$.0000000
Log. x. . . .	$\bar{1}$.4685311
Or . .	$x=.2941176=\frac{5}{17}$

the positive root.

And if cot. $\frac{1}{2}z$ be taken, instead of tan. $\frac{1}{2}z$, the other value of x will be found $=-.4532085$; or $\frac{1695}{12716}\div-\frac{5}{17}=-\frac{339}{748}$, the negative root.

Demonstration of the Rule of Three.

Four numbers of the same kind are said to be proportional when the first contains the 2d as often as the 3d contains the 4th. If $\frac{A}{B}=\frac{C}{D}$; then A : B : : C : D, or B : A : : D : C. If 4 numbers are proportional, A : B : : C : D, the product of the extreme is equal to the product of the means, viz. $A \times D = B \times C$. Let $\frac{A}{B}=\frac{C}{D}$; these fractions reduced, (Case 3, page 31,) become $\frac{AD}{BD}=\frac{BC}{BD}$; but when *equal* fractions have the same denominator, their numerators are equal; therefore $AD = BC$. Hence we deduce a demonstration of the rule of three. If A : B : : C : D; then $A\times D=B\times C$. Now if (p. 60, art. 14,) $\therefore D=\frac{B\times C}{A}$, which is the common rule as given in my Arithmetic.

Cor. $D = \frac{B}{A}\times C=\frac{C}{A}\times B = C\div\frac{A}{B} = B \div\frac{A}{C}$. The rule of 3 inverse may be made a rule of three direct, by making the 3d term the first, and by proceeding forward to the other two terms; $\therefore$ the above demonstration will serve for both rules.

Demonstration of the Double Rule of Three, or of Compound Proportion.

The rule of compound proportion is easily illustrated. Let C be any cause, and E the effect produced; and let cx be any other similar cause and ey the effect produced, then it is plain that C : E : : cx : ey, $\therefore E\times cx=C\times ey$; hence x (an unknown part of the 2d cause) $=\frac{C\times ey}{E\times c}$, and $y=\frac{E\times cx}{C\times e}$ (= an unknown part of the second effect. Hence the rule is manifest.

Harmonical Progression.

Three quantities are said to be in harmonical proportion when the first is to the third as the difference of the first and second is to the difference of the second and third.

Any number of magnitudes a, b, c, d, e, f, &c. are said to be in harmonical proportion, if $a : c :: a-b : b-c$,

And $b : d :: b-c : c-d$,

And $c : e :: c-d : d-e$, &c.

The reciprocals of quantities in harmonical progression are in arithmetical progression. Let a, b, c, d, &c. be in harmonical progression; then since $a : c :: a-b : b-c$, $\therefore ab-ac=ac-bc$, Divide this equation by abc : $\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}$. Again $b : d :: b-c$:

$c-d$; $\therefore bc-bd=bd-dc$. Divide this equation by bcd, $\frac{1}{d}-\frac{1}{c}=\frac{1}{c}-\frac{1}{b}$, or $=\frac{1}{b}-\frac{1}{a}$; $\therefore$ the quantities $\frac{1}{a}$; $\frac{1}{b}$; $\frac{1}{c}$; $\frac{1}{d}$; &c. having a common difference, are in arithmetical progression.

From these definitions it follows, that in three harmonical proportionals, a, b, c, any two being given, the third may be found; since $ab-ac=ac-bc$, or $ab+bc=2ac$; $\therefore b=\frac{2ac}{a+c}$; that is, a harmonical mean between 2 quantities is equal to twice their product divided by their sum. Also, $c=\frac{ab}{2a-b}$ = a third harmonical proportion to a and b.

In a similar manner, if any three out of four harmonical proportions, a, b, c, d, be given, the other may be found; for since $a:d::a-b:c-d$, $\therefore ac-ad=ad-bd$; and from this equation I get $b=\frac{2ad-ac}{d}$; $c=\frac{2ad-bd}{a}$; $d=\frac{ac}{2a-b}$

The following is a curious property belonging to quantities in harmonical proportion, that their reciprocals are in arithmetical progression. Thus, if a, b, c, d, e, &c. be in harmonic proportion, then from what has been said above, I shall have $bc+ab=2ac$; $cd+bc=2bd$; $de+cd=2ce$, &c. Dividing the first of these equations by abc, the 2d by bcd, the 3d by cde, &c. I shall have $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$; $\frac{1}{b}+\frac{1}{d}=\frac{2}{c}$; $\frac{1}{c}+\frac{1}{e}=\frac{2}{d}$; $\therefore \frac{1}{a}$; $\frac{1}{b}$; $\frac{1}{c}$; $\frac{1}{d}$; $\frac{1}{e}$; &c. are in arithmetical progression. Thus, of five musical strings of equal tension and thickness be made to sound together, the most perfect harmony they can produce, will be when their lengths are as $\frac{1}{2}$; $\frac{1}{3}$; $\frac{1}{4}$; $\frac{1}{5}$; $\frac{1}{6}$; or as 30, 20, 15, 12, 10; and hence this proportion is called harmonic, or musical proportion.

Problem 1. To insert any number (m) of harmonic means between two given quantities (a) and (b).

Since the reciprocals of quantities in harmonic progression are in arithmetical progression, therefore it will be best to insert (m) arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$; these means inverted will be the harmonic means required. Let x = the common difference of this arithmetic progression.

then $\frac{1}{a}+(m+1)x=\frac{1}{b}$; [p. 186.] & $x=\frac{a-b}{(m+1)ab}$; $\therefore$ the series $\frac{1}{a}$; $\frac{1}{a}+\frac{a-b}{(m+1)ab}$; $\frac{1}{a}+\frac{2(a-b)}{(m+1)ab}+\frac{1}{a}+\frac{3(a-b)}{(m+1)ab}$ &c..... $\frac{1}{b}$ is arith.;

and the reciprocals a; $\dfrac{(m+1)ab}{a+mb}$; $\dfrac{(m+1)ab}{2a+(m-1)b}$; $\dfrac{(m+1)ab}{3a+(m-2)b}$; are in harmonic progression. Insert 6 harmonic means between 1 and 20. Here $m=6$, $a=1$, $b=20$; $\therefore$ the means are $\frac{140}{121}$; $\frac{70}{51}$; $\frac{140}{83}$; $\frac{35}{16}$; $\frac{28}{9}$; $\frac{70}{13}$.

Prob. 2. Having given two consecutive terms of an harmonical series to determine the law and limit of continuation.

Let t, t', be two consecutive terms of the harmonic progression, and let a, $a-x$, be the two consecutive terms of the corresponding arithmetic progression; then $\dfrac{1}{t}=a$; $\dfrac{1}{t'}=a-x$; therefore $x=\dfrac{t'-t}{tt'}$, and since the (n)th or general term of the arithmetic progression, reckoning from $\dfrac{1}{t}$ is $(a\mp nx)$, therefore the (n) term of the harmonic progression is $\dfrac{1}{a\mp nx}=\dfrac{tt'}{t'\mp n(t'-t)}$, which is indefinite when $t'-n(t'-t)=0$, or when $n=\dfrac{t'}{t'-t}$.

Given 3, 4, two consecutive terms of an harmonic progression, to show the law of continuation both ways. The general law is $\dfrac{12}{4\mp n}$; $\therefore$ the next term of the progression after 4, is $\dfrac{12}{4-2}$, or 6; the next is $\dfrac{12}{4-3}$ or 12; the next $\dfrac{12}{4-4}$ or $\dfrac{12}{0}$, a quantity indefinitely great. The term preceding 3 is $\dfrac{12}{4+1}$ or $\dfrac{12}{5}$; the next preceding term $\frac{12}{6}$ or 2, &c. hence the series cannot be carried beyond 12, but may be extended to any length the other way.

An arithmetic mean between two quantities is greater than a geometric, and a geometric is greater than a harmonic mean. Let p and q be the two quantities, then since $(p-q)^2$ is positive, whether p be greater or less than q, $\therefore$ $p^2-2pq+q^2$ is positive, or p^2+q^2 greater than $2pq$, $\therefore$ $p^2+2pq+q^2$ greater than $4pq$ or $p+q$ greater than $2\sqrt{(pq)}$; $\therefore$ $\frac{1}{2}(p+q)$ is greater than $\sqrt{(pq)}$. But $\frac{1}{2}(p+q)$ is an arith. mean between a and b, and $\sqrt{(pq)}$ is a geo. mean, $\therefore$ the arith. is greater than the geo. mean. Again $\dfrac{2pq}{p+q}$ is an harmonic mean between p and q. Also $\dfrac{p+q}{2} : \sqrt{(pq)} :: \sqrt{(pq)} : \dfrac{2pq}{p+q}$ Or arith. mean : geo. mean :: geo. mean : harmonic mean. But since the first term has been proved to be greater than the second, $\therefore$ by Euclid, book 5, p. A., the 3d term is greater than the fourth.

On Ratios, Proportion, and Variation.

73. By Ratio is meant the relation which one quantity bears to another, with respect to magnitude. It is evident that this relation can exist only between quantities of a similar kind; thus, a number must be compared with a number, a line with a line, &c. &c.; and it would be absurd to compare a certain number of feet with a certain number of dollars, &c. &c.

74. There are two ways in which the magnitude of quantities may be compared. In the first place, they may be compared with regard to their difference; and then the question asked is, "How much one quantity is greater or less than another." The relation which quantities bear to each other in this respect, is called their arithmetical ratio. The other way in which they may be compared, is, by inquiring, "How often one quantity is contained in the other." This relation between quantities is called their geometrical ratio. The term ratio, when simply applied, is generally understood in the latter sense, and it is in this sense that the word will be made use of in the present chapter.

75. In considering how often one quantity is contained in another, the natural process is to divide the one by the other. Thus, in comparing the number 12 with the numbers 4 and 3, we know that 4 is contained in 12 three times, and that 3 is contained in the same number four times; from which we infer, that the ratio of 12 : 3* is greater than the ratio of 12 : 4, the magnitude of a ratio being measured by the number of times one quantity is contained in another. For the same reason, the ratio of 11 : 7 is said to be less than the ratio of 11 : 5. When the ratio is thus expressed, the first term of it is called the antecedent, the last term the consequent, of that ratio.

76. From this mode of estimating the magnitude of a ratio, it appears that when the consequent of a ratio is not an aliquot part of the antecedent, the value of the ratio must be expressed by a fraction whose numerator is the antecedent, and denominator the consequent, of that ratio. Thus, the magnitude of the ratio of 15 : 7 is expressed by the fraction $\frac{15}{7}$, and of the ratio 4 : 13, by the fraction $\frac{4}{13}$. When the antecedent of a ratio is greater than the consequent, it is called a ratio of greater inequality; when the antecedent is less than the consequent, a ratio of lesser inequality;

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* In expressing the ratio of two quantities, the word *to* is generally supplied by two dots; thus, the ratio of a to b is expressed by $a : b$.

and if the two terms of a ratio be the same, then it is said to be a ratio of equality.

77. The foregoing definitions evidently apply only to those instances in which the consequent of a ratio is contained a certain number of times in the antecedent, or to those in which the magnitude of the ratio may be expressed by some definite fraction. It does not, therefore, comprehend such ratios as $\sqrt{2}:5$; $\sqrt{3}:\sqrt[3]{7}$; $4:\sqrt[4]{3}$, &c., where the values of the quantities $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{7}$, $\sqrt[4]{3}$, &c., can only be expressed in decimal fractions which do not terminate. The ratio which exists between quantities of this latter kind, when the radical quantity is expressed by a decimal fraction, is called their approximate ratio.

78. Proportion consists in the inequality of ratios; thus, since 4 is contained in 12 the same number of times that 6 is in 18, the ratio of $12:4$ is said to be equal to the ratio of $18:6$, or, in other words, that $12:4::18:6$.* Of the four terms of which every proportion consists, the first and last terms are called the extremes, and the second and third the means, of that proportion.

79. If there be a set of quantities related together in the following manner, viz. $a:b::b:c::c:d::d:e$, &c., where the consequent of every preceding ratio is the antecedent of the following one, then the quantities a, b, c, d, e, &c. are said to be in continued proportion; and if only three quantities be concerned, as in the proportion $a:b::b:c$, then b is said to be a mean proportional between the two extremes a and c.

80. Since the proportion $a:b::c:d$ expresses the equality of the ratios $a:b$ and $c:d$, and since the magnitude of the ratio $a:b$ is measured by the fraction $\frac{a}{b}$, and that of the ratio $c:d$ by the fraction $\frac{c}{d}$, it follows that $\frac{a}{b}=\frac{c}{d}$, or that when four quantities are proportional, the quotient of the first divided by the second is equal to the quotient of the third divided by the fourth; and *vice versa*, if there be four quantities, a, b, c, d, such, that $\frac{a}{b}=\frac{c}{d}$, then those four quantities are proportional, or $a:b::c:d$.

On the Comparison and Composition of Ratios.

81. *On the comparison of ratios.*

I. Since the ratio of $a:b$ may be expressed by the fraction

* In stating a proportion, the words *is to* and *to* are generally supplied by two dots, and the word *as* by four dots; thus, the proportion a is to b as c to d, is expressed by $a:b::c:d$.

$\frac{a}{b}$, let the numerator and denominator of this fraction be multiplied by any quantity m, (m being either integral or fractional,) then $\frac{ma}{mb}=\frac{a}{b}$, and therefore the ratio of $ma:mb$ is the same with the ratio of $a:b$; from which we infer, that if the terms of a ratio be multiplied or divided by the same quantity, it does not alter the value of the ratio. Hence also it appears, that a ratio is reduced to its lowest terms by dividing its antecedent and consequent by their greatest common measure.

II. Ratios are compared together by reducing the fractions by which their values are respectively represented, to a common denominator. Thus, the ratio of $8:5$ is represented by the fraction $\frac{8}{5}$, and the ratio of $9:6$ by the fraction $\frac{9}{6}$; reduce these fractions to others of the same value, having a common denominator, and they become $\frac{48}{30}$ and $\frac{45}{30}$ respectively; and since $\frac{48}{30}$ is greater than $\frac{45}{30}$, the ratio $8:5$ is greater than the ratio of $9:6$.

III. A ratio of greater inequality is diminished, and a ratio of lesser inequality is increased, by adding the same quantity to both its terms. Let $a+b:a$ represent a ratio of greater inequality, and let x be added to each of its terms, and it becomes the ratio of $a+b+x:a+x$. Now the ratio of $a+b:a=\frac{a+b}{a}$, and that of $a+b+x:a+x=\frac{a+b+x}{a+x}$; let these fractions be reduced to others of the same value, having a common denominator, and they become $\frac{a^2+ab+ax+bx}{a(a+x)}$ and $\frac{a^2+ab+ax}{a(a+x)}$, respectively; and since $a^2+ab+ax+bx$ is evidently greater than $a^2+ab+ax$, the ratio of $a+b:a$ is greater than the ratio of $a+b+x:a+x$; i. e. the ratio of $a+b:a$ has been diminished by adding x to each of its terms. Next, let $a-b:a$ represent a ratio of lesser inequality; then, proceeding with the fractions $\frac{a-b}{a}$ and $\frac{a-b+x}{a+x}$, as in the former instance, the resulting fractions are $\frac{a^2-ab+ax-bx}{a(a+x)}$ and $\frac{a^2-ab+ax}{a(a+x)}$; and since $a^2-ab+ax-bx$ is less than $a^2-ab+ax$, the ratio of $a-b:a$ is less than the ratio of $a-b+x:a+x$, and consequently the ratio of $a-b:a$ has been increased by adding x to each of its terms. In the same manner it might be shown that a ratio of greater inequality is increased, and a ratio of lesser inequality is diminished, by subtracting the same quantity from each of its terms.

82. *On the composition of ratios.*

I. Ratios are compounded together by multiplying their antecedents together for a new antecedent, and their consequents together for a new consequent. Thus, if the ratio of $a : b$ be compounded with the ratio of $c : d$, the resulting ratio is that of $ac : bd$; or if the ratios $4 : 3$, $5 : 2$, and $7 : 1$, be compounded together, there results the ratio of $4\times5\times7 : 3\times2\times1$, or of $140 : 6$, or (dividing each term by 2) of $70 : 3$.

II. If the same ratio be compounded with itself once, twice, thrice, &c., the resulting ratios are those of $a^2 : b^2$, $a^3 : b^3$, $a^4 : b^4$; &c. &c. The ratio of $a^2 : b^2$ is called the duplicate ratio of $a : b$; $a^3 : b^3$, the triplicate; $a^4 : b^4$, the quadruplicate, &c. &c.; and as these ratios receive their denominations from the indices of the several powers of a and b, the ratio of $\sqrt{a} : \sqrt{b}$ is called the subduplicate ratio of $a : b$; the ratio of $\sqrt[3]{a} : \sqrt[3]{b}$, the subtriplicate, &c. &c.

III. If a set of ratios, whereof the consequent of the preceding ratio is the same with the antecedent of the succeeding one, be compounded together, the resulting ratio is that of the first antecedent to the last consequent. Thus, when the ratios of $a : b$, $b : c$, $c : d$, $d : e$, are compounded together, the resulting ratio is that of $abcd : bcde$, or, (dividing by bcd) that of $a : e$, or of the first antecedent : the last consequent; and the same will be the case whatever be the number of ratios.

IV. A ratio of greater inequality compounded with another ratio, increases it; and a ratio of lesser inequality compounded with another ratio, diminishes it. Thus, let $1+n : 1$ represent a ratio of greater inequality, and let it be compounded with the ratio $a : b$, the resulting ratio is that of $a + na : b$, which is evidently greater than the ratio of $a : b$. On the other hand, let $1 - n : 1$ represent a ratio of lesser inequality, and let it be compounded with the ratio of $a : b$, then the resulting ratio is that of $a - na : b$, which is evidently less than the ratio of $a : b$.

1. Reduce the ratio of $360 : 315$, and $1595 : 667$, to their lowest terms.

$360 : 316, = \dfrac{360}{315} \div 45$ the greatest common measure $\dfrac{8}{7} = 8 : 7$,

and $1595 : 667, = \dfrac{1595}{667} \div 29$, greatest common measure, $= \dfrac{55}{33}$ $= 55 : 23$.

2. Reduce the ratio $a^3+2a^2x : a^3$ to its lowest terms.

Let $a^3+2a^2x : a^3 = \dfrac{a^3+2a^2x}{a^2} \div a^2 = \dfrac{a+2x}{1}$. Hence $a+2x : 1$,

3. Which is the greater, the ratio of 16 : 15, or that of 17 : 14?

$\frac{16}{15}\times14=\frac{227}{210}$; $\frac{17}{14}\times15=\frac{255}{210}$ $\therefore$ is the greatest.

4. Which is the least of the three ratios, 20 : 17, 22 : 18, or 25 : 23? and which is the greatest of the three ratios, 8 : 7, 6 : 5, and 10 : 9?

$\frac{20}{17}$; $\frac{22}{18}$; $\frac{25}{23}$ $\times$ by the least common multiple of the denominators, and $\frac{20}{17}=4140 : 3519$; $\frac{22}{18}4301 : 3519$; $\frac{25}{23}=3825 : 3519$ least, and $\frac{8}{7}=360 : 315$; $\frac{6}{5}=378 : 315$ $\therefore$ greatest, $\frac{10}{9}=$ $350 : 315$.

5. Which is the greater, the ratio of $a+2 : \frac{1}{3}a+4$, or that of $a+4 : \frac{1}{2}a+5$? Ans. the ratio of $a+4 : \frac{1}{2}a+5$.

$\frac{a+2}{\frac{1}{3}a+4}$ and $\frac{a+4}{\frac{1}{2}a+5}$ $\times$ by the alternate denominator, and $\frac{a+2}{\frac{1}{3}a+4}$ $=\frac{a^2}{3}+6a+16 : \frac{a^2}{4}+\frac{9a}{4}+20$. Hence $a+4 : \frac{1}{2}a+5$ the greatest.

6. Compound together the ratios of 11 : 3, 7 : 2, and 5 : 9.

$$\frac{11\times7\times5}{3\times2\times9}=\frac{385}{54}=385 : 54.$$

7. Compound together the ratios of 15 : 12, 6 : 7, and 9 : 4; and then reduce the resulting ratio to its lowest terms.

$\frac{15\times6\times9}{12\times7\times4}=\frac{810}{336}\div6$ common measure, $=\frac{135}{56}=135 : 56$.

8. Express in the simplest terms the ratio compounded of $a^2-x^2 : a^2$, $a+x : b$, and $b : a-x$.

$\frac{(a^2-x^2)\times(a+x)\times b}{a^2\times b\times(a-x)}=\frac{(a+x)\times(a+x)}{a^2}=(a+x)^2 : a^2$, by cancelling those quantities which are common to the terms.

9. If the ratios of $x+y : a$, $x-y : b$, and $b : \frac{x^2-y^2}{a}$ be compounded together, show that the resulting ratio is a ratio of equality.

$$\frac{(x+y)\times(x-y)\times b}{a\times b}\times\frac{a}{x^2-y^2}=\frac{x^2-y^2}{x^2-y^2}=1 : 1.$$

10. If the ratios of $3a+2 : 6a+1$, and of $2a+3 : a+2$ be compounded together, is the resulting ratio a ratio of greater or lesser inequality?

$$\frac{(3a+2)\times(2a+3)}{(6a+1)\times(a+2)}=\frac{6a^2+13a+6}{6a^2+13a+2}=6a^2+13a+6 : 6a^2+13a+2$$
$\therefore$ a ratio of greater inequality.

11. What are the least numbers in the ratio compounded of the three following ratios, viz. the ratio of $7 : 5$, the duplicate ratio of $4 : 9$, and the triplicate ratio of $3 : 2$?

$$\frac{7\times4\times3}{5\times9\times2}=\frac{14}{15}, \therefore 14 \text{ and } 15.$$

12. Compound the subduplicate ratio of $x^2 : y^2$ with the quadruplicate ratio of $\sqrt{x} : \sqrt{y}$.

$\frac{x}{y}=$ subduplicate ratio of $x^2 : y^2$, and $\frac{x^2}{y^2}=$ quadruplicate ratio of $\sqrt{x} : \sqrt{y}$; $\therefore \frac{x\times x^2}{y\times y^2}=x^3 : y^3$.

On Proportion.

83. The most useful Theorems relating to proportional quantities are the following:

Th. 1. If four quantities be proportional, the product of the extremes will be equal to the product of the means; for let $a : b :: c : d$, then, by Art. 80, $\frac{a}{b}=\frac{c}{d}$, $\therefore ad=bc$. Hence, also, it follows, that if any three terms of a proportion be known, the fourth may be found; for, from the equation $ad = bc$, we have $a=\frac{bc}{d}$; $b=\frac{ad}{c}$; $c=\frac{ad}{b}$; and $d=\frac{bc}{a}$.

Th. 2. The converse of the foregoing theorem is also true; viz.: If the product of any two quantities be equal to the product of two others, those four quantities will constitute a proportion, provided that the terms of one product be made the means, and the terms of the other product be made the extremes, of such proportion. Thus, if the four quantities a, b, c, d, be such that $ad=bc$, then (dividing by bd) $\frac{a}{b}=\frac{c}{d}$; $\therefore$ by Art. 80, $a : b :: c : d$.

Th. 3. If three quantities be proportional, the product of the two extremes is equal to the square of the mean; for, if $a : b :: b : c$, then, by theorem 1, $ac=b^2$. Hence also it follows, that a mean proportional between any two quantities is equal to the square root of their product; for, let x be a mean proportional between a and c, then $a : x :: x : c$; $\therefore x^2 = ac$, and $x = \sqrt{ac}$.

TH. 4. If four quantities be proportional, they will also be proportional when taken inversely or alternately; thus, if $a:b::c:d$, then $\frac{a}{b}=\frac{c}{d}$; invert the fractions, then $\frac{b}{a}=\frac{d}{c}$; $\therefore\ b:a::d:c$. Again, since $ad=bc$, then (dividing by cd) we have

$$\frac{ad}{cd}=\frac{bc}{cd}, \text{ or } \frac{a}{c}=\frac{b}{d};\ \therefore a:c::b:d.$$

TH. 5. If there be six proportional quantities, and the first be to the second as the third to the fourth, and the third to the fourth as the fifth to the sixth, then will the first be to the second as the fifth to the sixth. For let $a:b::c:d$, and $c:d::e:f$: then $\frac{a}{b}=\frac{c}{d}$; and $\frac{c}{d}=\frac{e}{f}$; $\therefore \frac{a}{b}=\frac{e}{f}$, or, by Art. 80, $a:b::e:f$.

TH. 6. If four quantities be proportional, then the sum or difference of the first and second will be to the second as the sum or difference of the third and fourth is to the fourth. For let $a:b::c:d$, then $\frac{a}{b}=\frac{c}{d}$; add 1 to, or subtract it from, each side of the equation, then $\frac{a}{b}\pm1=\frac{c}{d}\pm1$; $\therefore \frac{a\pm b}{b}=\frac{c\pm d}{d}$ consequently, by Art. 80, $a\pm b:b::c\pm d:d$.

TH. 7. If four quantities be proportional, the first is to the sum or difference of the first and second as the third to the sum or difference of the third and fourth. For, by theorem 6, $a\pm b:b::c\pm d:d$, and alternately $a\pm b:c\pm d::b:d$; but, by theorem 4, $b:d::a:c$; hence, by theorem 5, $a\pm b:c\pm d::a:c$, and alternately $a\pm b:a::c\pm d:c$; $\therefore$ inversely $a:a\pm b::c:c\pm d$.

TH. 8. If four quantities be proportional, then the sum of the first and second is to their difference as the sum of the third and fourth is to their difference. For, by theorem 6, $\frac{a+b}{b}=\frac{c+d}{d}$; and $\frac{a-b}{b}=\frac{b-d}{d}$; invert the last two fractions, then $\frac{b}{a-b}=\frac{d}{c-d}$ hence $\frac{a+b}{b}\times\frac{b}{a-b}=\frac{c+d}{d}\times\frac{d}{c-d}$, or $\frac{a+b}{a-b}=\frac{c+d}{c-d}$; $\therefore$ by Art. 80, $a+b:a-b::c+d:c-d$.

TH. 9. If four quantities be proportional, and any equimultiples or equal parts whatever be taken of the first and second, and also of the third and fourth, then will the resulting quantities taken in the same order, be still proportional. For let $a:b::c:$

d; then, by Case I, Art. 82, the ratio of $ma : mb$ is the same with the ratio of $a : b$; and for the same reason, the ratio of $nc : nd$ is the same with the ratio of $c : d$; hence, (Article 78,) $ma : mb :: nc : nd$, where m and n may be any quantities whatever, either integral or fractional.

Th. 10. The same theorem is true, if any equimultiples or equal parts whatever be taken of the first and third, and also of the second and fourth; for since $\frac{a}{b}=\frac{c}{d}$, multiply each side of the equation by $\frac{m}{n}$, then $\frac{ma}{nb}=\frac{mc}{nd}$; $\therefore ma : nb :: mc : nd$, where m and n may be any quantities whatever, either integral or fractional.

Th. 11. If four quantities be proportional, any powers or roots of those quantities will also be proportional. For since $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a^n}{b^n}=\frac{c^n}{d^n}$; $\therefore a^n : b^n :: c^n : d^n$, where n may be any number, either integral or fractional.

Th. 12. If the corresponding terms of two sets of proportionals be multiplied together, or divided by each other, the resulting quantities, taken in order, will still be proportional. Thus, let

$$\left.\begin{array}{c} a : b :: c : d \\ \text{and} \\ e : f :: g : h \end{array}\right\} \left.\begin{array}{c} \text{then } \frac{a}{b}=\frac{c}{d} \\ \\ \text{and } \frac{e}{f}=\frac{g}{h} \end{array}\right\} \text{hence } \frac{ae}{bf}=\frac{cg}{dh}, \text{or } ae : bf :: cg : dh.$$

Again, by theorem 1, $ad=bc$, and $eh=fg$; $\therefore \frac{ad}{eh}=\frac{bc}{fg}$; hence by theorem 2, $\frac{a}{e}:\frac{b}{f}::\frac{c}{g}:\frac{d}{h}$. The same will evidently be true of any number of proportions.

Th. 13. If there be two rows of proportional quantities, wherein the second and fourth terms of the first row are the same with the first and third terms of the second row, then will the remaining quantities, taken in order, be proportional. Thus,

let $a : b :: c : d$,

and $b : e :: d : f$; then, by theorem 12, $ab : be :: cd : df$,

or, (reducing each ratio to its lowest terms,) $a : e :: c : f$.

Th. 14. If there be a set of proportional quantities, $a : b :: c : d :: e : f :: g : h$, &c., then will the first be to the second as the sum of all the antecedents to the sum of all the consequents. For, since $ab=ba$, and (by theorems 1 and 5,) $ad = bc$,

$af=be$, $ah=bg$, &c., we have $ab+ad+af+ah+$&c.$=ba+bc+be+bg+$&c., or $a(b+d+f+h+$&c.$)=b(a+c+e+g+$&c.$)$ $\therefore$ (by Th. 2,) $a : b :: a+c+e+g+$&c. : $b+d+f+h+$&c.

Th. 15. If $a : b :: b : c :: c : d :: d : e$ &c., as in article 79, then $a : c :: a^2 : b^2$, or in the duplicate ratio of $a : b$;
$a : d :: a^3 : b^3$, or in the triplicate ratio of $a : b$;
$a : e :: a^4 : b^4$, or in the quadruplicate ratio of $a : b$;
&c., &c., &c.

For $a : b :: a : b$;
and $b : c :: a : b$;
$\therefore$ by Th. 12, $a : c :: a^2 : b^2$.

Again, $a : c :: a^2 : b^2$;
but $c : d :: a : b$;
$\therefore$ Th. 12, $a : d :: a^3 : b^3$.

Moreover, $a : d :: a^3 : b$
but $d : e :: a : b$
$\therefore$ by Th. 12, $a : e :: a^4 : b^4$, &c. &c. &c.

Arithmetical Progression is when a series of numbers or quantities increase or decrease by the same common difference.

Thus, 1, 3, 5, 7, 9, &c. and a, $a+d$, $a+2d$, $a+3d$, &c., are increasing series in arithmetical progression, the common differences of which are 2 and d.

And 15, 12, 9, 6, &c. and a, $a-d$, $a-2d$, $a-3d$, &c. are decreasing series in arithmetical progression, the common differences of which are 3 and d.

The most useful properties of arithmetical proportion and progression are contained in the following theorems:

1. If four quantities are in arithmetical proportion, the sum of the two extremes will be equal to the sum of the two means.

Thus, if the two proportionals be 2, 5, 7, 10, or a, b, c, d; then will $2+10=5+7$, and $a+d=b+c$.

2. And if three quantities be in arithmetical proportion, the sum of the two extremes will be double the mean.

Thus, if the proportionals be 3, 6, 9, or a, b, c, then will $3+9=2\times6=12$, and $a+c=2b$.

3. Hence, an arithmetical mean between any two quantities is equal to half the sum of those quantities.

Thus, an arithmetical mean between 2 and 4 is $=$

$$\frac{2+4}{2}=3;\text{ and between 5 and 6, it is }=\frac{5+6}{2}=5\tfrac{1}{2}.$$

And an arithmetical mean between a and b is $\frac{a+b}{2}$.

4. In any continued arithmetical progression, the sum of the two extremes is equal to the sum of any two terms that are equally distant from them, or to double the middle term, when the number of terms is odd.

Thus, if the series be 2, 4, 6, 8, 10, then will $2+10=4+8=2\times6=12$.

If two or more arithmetical means between any two quantities be required, they may be expressed as below.

Thus, $\frac{2a+b}{3}$ and $\frac{a+2b}{3}$ = two arithmetical means between a and b, a being the less extreme and b the greater.

And $\frac{na+b}{n+1}$, $\frac{(n-1)a+2b}{n+1}$, $\frac{(n-2)a+3b}{n+1}$, &c. to $\frac{a+nb}{n+1}$ = any number (n) of arithmetical means between a and b; where $\frac{b-a}{n+1}$ is the common difference; which being added to a, gives the first of these means; and then to this last, gives the second; and so on.

And, if this series be a, $a+d$, $a+2d$, $a+3d$, $a+4d$, then will $a+(a+4d)=+d+(a+3d)=2(a+2d)$.

5. The last term of any increasing arithmetical series is equal to the first term plus the product of the common difference by the number of terms less one; and if the series be decreasing, it is equal to the first term minus that product.

Thus, the last term (l) of the series a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c. continued to n terms, is $l=a+(n-1)d$.

And the last term of the series a, $a-d$, $a-2d$, $a-3d$, $a-4d$, &c. to n terms, is $l=a-(n-1)d$.

6. The sum of any series of quantities in arithmetical progression is equal to the sum of the two extremes multiplied by half the number of terms. Thus, the sum of the series 2, 4, 6, 8, 10, 12, is $=(2+12)+\frac{6}{2}=14\times3=42$.

And if the series be $a+(a+)+(a+2d)+(a+3d)+$, &c. to n terms, we shall have $s=(a+l)\times\frac{1}{2}n$, where l is the last term, n the number of terms, and s the sum of the series.

Or, the sum of any increasing arithmetical series may be found by adding the product of the common difference by the number of terms less one, to twice the first term, and then multiplying the result by half the number of terms.

And, if the series be decreasing, its sum will be found by subtracting the product of the common difference by the number of terms less one, from twice the first term, and then multiplying the result by half the number of terms, as before.*

* The sum of any number of terms (n) of the series of natural numbers 1, 2, 3, 4, 5, 6, 7, &c. is $=\frac{n(n+1)}{4}$. Thus, $1+2+3+4+5+$, &c. continued to 100 terms, is $=\frac{100\times101}{2}=50\times101=5050$.

Thus, if the series be $a+(a+d)+(a+2d)+(a+3d)+(a+4d)$ $+$, &c. continued to n terms, we shall have $s=\{2a+(n-1)d\}\times\frac{n}{2}$. And, if the series be $a+(a-d)+(a-2d)$ $+(a-3d)+(a-4d)+$, &c. continued to n terms, we shall have

$$s=\{2a-(n-1)d\}\times\frac{n}{2}.$$

CASE I. The sum of a series of quantities in arithmetical progression is found by multiplying the sum of the first and last terms by half the number of terms.

Let $a=$ first term, $d=$ common difference,
$n=$ number of terms, and $s=$ sum of the series.

$$a+a+d+a+2d+a+3d \text{ \&c.} \ldots\ldots\ldots a+(n-1)d=s$$
$$a+(n-1)d+a+(n-2)d+a+(n-3)d+, \text{ \&c.} \ldots . a=s$$

By addition $2a+(n-1)d+2a+(n-1)d+$&c. to n terms $=2s$

$$\therefore \{2a+(n-1)d\} \,.\, n=2s.$$

Whence $s=\{2a+(n-1)d\}\cdot\frac{n}{2}=\{a+a+(n-1)d\}\frac{n}{2}$

$$=\{\text{first term} + \text{last term}\}.\frac{\text{No. of terms}}{2}.$$

Cor. 3. Any three of the four quantities s, a, n, d, being known, the fourth may be found.

If s, d, n be given, ——— $a=\frac{2s-dn(n+1)}{2n}$.

$s=\{2a+(n-1)d\}\frac{n}{2}$; by actual multiplication, $2an+dn^2-nd=2s$, or $2an=2s-dn^2+dn$; $\therefore a=\frac{2s-n^2d+dn}{2n}$.

If a, s, n be given, we find d $\therefore d=\frac{2}{n}.\frac{s-an}{n-1}$.

$bn^2-bn=2s-2an$, or $(n^2-n)d=2s-2an$; $\therefore d=\frac{2s-2an}{n^2-n}$.

If s, d, a be given ——— $n=\frac{\sqrt{\{(2a-d)^2+8sd\}}-(2a-d)}{2d}$.

To find n, we have, by transposition, $dn^2+2an-dn=2s$, or $dn^2+(2a-d)n=2s$; $\therefore n^2+\frac{2a-d}{d}\times n=\frac{2s}{d}$; and, by case 1, quadratics, we have $n=\frac{d-2a}{2d}\pm\frac{1}{2d}\sqrt{\{(2a-d)^2+8ds\}}$.

Cor. 2. If the series be a decreasing one, d is negative, and $s=\{2a-(n-1)d\}\cdot\frac{n}{2}$.

1. The first term of an increasing arithmetical series is 3, the common difference 2, and the number of terms 20. Required the sum of the series.

First, $3+2(20-1)=3+2\times19=3+38=41$, the last term, and $(3+41(\times\frac{20}{2}=44\times\frac{20}{2}=44\times10=440$, the sum. Or, $\{2\times2+2(20-1)\times\frac{20}{2}=(6+2\times19)\times10=(6+38)\times10=44\times10=440$, as before.

2. The first term of an increasing arithmetical series is 100, the common difference 3, and the number of terms 34. Required the sum of the series.

First, $100-3(34-1)=100-3\times33=100-99=1$, the last term, and $(100+1)\times\frac{34}{2}=101\times\frac{34}{2}=101\times17=1717$, the sum required. Or, $\{2\times100-3(34-1\})\times\frac{34}{2}=(200-3\times33)\times17=(200-99)\times17=101\times17=1717=$ sum, as before.

3. Required the sum of the odd numbers 1, 3, 5, 7, 9, &c. continued to 101 terms.

Here the formula $s=\{2a+(n-1)a\}\frac{n}{2}$ becomes $\{2+100\times2\}\times\frac{101}{2}=202\times\frac{101}{2}=101^2=10201$.

4. How many strokes do the clocks of Venice, which go on to 24 o'clock, strike in the compass of a day?

Here $s=(a+l)\frac{n}{2}=(1+24)\times12=300$, Ans.

5. The first term of a decreasing arithmetical series is 10, the common difference $\frac{1}{3}$, and the number of terms 21. Required the sum of the series. Ans. 140.

Here the formula, $l=a+(n-1)d$, becomes $l=2+(365-1)2=2+728=730$, Ans.

6. One hundred apples being placed on the ground, in a straight line, at the distance of a yard from each other; how far will a person travel, who shall bring them, one by one, to a basket placed at the distance of a yard from the first apple?

Here the 1st term is $1+1=2$, and the last $100+100=200$, the number of terms 100. Therefore $(a+l)\frac{n}{2}=(2+200)50=10100$ yards, or 5 miles 1300 yards, Ans.

7. Required the 365th term of the series of even numbers 2, 4, 6, 8, 10, 12, &c. Ans. 730.

Also, the sum of any number of terms (n) of the series of odd numbers, 1, 3, 5, 7, 9, 11, &c. is $=n^2$. Thus, $1+3+5+7+9+$, &c., continued to 50 terms, is $=50^2=2500$.

To this we may add, that if any three of the quantities a, d, n, s, be given, the fourth may be found from the equation

$s=\{2a\pm(a-1)d\}\times\frac{n}{2}$, or $(a+l)\times\frac{n}{2}$, where the upper sign $+$ is to be used when the series is increasing, and the lower sign $-$ when it is decreasing; also the last term $l=a\pm(n-1)d$, as above.

Here $s=\{2a-(n-1)d\}\frac{n}{2}=\{20-20\times\frac{1}{3}\}\times\frac{21}{2}=(20-\frac{20}{3})\times\frac{21}{2}=\frac{40}{3}\times\frac{21}{2}=140$, Ans.

The following problems and theorems contain the whole practice of arithmetical progression, with the rules already given.

Given l or a = the first or least term, g = the greatest term, n = the number of terms, s = the sum of the term or series, and b or d = the common excess or difference.

Case 1. Given l, g, and n, to find s and d.

Theo. 1. $(l+g)\frac{n}{2}=s$. Theo. 2. $\frac{g-l}{n-1}=d$.

Problem 1. Given the least term, the greatest term, and the number of terms, of an arithmetical progression, to find the sum of the terms.

Rule. To the least term add the greatest, multiply the sum by half the number of terms, and the product will be the sum of the terms.

Prob. 2. Given the least term, the greatest term, and the number of terms, to find the common excess or difference.

Rule. Divide the difference between the greatest and the least term by the number of terms less unity, and the quotient will be the common excess, or difference.

Case 2. Given l, g, and s, to find n and d.

Theo. 3. $\frac{2s}{g+l}=n$. Theo. 4. $\frac{(g+l)\times(g-l)}{2s-(g+l)}=d$.

Case 3. Given l, g, and d, to find n and s.

Theo. 5. $\frac{g-l}{a}+1=n$. Theo. 6. $\frac{(g-l)}{d}+1\times\frac{g+l}{2}=s$.

Prob. 3. Given the least term, the greatest term, and the common exeess, or difference, to find the number of terms.

Rule. Divide the difference between the greatest and the least term, by the common excess, or difference, the quotient, increased by an unit, will give the number of terms.

Prob. 4. Given the greatest term, the number of terms, and the common excess, or difference, to find the least term.

Rule. Multiply the common excess, or difference, by the num-

Note 1. If three numbers are in arithmetrical progression, the sum of the extremes will be equal to double the mean; and the product of the extremes, increased by the square of the common difference, will be equal to the square of the mean. Thus, if 5, 7, 9, are in arithmetical progression, then will $5+9=7\times2$, and $(9+5)+(2+2)=7\times7$.

2. If four numbers are in arithmetical progression, the sum of

ber of terms less 1; subtract the product from the greatest term, and the remainder will be the least term.

Case 4. Given l, n, and s, to find g and d.

Theo. 7. $\frac{2s}{n}-l=g$. Theo. 8. $\frac{(s-ln)\times 2}{(n-1)\times n}=\frac{2s-2ln}{n^2-n}=d$.

Case 5. Given l, n, and d, to find g and s.

Theo. 9. $l+(n-1)\times d=g$. Theo. 10. $\{(nd-d)+2l\}\times\frac{n}{2}=s$.

Prob. 5. Given the number of terms, the common excess, or difference, and the sum of the terms, to find the least term.

Rule. Divide the sum of the terms by the number of terms; and, from the quotient subtract half the product of the common excess, or difference, by the number of terms less 1; the remander will be the least term.

Case 6. Given l, s, and d, to find g and n.

$\frac{1}{2}\sqrt{\{8ds+(2l-d)^2\}}-\frac{1}{2}d=g$. $\frac{\sqrt{\{8ds+(2l-d)^2\}}-(2l-d)}{2d}=n$.

Case 7. Given g, n, and s, to find l and d.

Theo. 13. $\frac{2s}{n}-g=l$. Theo. 14. $\frac{(ng-s)\times 2}{(n-1)\times m}=\frac{2ng-2s}{n^2-n}=d$.

Prob. 6. Given the least term, the number of terms, and the common excess, or difference, to find the greatest term.

Rule. Multiply the number of terms by the common excess, or difference, and to that product add the least term; from this sum subtract the common excess, or difference, and the remainder will be the greatest term.

Case 8. Given g, n, and d, to find l and s.

Theo. 15. $g-(n-1)d=l$. Theo. 16. $\{(2g+d)-nd\}\times\frac{n}{2}=s$.

Case 9. Given g, s, and d, to find l and n.

Theo. 17. $\frac{1}{2}\sqrt{\{(2g+d)^2-8ds\}}+\frac{1}{2}d=l$. 18. See p. 186, cor. 1.

Case 10. Given n, s, and d, to 9nd l and g.

Theo. 19. $\frac{s}{n}-\{(n-1)\times\frac{1}{2}d\}=l$. 20. $\frac{s}{n}+\{(n-1)\times\frac{1}{2}d\}=g$.

---

the two extremes will be equal to the sum of the means. Thus, if 2, 5, 8, 11, are in arithmetical progression,

Then will $2+11=5+8$.

3. If a series of numbers (consisting of any number of terms) are in arithmetical progression, the sum of the extremes will always be equal to the sum of any two means equidistant from the extremes; or to double the mean if the terms be odd.

Thus, if 3, 5, 7, 9, 11, 13, &c. are in arithmetical progression,

Geometical Progression.

85 When a series of numbers increase by a common multiplier, or decrease by a common divisor, those numbers are said to be in geometrical progression; such as 2, 4, 8, 16, *&c.* *&c.* The first and last terms are usually called the extremes, and the common multiplier or divisor the ratio.

Note 1. If three numbers be in geometrical progression, the product of the two extremes will be equal to the square of the mean. Thus, if 3, 9, 27, be in geometrical progression,

Then will $3\times27=9\times9$.

2. If four numbers be in geometical progression, the product of the two extremes will be equal to the product of the means.

Thus, if 2, 4, 8, 16, be in geometrical progression,

Then will $2\times16=4\times8$.

3. If a series of numbers, consisting of any number of terms, be in geometrical progression, the product of the two extremes will be equal to the product of any two means equidistant from the extremes; or to the square of the mean, if the terms be odd.

Thus, if 1, 2, 4, 8, 16, 32, *&c.* be in geometrical progression,

Then will $1\times32=2\times16=4\times8$.

Or, if 1, 2, 4, 8, 16, *&c.* be in geometrical progression,

Then will $1\times16=2\times8=4\times4$.

4. If, out of any series of numbers in geometrical progression, there be taken any series of equidistant terms, that series will likewise be in geometrical progression.

Thus, if 2, 4, 8, 16, 32, 64, *&c.* be in geometrical progression,

Then will 4, 16, 64, *&c.* be in geometrical progression.

PROBLEM I. Given the number of terms the ratio, and either

Then will $3+13=5+11=7+9$. Or 1, 4, 7, 10, 13, &c. are in arithmetical progression, then will $1+13=4+10=7\times2$.

4. If, out of any series of numbers in arithmetical progression, there be taken any series of equidistant terms, that series will also be in arithmetical progression.

Thus, if 2, 4, 6, 8, 10, 12, 14, &c. are in arithmetical progression, then will 4, 8, 12, &c. be in arith. prog.

5. In any series of numbers in arithmetical progression the common excess, or difference, is as often repeated as there are terms in the progression, wanting one; viz. every term, except the first, is continually increased or diminished by the common excess or difference.

6. The rules for an ascending or descending series are the same; for a descending series becomes an ascending one by beginning at the least term.

of the extreme terms, of a limited geometrical series, to find the other extreme.

Rule. Write down a few terms of a geometrical series, beginning with, and formed by the given ratio; over which place the arithmetical series 1, 2, 3, 4, 5, &c. as indices; observe what figures of these indices, when added together, will give a number an unit less than that expressing the number of terms; and find the product of the terms in the geometrical series which stand under these indices. This product multiplied by the first term given in the question, or the first term divided by this product, according as the progression is increasing or decreasing, will give the term sought.

Prob. II. Given one extreme, the ratio, and the number of terms, of a geometrical series, to find the sum of the terms.

Rule. Find the other extreme by Problem 1. Then divide the difference between the extremes by the ratio less 1; the quotient increased by the greater extreme will give the sum of the terms.

Prob. III. In any series of numbers in geometrical progression, decreasing ad infinitum, given the first term and the ratio, to find the sum of the series.

Rule. Subtract the second term from the first; the square of the first term, divided by this difference, will give the sum of the series.

If $l=$ the least term,	$s=$the sum of the terms.
$g=$ the greatest;	$r=$the ratio.
$n=$ the number of terms.	$log=$logarithm of any letter.

Then will the following theorems exhibit all the possible cases of geometrical progression, including those already given.

86. If a series of quantities increase or decrease by continual multiplication or division by the same quantity, then those quantities are said to be in geometrical progression. For the exponents of r are a series of numbers in arithmetical progression, beginning with zero, or nought, and increasing by unity in each succeeding term, as 0, 1, 2, 3; and therefore the exponent of the nth term is $n-1$; consequently, if the nth term be represented by l, then will $l=ar^{n-1}$.

To find the sum of n terms of a geometrical progression, let $a=$ first term, r the common ratio, $n=$ the number of terms, and $s=$ the sum of (n) terms. Thus, the numbers 1, 2, 4, 8, 16, &c. which increase by continual multiplication by 2, and the numbers 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c. which decrease by

continual division by 3, or multiplication by $\frac{1}{3}$, are in geometrical progression, as before.

87. In general, if a represents the first term, r the common multiple or ratio, then may the series itself be represented by a, ar, a^2r, ar^3, ar^4, &c., which will evidently be an increasing or a decreasing series, according as r is a whole number or a proper fraction. In the foregoing series, the index of r in any term is less by unity than the number which denotes the place of that term in the series. Hence, if the number of terms in the series be denoted by n, the last term will be ar^{n-1}.

88. Let s be the sum of the series a, ar, ar^2, ar^3, &c.; then
$a + ar + ar^2 + ar^3 +$ &c. .... $ar^{n-2} + ar^{n-1} = s$.

Multiply the equation by r, and it becomes
$ar + ar^2 + ar^3 + ar^4 +$ &c..... $ar^{n-2} + ar^{n-1} + ar^n = rs$.

Subtract the upper equation from the lower, or the series which is the value of s from the series which is the value of rs, and we have

$$ar^n - a = rs - s, \text{ or } (r-1)s = ar^n - a; \text{ and } s = \frac{ar^n - a}{r-1}.$$

If r is a proper fraction, then r and its powers are less than 1. For the convenience of calculation, therefore, it is better in this case to transform the equation, into $s = \frac{a - ar^n}{1-r}$, by multiplying the numerator and denominator of the fraction $\frac{ar^n - a}{r-1}$ by -1.

89. If l be the last term of a series of this kind, then $l = ar^{n-1}$; $\therefore rl = ar^n$; hence $s = \left\{ \frac{ar^n - a}{r-1} \right\} = \frac{rl - r}{r-1} = l + \frac{l-a}{r-1}$.

From this equation, therefore, if any three of the four quantities s, a, r, l, be given, the fourth may be found. For $s = \frac{rl-a}{r-1}$; $a = rl - (r-1)s$; $r = \frac{s-a}{s-l}$, and $l = \frac{(r-1)s+a}{r}$. The value of n cannot be found from the equation $s = \frac{ar^n - a}{r-1}$, except by means of logarithms, thus: See page 193, Art. 92.

Find the sum of the series 1, 3, 9, 27, &c. to 12 terms.

Here $a=1$, $r=3$, $n=12$. and

$$s = \frac{ar^n - a}{r-1} = \frac{1 \times 3^{12} - 1}{3-1} = \frac{81^3 - 1}{2} = \frac{531441 - 1}{2} = \frac{531440}{2} = 265720.$$

90. Let l be the last term of an arithmetic series, whose first term is a, common difference d, and number of terms n; then

$l=a+(n-1)d$; $\therefore (n-1)d=l-a$, or $d=\frac{l-a}{n-1}$. Now the number of intermediate terms between the first and the last is $n-2$; let $n-2=m$, then $n-1=m+1$. Hence $d=\frac{l-a}{m+1}$, which gives the following rule for finding any number of arithmetic means between two numbers. Divide the difference of the two numbers by the given number of means increased by unity, and the quotient will be the common difference. Having the common difference, the means themselves will be known. See page 290.

91. The general expression for the sum of a geometric series whose common ratio (r) is a fraction, is $s=\frac{a-ar^n}{1-r}$. Suppose now n to increase indefinitely; then r^n (r being a proper fraction) will decrease indefinitely; therefore ar^n will decrease indefinitely with respect to a, or a will be the limit of $a-ar^n$, and $\frac{a}{1-r}$ the limit of $\frac{a-ar^n}{1-r}$, or s; and consequently $\frac{a}{1-r}$ will express the value of the series when the number of its terms is supposed to be indefinitely increased, or, as it is commonly called, the sum of the series ad infinitum. See page 312.

92. If the sum of the series, the common ratio, and the first term be given, the number of terms may be found thus:

Since $rs-s=ar^n-a$, or $ar^n=rs-s+a$, and $r^n=\frac{rs-s+a}{a}$.

$$\therefore \log. r^n \text{ or } n\times\log. r=\log. (rs-s+a)-\log. a.$$

$$\text{Hence } n=\frac{\log. (rs-s+a)-\log. a.}{\log. r}.$$

1. The sum of a geometric series is 6560, its first term 2, and common ratio 3. What is the number of terms?

Here $s=6560$, $a=2$, $r=3$; $\therefore n=\frac{\log. (rs-s+a)-\log. a}{\log. r}$

$$=\frac{\log. 13122-\log. 2}{\log. 3.}=\frac{3.8169700}{.4771213}=8, \text{ Ans.}$$

2. A person undertakes a journey of 364 miles, going one mile the first day, three the second, nine the third, and so on. When will he arrive at his journey's end?

Here $s=364$, $a=1$, $r=3$; $\therefore n=\frac{\log. (rs-s+a)-\log. a}{\log. r}$

$$=\frac{\log. 728-\log. 3}{\log. 1}=\frac{2.862728}{.477121}=6, \text{ days Ans.}$$

Cubic Equations.

Let $dx^3+cx^2+bx=N$ be any cubic equation, and let, as in the solution of quadratics $x=r+y$, or $y+r=x$, then

$$\left.\begin{array}{r} by+br=bx \\ cy^2+2cry+cr^2=cx^2 \\ dy^3+3dry^2+3dr^2y+dr^3=dx^3 \end{array}\right\}=N$$

$$dy^3+c'y^2+b'y+A=N \quad \text{Sum.}$$

Or, $dy^3+c'y^2+b'y=(N-A)=N'$

Again, $y=s+z$, then

$$\left.\begin{array}{r} c'z+b's=b'y \\ b'z^2+2b'sz+b's^2=b'y^2 \\ dz^3+3dsz^2+3ds^2z+ds^2=dy^3 \end{array}\right\}=N$$

$$dz^3+c''z^2+c''z+A'=N' \quad \text{Sum.}$$

See page 144-5

Given $bx^2+ax=n$

b	a
br	$+a$
$2br$	$+a=a'$
b	a'
bs	$+a'$
$2bs$	$+a'=a''$.

Or $dz^3+c''z^2+b''z=(N'-A')=N'$ Here it is evident that these equations are similar to the first, and thus we may successively find the values of r, s, t, &c. &c. the figures of the root. Now to find the denominators, put down the coefficients in a line thus,

d	c	b	d	c'	b'
	$dr+c$,	r^2+cr+b		$ds+c'$,	$ds^2+c's+b'$,
	$2dr+c$,	$3r^2+2cr+b=b$,		$2ds+c'$,	$3ds^2+2c's+b'$,
	$3dr+c=c'$.			$3ds+c'$.	
	&c.	&c.			&c.

And if we suppose $d=1$, then the equation will be $x^3+cx^2+bx=n$. And then the values of r, s, t, &c. are as follows:

$$\frac{N}{r^2+cr+b}=r;\ \frac{N'}{s^2+c's+b'}=s;\ \frac{N''}{t^2+c''t+b''}=t, \text{ \&c. \&c.}$$

Now, by substituting the values of c', b', c'', b'', &c. we have

1. $$\frac{N}{r^2+cr+b}=\frac{N}{r(r+c)+b}=r.$$

2. $$\frac{N'}{s^2+(b+3r)s+3r+2br+b}=\frac{N'}{s(s+3r+c)+3r^2+2cr+b}=s.$$

3. $$\frac{N''}{t^2+c''t+b''}=\frac{N''}{t(t+c+3(r+s))+3s^2+2c's+b}=t.$$

4. $$\frac{N'''}{u^2+c'''u+b'''}=\frac{N'''}{u\{u+c+3(r+s+t)\}+3t^2+2c''t+b''}=u.$$

Hence it is evident how these formulas may be continued at pleasure, and all the quantities contained in the same are known; and since, in the second formula, $3r^2+2cr+b$ form a part of the divisor, this may be used as a trial divisor for finding s, and so for the rest. An example will render these formulas very easy.

Given $x^3+9x^2+4x=80$, to find x.

Here 2 is the first figure of the root found by trial.

$c=9,\ b=4,\ N=80,\ r=2.$

	$b = 4$	$c = 9.$	Root
	$r(r+c) = 22$	80(2.47213596	
First divisor	$= r^2+cr+b = 26$	52 $r.\,stu$ &c.	
	$r^2 = 4.$	$28 = N'$	
$s^2+c's = s(s+3r+b) =$	6.16	23.264	
2d divisor is	$= s^2+c's+b' = 58.16$	$4.736 = N'$	
	$s^2 = .16$	4.593323	
	$t(t+c+3(r+s) = 1.1389$	.142677	*398
3d divisor is	$t^2+c''t+b'' = 65.6189$	.133591	334
And so on,	49	9086	64
(9+(2.47×3)+.002)2 = 16·4|12	.032824	6683	60
Fourth divisor is	66,795|5	2403	4
16.3161 × 1 ········1|6·4161	2	2005	4
Fifth divsior is.	|6|6·|8|3|0	*398	

Hence, from these formulas, we derive the following rule. Put down b the coefficient of x, and a little to the right place the absolute number, which is to be considered as a dividend, the figures of the root forming the quotient. Place the first figure of the root found by trial in the quotient, above which write the coefficient of x^2, observing that its unit's place be over the unit's place of the quotient. Multiply the value of the quotient figure, taking in those above by that value; add the product to b, and the sum is the first divisor. Write the square of the quotient figure just found under the first divisor. Find now the next figure of the root, and to its value (including those above it) prefix three times the preceding, taking in the value of the figure above it. Multiply the result by the last found figure, add the product to the three sums immediately above, and we shall have the *second* divisor; and in the same manner are the succeeding divisors to be obtained.

Given $x^3+4x^2+2x=2328$, to find the value of x.

Here the highest denomination of the root found by trial is 10.

	b	$4 = c$
	2	2328 (12 root.
14....140		142
Divisor	142	908
	100	908
36.... 72		
Divisor	454	

Here it must be observed, as in quadratics that the 1 in the quotient must be considered 10 thus, $(r+a)r = (10+4)\times 10 = 140$; and so if were in the hundred's place it must be considered as 100, &c.

Biquadratic Equations.

For the solution of equations of a higher order, the principle is

nearly the same. Let $Ex^4+Dx^3+Cx^2+Bx=N$ be any biquadratic equation; let r be the first figure in the root; then we shall have $\frac{N}{Er^3+Dr^2+Cr+B}=r$. Now let y denote the remaining figures of the root, then $y+r=x$, and by substitution,

$$\begin{array}{rrrrrl}
 & & & By & +\ Br & = Bx \\
 & & Cy^2 & +\ 2Cry & +\ Cr^2 & = Cx^2 \\
 & Dy^3 & +\ 3Dry^2 & +\ 3Dr^2y & +\ Dr^3 & = Dx^3 \\
Ey^4 & +\ 4Ery^3 & +\ 6Er^2y^2 & +\ 4Er^3y & +\ Er^4 & = Ex^4 \\
\hline
Ey^4 & +\ D'y^3 & +\ C'y^2 & +\ B'y & +\ A & = N.
\end{array}$$

Or, $Ey^4+D'y^3+C'y^2+B'y=N'$, an equation similar to the one proposed; and if S be the first figure of the root, then we shall have $\frac{N'}{ES^3+D'S^2+C'S+B'}=S$. Now to find the denominators, put down the coefficients in a line, thus;

$$\begin{array}{llll}
E & D & C & B \\
 & Er+D, & Er^2+Dr+C, & Er^3+Dr^2+Cr+B \\
 & 2Er+D, & 3Er^2+2Dr+C, & 4Er^3+3Dr^2+2Cr+B=B' \\
 & 3Er+D, & 6Er^2+3Dr+C=C' & \\
 & 4Er+D=D', & &
\end{array}$$

$$\begin{array}{llll}
E' & D' & C' & B' \\
 & Es+D', & Es^2+D's+C', & Es^3+D's^2+C's+B' \\
 & 2Es+D', & 3Es^2+2D's+C', & 4Es^3+3D's^2+2C's+B' \\
 & 3Es+D', & 6Es^2+3D's+C', & \\
 & 4Es+D, & \&c. & \&c.
\end{array}$$

Hence it is evident how these lines are formed, viz. by multiplying the succeeding terms by r, s, &c. and adding them respectively. By this means figurative numbers of the 2nd, 3d, &c. degrees are formed; (see Simpson's Alg.) and so we may proceed to an equation of any degree. Thus, let there be given the equation, $Fx^5+Ex^4+Dx^3+Cx^2+Bx=N$. Let r be the first figure of the root. Then $r+y=x$, or $y+r=x$.

$$\begin{array}{rrrrrrl}
 & & & & By & +\ Br & = Bx \\
 & & & Cy^2 & +\ 2Cry & +\ Cr^2 & = Cx^2 \\
 & & Dy^3 & +\ 3Dry^2 & +\ 3Dr^2y & +\ Dr^3 & = Dx^3 \\
 & Ey^4 & +\ 4Ery^3 & +\ 6Er^2y^2 & +\ 4Er^3y & +\ Er^4 & = Ex^4 \\
Fy^5 & +\ 5Fry^4 & +\ 10Fr^2y^3 & +\ 10Fr^3y^2 & +\ 5Fr^4y & +\ Fr^5 & = Fx^5 \\
\hline
Fy^5 & +\ E'y^4 & +\ D'y^3 & +\ C'y^2 & +\ B'y & +\ A & = N
\end{array}$$

Or, $Fy^5+E'y^4+D'y^3+C'y^2+B'y=N'$, an equation similar to the proposed one whose root s must be such, that if N' be divided by $Fs^4+E's^3+D's^2+C's+B'$, the quotient will be s. Now, to find the denominator of these fractions, multiply by r, s, &c., as before. Thus,

$$\begin{array}{llllllllllllll}
F & E & & & D & & & & C & & & & & B \\
Fr + & E, & Fr^2 + & Er + & D, & Fr^3 + & Er^2 + & Dr + & C; & Fr^4 + & Er^3 + & Dr^2 + & Cr + & B \\
2Fr + & E, & 3Fr^2 + & 2Er + & D, & 4Fr^3 + & 3Er^2 + & 2Dr + & C; & 5Fr^4 + & 4Er^3 + & 3Dr^2 + & 2Cr + & B = B' \\
3Fr + & E, & 6Fr^2 + & 3Er + & D, & 10Fr^3 + & 6Er^2 + & 3Dr + & C = C' \\
4Fr + & E, & 10Fr^2 + & 4Er + & D = D', \\
5Fr + & E = E'.
\end{array}$$

$$\begin{array}{llllllllllllll}
F & E' & & & D' & & & & C' & & & & & B' \\
Fs + & E' & Fs^2 + & E's + & D' & Fs^3 + & E's^2 + & D's + & C' & Fs^4 + & E's^3 + & D's^2 + & C's + & B' \\
2Fs + & E' & 3Fs^2 + & 2E's + & D' & 4Fs^3 + & 4E's^2 + & 2D's + & C' & 5Fs^4 + & 4E's^3 + & 3D's^2 + & 2C's + & B' = B' \\
3Fs + & E' & 6Fs^2 + & 3E's + & D' & 10Fs^3 + & 6E's^2 + & 3D's + & C' \\
4Fs + & E' & 10Fs^2 + & 4E's + & D' \\
5Fs + & E', & & \&c. & & & \&c. & & & & \&c.
\end{array}$$

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From these principles being found to hold good in all equations whatever, we can determine the following general rule. RULE. Arrange the coefficients of the given equation in a row, commencing with that of the first term.

Add the product of the first root figure, found by trial, and the first coefficient to the second coefficient; the product of the sum and same figure to the third coefficient, and so on to the last coefficient, and the last sum will be the divisor. Repeat this process with the first coefficient and these sums, and the number under the last sum will be the trial divisor for the next figure. Perform a similar process with the first coefficient and these second sums, stopping under the $n - 1$th coefficient. Perform again a similar process with the same first coefficient and these last sums, stopping here under the preceding, or $n - 2$th coefficient, and so on till the last sum falls under the second coefficient. Find now from the trial divisor the next figure of the root, and proceed with the last set of sums, and this new figure exactly the same as with the original coefficients and the first figure in finding the preceding divisor, and the next divisor will be obtained; and in a similar manner are the other divisors to be determined. Lastly, let us take the general equation. Let $Bx^n + \ldots . Ex^4 + Dx^3 + Cx^2 + Bx = N$ be an equation of any degree whatever, and let r be such a number in either of the series 0, 1, 2, 3, &c.; or

series 0, 1, 2, 3, ; or 10, 20, 30, &c. ; that when it and the next succeeding number are separately substituted for x in the equation, the results shall be the one less, and the other greater than N; then r will be the first figure of one of the roots of the equation ; and if N be divided by $Rr^{n-1}+\ldots Er^3+Dr^2+Cr+B$, the quotient must be r. Suppose such a value of r is found, and let y represent the succeeding figures of the root ; then $y+r=x$, and therefore

$$
\begin{array}{rl}
 & By + Br = Bx, \\
 & Cy^2 + 2Cry + Cr^2 = Cx^2, \\
 & Dy^3 + 3Dry^2 + 3Dr^2y + Dr^3 = Dx^3, \\
 & Ey^4 + 4Ery^3 + 6Er^2y^2 + 4Er^3y + Er^4 = Ex^4, \\
 & Ry^n + \ldots \\
\hline
 & Ry^n + \ldots E'y^4 + D'y^3 + C'y^2 + B'y + A = N,
\end{array}
$$

and, by transposition,

$$Ry^n + \ldots E'y^4 + D'y^3 + C'y^2 + B'y = N - A = N',$$

an equation similar to the one proposed, the first figure s, in the root y of which must be such, that if N' be divided by $Rs^{n-1}+\ldots E's^3+D's^2+C's+B'$, the quotient will be s; if, therefore, we suppose s to be found, and if z be put for the remaining figures of the root, we shall, by proceeding as before, get another equation $Rz^n+E''z^3+C''z^2+B''z=N''$; also similar to the first ; and if we continue this process, we may obtain, one by one. all the figures of the root x, and it is evident that each divisor will be similarly formed from the coefficients of the corresponding equation, and the new figure of the root.

Now it is obvious that $B'x$ is the nth term in the equation $Ry^n+\ldots E'y^4+D'y^3+C'y^2+B'y=N'$, or that the column represented by B' is the nth column from the left, and that it consists of n terms ; the column represented by C' is the $n-1$th, and consists of $n-1$ terms, &c. ; also each of these columns, omitting the numeral parts, is equal to the preceding multiplied by r, plus the corresponding coefficient in the proposed equation ; consequently, since the first column is simply R, if the second, third, &c. coefficients of the proposed equation be Q, P, &c. respectively, then the first, second, third, &c. columns, without the numeral coefficients, will be R, $Rr+Q$, Rr^2+Qr+P, &c. ; but if the first term in this series be multiplied by r, and the product added to the second, and the result be multiplied by r, and the product added to the third &c. &c. that the proper numeral coefficients will be obtained, because the above columns of numeral coefficients are the same as the several binomial columns, except that the above are written in a reverse order, that is, from right to left, and in this reverse order they will be produced by adding the series, as above. See Emerson Algebra Ed. 1764 page 287.

Given $x^3+90x^2+2700x=21168$, to find x. $+90$

2700	21168(6.384761	
96....576	19656	[967
Div. 3276	1512	
36	1176.147	
108.3..32.49	335.853	
Div. 3920.49	316.943072	
9	18.909928	
108.98..8.7184	15.883799	
Div. 3961.7884	3.026129	
64	2.780024	
10|9.1|44..4366	246105	
Div. 3970.9498	238292	
77	267*	7813
Div. 3971.463	288	3972
Div. 3|9|7|1.|54	29	3841
	28	3574*

Given $x^3+2x^2-3x=9$, to find x. $+2$

—3	9(1.9394650585
3..3	8.379
Div. 0	.621
1	469857
5.9..5.31	151143
Div. 9.31	143684019
81	7458981
7.73..2319	6415316
Div.15.6619	1043665
9	962514
7.799..70191	81151
Div.15.964891	80212
81	939
7.8|17...313	802
Div.16.03829	137
5	128
16.0419	9
1,6.0,4,2,4	8

Given $x^3+x^2=500$, to find x to 10 or 12 places. $+1$

0	500(7.61727975	
8......56	392	[594
Div. 56	108	
49	104.736	
22.6.13.56	3.264	
Div.174.56	1.887181	
36	1.376819	
23.81..23.81	1.323862113	
Div.188.7181	52956887	
1	37858967	
23.837..166859	15097920	
Div.189.123159	13251090	
49	1846830	
23|85|12..4770	1703729	
Div.189.294837	1124	143101
167	946	132512
Div.189.30128	178	10589
2	170	9465
Div.1,8,9.3,0,3,2		

Given $x^3+10x^2+5x=260$, to find x. $+10$

5	260(4.11798686	
14.....56	244	
Div. 61	16	
16	13.521	
22.1....2.21	2.479	
Div. 135.21	1.376531	
1	1.102469	
22.31.....2231	966091	
Div. 137.6531	136378	
1	124372	
2|2.3|37...1570	120	12006
Div. 138.013	111	11057
21	9	949
138.191	8	829
1,3,8,.2,1		120

A and B between them owe \$240. A pays eight dollars a day, and B pays the first day \$1, the second 2, the third 3, and so on. In how many days will they clear the debt, and how much did each of them owe? Ans. 15 days.

Given $x^8 + 10x^7 + 21x^6 - 55x^5 - 100x^4 + 525x^3 + 804x^2 - 630x = 216$ to find x.

<table>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>root</td></tr>
<tr><td>10</td><td>21</td><td>—55</td><td>—100</td><td>525</td><td>804</td><td>—630</td><td>216(.79128784748</td></tr>
<tr><td>.7</td><td>7.49</td><td>19.943</td><td>— 24.5399</td><td>—87.17793</td><td>306.475449</td><td>777.3328143</td><td>103.13297001</td></tr>
<tr><td>10.7</td><td>28.49</td><td>—35.057</td><td>—124.5399</td><td>437.82207</td><td>1110.475449</td><td>147.3328143</td><td>112.86802999</td></tr>
<tr><td>7</td><td>7.98</td><td>25.529</td><td>— 6.6696</td><td>—91.84665</td><td>242.182794</td><td>946.8607701</td><td>111.09195214</td></tr>
<tr><td>11.4</td><td>36.47</td><td>— 9.528</td><td>—131.2095</td><td>345.97542</td><td>1352.658243</td><td>1094.1935844</td><td>1.77507785</td></tr>
<tr><td>.7</td><td>8.47</td><td>31.458</td><td>15.3510</td><td>—81.10095</td><td>185.412129</td><td>140.1614394</td><td>1.37786939</td></tr>
<tr><td>12.1</td><td>44.94</td><td>21.930</td><td>—115.8585</td><td>264.87447</td><td>1538.070372</td><td>1234.355023|8</td><td>.49720836</td></tr>
<tr><td>.7</td><td>8.96</td><td>37.730</td><td>41.7620</td><td>—51.86755</td><td>19.278955</td><td>141.917499</td><td>27595723</td></tr>
<tr><td>12.8</td><td>53.90</td><td>59.660</td><td>— 74.0965</td><td>213.00692</td><td>1557.34932|7</td><td>1376.272523</td><td>12125114</td></tr>
<tr><td>7</td><td>9.45</td><td>44.345</td><td>72.8035</td><td>1.20369</td><td>19.51177</td><td>1.596962</td><td>11041868</td></tr>
<tr><td>13.5</td><td>63.35</td><td>104.005</td><td>— 1.2930</td><td>214.2016|1</td><td>1576.86110</td><td>1377.86948|5</td><td>1083245</td></tr>
<tr><td>7</td><td>9.94</td><td>51.303</td><td>14.6673</td><td>2.5869</td><td>19.8748</td><td>1.59719</td><td>966260</td></tr>
<tr><td>14.2</td><td>73.29</td><td>155.308</td><td>13.3743</td><td>216.797|5</td><td>1596.7359</td><td>1379.46667</td><td>116985</td></tr>
<tr><td>7</td><td>10.43</td><td>7.662</td><td>15.368</td><td>4.033</td><td>.2265</td><td>31949</td><td>100430</td></tr>
<tr><td>14.9</td><td>83.72</td><td>162.97|0</td><td>28.74|3</td><td>220.83|1</td><td>1596.062|4</td><td>1379.7861|6</td><td>6555</td></tr>
<tr><td>7</td><td>1.41</td><td>7.79</td><td>16.07</td><td>5.54</td><td>226</td><td>1380.1057</td><td>5522</td></tr>
<tr><td>15.6</td><td>85.1|3</td><td>170.7|6</td><td>44.8|1</td><td>226.37</td><td>1597.18|9</td><td>1278</td><td>1033</td></tr>
<tr><td>1</td><td>1.4</td><td>7.8</td><td>16.8</td><td>8</td><td>1597.41</td><td>1380.133|5</td><td>966</td></tr>
<tr><td>1|5.|7</td><td>86.|5</td><td>178.|6</td><td>66.|6</td><td>2|2|6.4|5</td><td>5</td><td>1380.361</td><td>67</td></tr>
<tr><td></td><td>2</td><td>8.</td><td>18</td><td></td><td>1|5|9|7|4|6</td><td>11</td><td>55</td></tr>
<tr><td></td><td>8|8</td><td>18|7</td><td>7|5</td><td></td><td></td><td>1380.372</td><td>12</td></tr>
<tr><td></td><td>9</td><td>20</td><td></td><td></td><td></td><td>1380.38</td><td>11</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td></tr>
</table>

Given $x^3 - 27x = 36$, to find x.

		0 root.
	—17	36(5.7658216
5.....	25	—10
Div.	—2	46
	25	41.293
15.7...	10.99	4.707
Div.	58.99	4.289976
	49	417024
17.16...	1.0296	363096
Div.	71.4996	53928
	36	50903
	864	3025
Div.	72.6192	2909
	12	116
Div.	72.718	73
Div.	7,2.7,3	43
		43

Given $x^3 + 5x^2 + 29x = 1829$, to find x.

		+5
	29	1829(10.09024
15....	150	1790 [296
Div.	179	39
	100	38.894229
35.09...	3.1581	105771
Div.	432.1581	87066
	81	18705
35|2702....	71	17414
Div.	435.3314	1291
	4,3,5,.3,4	871
		420
		392
		28
		26

10.09024296 root true to the last.

Given $x^3 - 9x = 12$, to find x.

	—9		12(3.5223339
3....	9		0
Div.	0		12
	9		11.375
9.5	..4.75		625
Div.	22.75		559208
	25		65792
10.52 .	. 2104		56384648
Div.	27.9604		9407352
	4		8464986
10.562	..21124		942366
Div.	28.192324		846603
	. 4		95763
	3169		84661
Div.	28.216621	96	11102
	32	85	8466
Div.	28.22011	11	2636
Div.	2,8.2,2,0,4	9	2540

Given $x^3 - 5x^2 + 2x = -12$, to find x.

			—5
	+2		—12(—1.22606
—6...	+6		— 8 [79
Div.	+8		— 4
	+1		— 3.328
—8.2....	1.64		—672
Div.	16.64		=557367
	4		—114633
—8.63....	2589		—113345
Div.	18.5789		—1288
	9		—1137
—8.696....	522	18	—151
Div.	18.8909	17	—133
	1,8,.9,4,3		18

Two farmers sell two sorts of corn: A sells 5 bushels; B re-receives in all for his \$36. Now, says B to A, if we add the number of my bushels to the number of your dollars, the sum will be 28. Says A to B, and if I add the square of my dollars to the square of your bushels, the sum will be 424. How many bushels did B sell, and how many dollars did A receive? See Index.

Ans. B sold 18 bushels, and A received \$10.

A. Given $x^6+2x^5+3x^4+4x^3+5x^2+6x=654321$, to find x.

3	4	5	6	654321(8.9569
80	664	5344	42792	342384
83	668	5349	42798	311937
144	1816	19872	201768	288370.478241
227	2484	25221	244566	23566.521759
208	3480	47712	75845.64249	20640.905917
435	5964	72933	320411.64249	2925.615842
272	5656	11339.9361	86879.31345	2514.350990
707	11620	84272.9369	407290.95594	411.264852
336	979.929	12259.6344	5527.16239	377.850317
1043	12599.929	96532.5705	412818.1183|3	33.414535
45.81	1021.887	13217.7510	5566.9694	29.395994
1088.81	13621.816	109750.3215	418388.0877	4.018541
46.62	1064.574	792.9263	673.4017	3.779571
1135.43	14686.390	110543.2478	419058.498|4	.238970
47.43	1107.990	796.141	673.989	209977
1182.86	15794.380	111339.38|8	419732.487	28993
48.24	64.146	799.36	101.198	25197
1231.10	15858.526	112138.75	419833.68|5	3796
49.05	64.29	96.36	101.21	3780
1280.15	15922.81	112235.1|1	419934.90	16
2.77	64.4	96.4	7.87	13
1282.9|2	15987.|2	112331.5	419942.7|7	3
2.8	65	96	7.9	3
1285.7	16052	112428	419950.7	Ans.
3	8	14	1.0	8.956979569036
128|8	1606|0	11244|2	419951.7	
1|2|9	1|6|0|7	1|1|2|4|5	4|1|9|9|3	

2 8 10 8 18 8 26 8 34 8 42 8 50.9 9 51.8 9 52.7 9 53.6 9 54.5 9 55.4

A man and his wife could drink a barrel of beer in 15 days; but after drinking together 6 days, the woman alone drank the remainder in 30 days. In what time could either alone drink the whole barrel?

Since the man and his wife had been drinking together 6 days, and they could drink the whole in 15 days, it is evident they drank $\frac{6}{15}$ or $\frac{2}{5}$ of the whole; and the woman drank $1-\frac{2}{5}$ or $\frac{3}{5}$ in 30 days, or the whole in 50 days, or 3 times in 150 days. And by the question, they could both drink the whole 10 times in 150 days, since the man alone could drink the whole 7 times in 150 days, or once in $21\frac{3}{7}$.

C. Given $16x^5 - 20x^3 + 5x = .5$, or $x^5 - 1.25x^3 + .3125x = .03125$, to find x, which gives the sine of 6°, and .5 being the sine of 30°.

<table>
<tr><td>0</td><td>—1.25</td><td>0</td><td>.3125</td><td>.03125</td><td>(.10452846</td></tr>
<tr><td>.1</td><td>.01</td><td>—.124</td><td>—124</td><td>.03001</td><td>[678535.</td></tr>
<tr><td>.1</td><td>—1.24</td><td>—.124</td><td>.3001</td><td>124</td><td></td></tr>
<tr><td>.1</td><td>.02</td><td>—.122</td><td>—246</td><td>1096086529024</td><td></td></tr>
<tr><td>.2</td><td>—1.22</td><td>—.246</td><td>.2755</td><td>143913470976</td><td></td></tr>
<tr><td>.1</td><td>.03</td><td>—.119</td><td>—1478367744</td><td>136166734100</td><td></td></tr>
<tr><td>.3</td><td>—1.19</td><td>—365</td><td>.274021632256</td><td>7746736876</td><td></td></tr>
<tr><td>.1</td><td>.04</td><td>4591936</td><td>—1498302976</td><td>5442744082</td><td></td></tr>
<tr><td>.4</td><td>—1.15</td><td>—369591936</td><td>.27252332928|0</td><td>2304022794</td><td></td></tr>
<tr><td>.1</td><td>002016</td><td>—4583808</td><td>—18986108</td><td>2177000312</td><td></td></tr>
<tr><td>.504</td><td>—1.147984</td><td>—374575744</td><td>.27233346820</td><td>127022482</td><td></td></tr>
<tr><td>4</td><td>2032</td><td>—4575616</td><td>—19014640</td><td>108848736</td><td></td></tr>
<tr><td>.508</td><td>—1.145952</td><td>—3791513610</td><td>.2721433218|0</td><td>18173746</td><td></td></tr>
<tr><td>4</td><td>2048</td><td>—57079</td><td>76177</td><td>16327301</td><td></td></tr>
<tr><td>.512</td><td>—1.143904</td><td>—37972215</td><td>.2621357041</td><td>1846445</td><td></td></tr>
<tr><td>4</td><td>2064</td><td>—57066</td><td>—76182</td><td>1632730</td><td></td></tr>
<tr><td>.516</td><td>—1.14184|0</td><td>3802928|1</td><td>.272128085|9</td><td>213715</td><td></td></tr>
<tr><td>4</td><td>26</td><td>—5706</td><td>—3047</td><td>190485</td><td></td></tr>
<tr><td>.520</td><td>—1.14158</td><td>—380863|4</td><td>.272125039</td><td>23230</td><td></td></tr>
<tr><td></td><td>26</td><td>—23</td><td>—3047</td><td>21770</td><td></td></tr>
<tr><td></td><td>—1.1413|2</td><td>—380886</td><td>.27212199|2</td><td>1460</td><td></td></tr>
<tr><td></td><td></td><td>— 23</td><td>—15</td><td>1361</td><td></td></tr>
<tr><td></td><td></td><td>—3'8|0|9|1</td><td>.27212184</td><td>99</td><td></td></tr>
<tr><td></td><td></td><td></td><td>.2|7|2|1|2|1|6|9</td><td>82</td><td></td></tr>
</table>

In an isosceles triangle, there is given the sum of the base and side of the inscribed square = 18 poles, and the difference of their areas = 84 square poles. To determine the triangle and square.

Put $a = 84$, $b = 18$, and x for the side of the inscribed square. Then will $b-x$ be the base of the triangle by the question. And $b - 2x$ the difference between the base and side of the square. Then, by similar triangles, $b-2x : b-x :: x : \frac{b-x}{2-2x} \times x =$ the perpendicular of the triangle. Consequently, $\frac{(b-x)^2}{b-2x} \times x$ is its double area; from which taking $2x^2$ the double area of the □, we have $\frac{(b-x)^2}{b-2x} \times x - 2x^2 = 2a$, by the question.

Hence $5x^3 - 72x^2 + 66x = 3024$. And $x = 7.534$.

* $\frac{(b-x)^2 \times x}{b-2x} - 2x^2 = 2a$, or $\frac{(b^2-2bx+x^2)x}{b-2x} - 2x^2 = 2a$, or $(b^2-2bx+x^2)x-(b-x)-2x^2 = 2a(b-2x)$, or $b^2x-2bx^2+x^3-2bx^2+4x^3 = 2ab-4ax$, or $5x^3-4bx^2+(b^2+4a)x = 2ab$. Hence $5x^3-72x^2+660x = 3024$. Ans. $x = 7.534$, &c.

Let the side of the square be $2x$, and the base of the triangle $2x+2y$. Then, by similar triangles, we have $y : 2x :: x : \frac{2x^2}{y}$; hence $\frac{2x^2}{y}+2x$ = the perpendicular. Now, by the question, $\frac{2x^3}{y}+2xy=84$, and $4x+2y=18$, or $y=9-2x$; this value of y substituted in the other equation, we have $5x^3-36x^2+165x=378$. Hence $x=3{\cdot}767$; then $2x=7{\cdot}534$ the side of the square, and $2x+2y$ or 10.465373 the base of the triangle. Otherwise,

Put $a=18$, $b=84$, and x for the base of the triangle. Then will $a-x$ be the side of the square, and $(a-x)^2$ its area. Again, by similar triangles, $2x-a : x :: a-x : \frac{ax-x^2}{2x-a}$ the perpendicular of the triangle; and therefore its area is $\frac{ax^2-x^3}{4x-2a}$. Then, by the question, $\frac{ax^2-x^3}{4x-2a}-(a-x)^2=b$. This reduced is $x^3-39{\cdot}6x^2+585{\cdot}6x=2937.6$; in which x is $=10{\cdot}4654$ the base of the triangle. Consequently $\frac{ax-x^2}{2x-a}=26.9009$ its perpendicular, and $a-x=7{\cdot}5344$ the side of the square, as before.

Given $x^3-39{\cdot}6x^2+585{\cdot}6x=2937{\cdot}6$, or $x^3-7{\cdot}2x^2+33x=75.6$ to, find one value of x, in each equation.

	— 7·2		— 39·6
+33	75·6(3·76719989	+585·6	2937·6(10·46537
—4.2..—12·6	61·2	— 29·6··· — 296·0	2896.0 304
Divisor 20·4	14·4	First divisor 289·6	41·6
9	12·985	100	35·968
+2·5···· 1·75	1·415	—9·2 .... 3·68	5 623
2d. Div. 18.55	1·261659	2d. Divisor. 89·92	5 153976
49	153344	16	478024
+3·96 ··· 2376	149082	— 8·34... —5004	426809
3d Div. 21.0276	4262	3d Divisor. 85·8996	51215
36	2133	36	51190
4·0|87 ···· 286	19 2130	—8·2|15....— 411	25
4th. Div. 21·297|4	17 1918	4th. Divisor. 85·3617	17
4.1011···· —4	211		8
—21|.327	192		

D. Given $(\frac{1}{5}x^2-15)^2+x\sqrt{x}=90$, or $\frac{1}{625}x^8-\frac{12}{25}x^6+\frac{234}{5}x^4-x^3-1620x^2=-18225$, to find x,

Or $x^8-300x^6+29250x^4-625x^3-1012500x^2=-11390625$, to find x.
Root, 10.5950369.

<table>
<tr><td>0</td><td>—300</td><td>0</td><td>29250</td><td>—625</td><td>—1012500</td><td>0</td><td>—11390625</td></tr>
<tr><td>10</td><td>100</td><td>—2000</td><td>—20000</td><td>92500</td><td>918750</td><td>—937500</td><td>—9375000</td></tr>
<tr><td>10</td><td>—200</td><td>—2000</td><td>9250</td><td>91875</td><td>—93750</td><td>—937500</td><td>—2015625</td></tr>
<tr><td>10</td><td>200</td><td>0</td><td>—20000</td><td>107500</td><td>—156250</td><td>—2500000</td><td>—1724211</td></tr>
<tr><td>20</td><td>0</td><td>—2000</td><td>—10750</td><td>—15625</td><td>—250000</td><td>—3437500</td><td>—291414</td></tr>
<tr><td>10</td><td>300</td><td>3000</td><td>10000</td><td>—7500</td><td>—231250</td><td>—10921</td><td>—276627</td></tr>
<tr><td>30</td><td>300</td><td>1000</td><td>—750</td><td>—23125</td><td>—481250</td><td>—344842|1</td><td>—14787</td></tr>
<tr><td>10</td><td>400</td><td>7000</td><td>80000</td><td>792500</td><td>459409</td><td>25868</td><td>—14679</td></tr>
<tr><td>40</td><td>700</td><td>8000</td><td>79250</td><td>769375</td><td>—2184|1</td><td>—318974</td><td>—108</td></tr>
<tr><td>10</td><td>500</td><td>12000</td><td>200000</td><td>149443</td><td>53920</td><td>11611</td><td>—88</td></tr>
<tr><td>50</td><td>1200</td><td>20000</td><td>279250</td><td>91881|8</td><td>5173|5</td><td>—30736|3</td><td>—20</td></tr>
<tr><td>10</td><td>600</td><td>18000</td><td>16636</td><td>15959</td><td>6242</td><td>1299</td><td>—17</td></tr>
<tr><td>60</td><td>1800</td><td>38000</td><td>29888|6</td><td>10784|0</td><td>11416</td><td>—29437</td><td>—3</td></tr>
<tr><td>10</td><td>700</td><td>1271</td><td>2019</td><td>1700</td><td>1885</td><td>80</td><td>—3</td></tr>
<tr><td>70</td><td>2500</td><td>3927|1</td><td>31910|7</td><td>1248|4</td><td>1290|1</td><td>—2935|7</td><td></td></tr>
<tr><td>10</td><td>40</td><td>130</td><td>209</td><td>181</td><td>153</td><td>9</td><td></td></tr>
<tr><td>8|0</td><td>254|0</td><td>405|7</td><td>340|0</td><td>142|9</td><td>1443</td><td>—2|6|26</td><td>Root.</td></tr>
<tr><td></td><td>4</td><td>418</td><td>21</td><td>19</td><td>15</td><td colspan="2">Ans. x=10.5950369.</td></tr>
<tr><td></td><td>25|8</td><td>43</td><td>361</td><td>162</td><td>1459</td><td></td><td>15.59</td></tr>
<tr><td></td><td>1</td><td></td><td></td><td>165</td><td>1|7</td><td></td><td></td></tr>
<tr><td></td><td>2|6</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Three men traded together with a stock of \$6500. A's money was in trade 3 months, B's 4 months, and C's 5 months. They divide their stock and gain, whereof A took for his stock and gain \$3900; B received for his stock and gain \$2700; and C's gain amounted to \$750. Find each man's stock.

Solution. Let x and y denote A and B's stock, respectively; then $3900-x$=A's gain, and $2700-y$; B's gain, and C's gain is $=750$. Hence the question may be solved in this manner, thus;

$(3900-x)\times4\times5=78000-20x$=A's gain and product,
$(2700-y)\times3\times5=40500-15y$=B's gain and product,
$750\ \times\ 3\times4=\ 9000$ =C's gain and product.
The sum, $127500-20x-15y$ of the products.

As $\left\{127500-20x-15y : 6500 :: \left\{\begin{array}{l}78000-20x : x\\ 40500-15y : y\end{array}\right.\right.$

$127500x - 20x^2 - 15xy = 6500(78000 - 20x$
$127500y - 20xy - 15y^2 = 6500(40500 - 15y$ } by the question.

or $25500x - 4x^2 - 3xy = 1300(78000 - 20x$
$25500y - 4xy - 3y^2 = 1300(40500 - 15y$ } By $\times$ and transposition we have

$51500x - 4x^2 - 3xy = 101400000$
$45000y - 4xy - 3y^2 = 52650000$ } and from this equation I have

$3xy = 51500x - 4x^2 - 10140000$, or $y = \dfrac{51500x - 4x^2 - 101400000}{3x}$

and by substituting this value of y in this equation, viz. $45000y - 4xy - 3y^2 = 52650000$, and then reducing, we have the final equation, (see 15th example, Multiplication,)

$x^3 - 34550x^2 + 226200000x = 395460000000$, to find x.

```
—34550                            —34550
+ 3000    22620000    395460000000($3017·82 3794 Root, A's.
—31550   —9465000     394650000000
Divisor.  131550000      810000000
            9000000      456446000
—25540··— 255400         353554000
Divisor.   45644600      316474263
                100       37079737
—25513 ···—178591         36009296
Divior.    45210609        1070441
                 49         899816
—25498|8.2····20399  4161   170625
Divisor.   45011619  4049   134971
—25496·58  ····51     112    35654
           449908|1    90    31493
           4|4|9|9|0|2 22     4161
                       18
```

```
Given x³+2x²—23x=70.
              +2
        —23·     70(5·134578
7 ...35          68
    —12          7.371
      25         2.629
17.1··· 1·71     2·278497
      73·71      .350503
         1       306161
17.33··5199      44342
    75·9499       3839
         9        6032
17.394··696       5364
    76.540|3
```

Given $x^3 - 7x = \pm 7$ to find 3 values of x.

```
— 7     —7(1.356895    —7      —7(1.69202146    —7      7(3·0489
  1     —6               1     — 6              3..9     6    169
— 6     —1             —6      —1                 2.     1
  1     — .903           1     — 1·104            9.    .814464
33.. ·99  —97          3·6.. 2· 16  ·104        9·04 ...3616·185528
— 3.01  —86625           — 1·84  ·100809          20·3616 166382
    9    10375                     3191              ·16   19154
3·95..1975  9049        4.89... 4401  3156888   9.128... 730  18791
—1·7325  —1326            1.1201   34112          20·797|8     353
     25  —1185          5·07··· 10144  31774    9.1449... 82   909
4·05...243                —1·578444               20·8791   19 144
                                                            19 125
                          1.588693                             19
```

Given $3x - x^3 = 1$, or $x^3 - 3x = -1$, to find x true to the last place.

```
                 0          Root.
  —3.        —1(.347296355
3  ,  .09     —0.873
 —2.91       —.127
    .09      —.107696
.94   376    —19304
 —2.6924     —18522077
     16       —781923
 1.027...7189 —527713
 —2.646011    —254210
       49     —237444
 1.04|12.. 208  —16766
 —2.63856,5     —15829
 1|0|4169....94  —14 —937
 2.|6|3|8|2|6|3  —13 —791
                   —146
                   —132
```

Given $x^3 + 24, 84x^2 - 67.613x = 3761.2758$, to find x to about 10 places of figures.

```
                          +24.84
     —67.613   3761.2758(11.1
34.84  348.4   2807.87           9733222
Div.   280.787  953.405
       100.     785.027
55.84..55.84    168.3788
Div.   785.027   84.7661
         1       83.6127
57.94... 5.794   77.283513
Div.   847.661    6.329187
          1       6.050544
58.23...5.2407     .278643
Div.   858.7057    .259437
          81        19206
58.41....4089       17296
Div.   864.3634      1910
          17         1730
       864.789        180
   8|6|4|.8|1         173
                        7
```

Given $x^3 - 15x^2 + 63x = 50$, to find x.

```
                   —15
        63.        50(1.02803923
—14  —14.          49        Root.
Div.  49.           1.
       1.          .715208
—11.98 —.2396      .284792
Div.   35.7604     283405952
         4           1386048
—11,932..—95456      1059900
Div.   35.425744      326148
         64           317966
—11.9|1597 —358    56  8182
Div.   35.329994   35  7066
   3|5|3|2|9|6     21  1116
                   21  1060
```

Given $x^3 + 9x = 30$, to find the value of x.

```
                 0.
         9       30(2·1808497702
2····  4         26    Root. 559
        13        4·
         4        2·161
6·1···· 61        1.839
Divisor 21·61     1.819232
          1        19768
6·38 ···· 5104    18609946112
Divisor 22·7404    1158053888
         64         930717101
6·5408 ····523264   227336787
Divisor 23·26243264 209414233
         64          17922554
6.5|42|44···· 26170  16287818
Divisor 23.26792752   1634736
6.54|2529 ···· 5888   1628782
Divisor 23.26824810      5954
6. 5 425477 ····458      4654
Divisor 23.26831156      1300
         46              1163
D.  2|3|.2|6|8|3|2        137
                          116
                           21
```

Given $x^3 - 33.1295x^2 - 372.4157x = -1590.3345$, or $x^3 - 6x^2 + 18x = 22$, to find x.

— 33.1295
—372·4157 —1590·3345(3·3652012 — 6
—30·1295 190·3885 —1388·4126 18 22(2·327481580
Divisor —462·8042 —201·9219 —4·· — 8 20 3
+9 —165·399465 D 10 2
—23.8295 —7.14885 —36·522435 4 1··827
Divisor — 551·33155 —33·586836 0·3··· ·09 ·173
+.09 —2·935599 D 6·09 125768
—23·1695 —1.3902 —2·806410 9 47232
Divisor —559.7806 —·129189 .92··· 184 44197783
+36 —·112280 D 6·2884 3034217
—115 —16909 4 2528464
Divisor 561·282 —16842 ·967 ··· 6769 505753
67 D 6·313969 505730
56 49 23
11 .98|14 ··· 393 19
Div. 6.321160 4
98228 ··· 78
|6|.3|2|1|6|3

Given $x^3 - 18\frac{1}{12}x = -29\frac{53}{108}$
Or $x^3 - 18.08\dot{3}x = -29.49\dot{0}7\dot{4}$. to find x.

—18.08$\dot{3}$ . . 0. root
2……4 —29.49$\dot{0}$74(2.333
—28.16$\dot{6}$ [333333
Div. —14.08$\dot{3}$
4. —1.324$\dot{0}$74
6.3…. 1.89 —1.257$\dot{9}$
Div. —4.19$\dot{3}$ —66074
9 —60162$\dot{9}$
6.93…..2079 —5911074
Div. —2.0054$\dot{3}$ —5326962
9 —584112
6.993…..20979 —525770
Div. —1.775654 —58342
9 —52507
6.99|93….210 —5835
—1.75256 —5251
2 6 —584
—1.7,5,0,2 5 —525
1 —59
—53

Given $x^3 - 24x^2 = -528.880073$, to find x.

0 —528.8800735*
—19. —95 —475.
Divisor —95 —53.880
25 —50.283
—8.7 —2.61 —3.597073
Divs. —167.61 —3.405832
9 —191241
—8.08..— 1616 —170461
Divs. —170.2916 —20780
4 —17047
—8,0.39 … — 80 —3733
Divs. — 170.460,9 —3409
—8 —324
Div. — 1,7,0.4,7,0 —171
—24. Ans. — 153
*$x = 5.321121924$, — 153
—19.

The root is 2.3333333333, true to 10 places, evidently showing that the true root is 2.3 or $2\frac{1}{3}$ exactly. The root is carried so far to show that the 3's are all repeaters.

2. Extract the root of the equation $x^5+6x^4-10x^3-112x^2-207x=110$. Here the first figure of the root is 4.

1	6	—10	—112	—207	110(4.46410161
	4	40	120	32	—700
	10	30	8	—175	810
	4	56	344	1408	667.05984
	14	86	352	1233	142.94016
	4	72	632	434.6496	133.46395
	18	158	984	1667.6496	9.47621
	4	88	102.624	477.4144	9.24089
	22	246	1086.624	2145.0640	23532
	4	10.56	106.912	79.3352	23158
	26.4	256.56	1193.536	2224.39912	374
	.4	10.72	111.264	80.389	232
	26.8	267.28	1304.800	2304.788	142
	.4	10.88	17.454	5.434	139
	27.2	278.16	1322.2513	2310.22|2	3
	.4	11.04	17.56	5.44	2
	27.6	289.20	1339.811	2315.66	
	.4	1.68	17.6	.14	
	2|8.|0	290.8|8	1357.4	2315.8|0	*1358.|6
		1.7	1.|2	.1	1
		2|9|2.|6	1358.|6*	2|2|3|1|5.9	1|3|6|0

Give $x^5+6x^4-10x^3-112x^2-207x=-110$ find the value of x.

1	26.3	246	984	1233	—110(4.36326146 root.
	3	7.89	76.167	318.0501	—700
	26.6	253.89	1060.167	1551.0501	+590
	.3	7.98	78.561	341.6184	465.31503
	26.9	261.87	1138.728	1892.6685	124.68497
	3	8.07	80.982	74-1897	118.01149
	27.2	269.94	1219.701	1966.858,2	6.67348
	.3	8.16	16.785	7.5202	6.13762
	27.5	278.10	1236.49,5	2042.060	53586
		1.65	16.88	3.814	40998
		279.75	1253.3,7	2945.87,3	12588
		1.7	17.0	3.81	12301
		281.4	1270.3	2049.68	287
		2.	.8	.25 this	205
		28,3	1,2,7,1	2049.93 add	82
				2,0,5.,0,2 sum	82
		18*		27	

Given $y^3-2.4y^2-3y=-.6$, to find y to 16 places.

```
                            —3.4
              — 3           —.6(.1691770039038507
— 3.3......— .33            — .333                root.
  Divisor.  — 3.33          — 267
                 1          — 229944
— 3.04.....  —1824          — 37056
  Divisor.  —3.8324         — 36336591
                36            — 719409
—2.911.........26199          — 406380629
  Divisor.  —4.037399         — 313028371
                  81          — 284560870787
— 2.8929.......—28929            — 28467500213
  Divisor.  — 4.06380629         — 28451646935
                     1              — 15853278
— 2.89263........2025841            — 12193624
  Divisor.  — 4.0642981541            — 3659654
                        49            — 3658087
—2.892843..........202574                — 1567
  Divisor.  —4.064520990                 — 1219
            —4.|0|6|4|5|4|1|247           — 348
                                          — 325
                                           — 23
                                           — 20
                                              3
```

Given $52x^4-753144$, to

```
     52
—441584
  52000
—389584
  52000
—337584
  52000
—285584
  52000
—233584
  20800
—212784
  20800
—191984
  20800
—171184
  20800
—150384
   2600
—147784
   2600
—145184
   2600
—142884
   2600
—139984
    416
—139568
    416
—139152
    416
—138736
    416
—138320
```

Given $x^3-240x^2+19200x=61115.5$, to find x.

```
                                —240          root.
            19200           61115.5(3.31888216121
— 237...    —711            55467
Divisor     18489            5648.5
add             9            5315.337
—230.7      — 69.21           333.163
Divisor     1771.79           176.463691
add               9           156.699309
—230.09     — 2.3009          141.1378224
Divisor     17646.3691         15.5614866
                     1         14.1121626
—230.0|62   — 1.8405            14493240
Divisor     17642.227|8         14112000
—23|0.0452    —1840               381240
Divisor     17640.20|33           352800
—|230.04352    —184                28439
Divisor     17640.0|009            17640
                 — 4               10800
Divisor     1.76|3|9.|9|8          10584
    Rem. 216—176=40—35=4.  215
```

$441584x^3 - 771997102.9985x^2 + 763080806290.03x = 1620020551$ find x to 18 or 20 places of decimals.

Root, 1458.015693276279949 6.

771997102.9985	763080806290.03	1620020551753144.0000
—389584000	382413102998.50	1145393909288530.0000
382413102.9985	1145493909288.53	474526642464614.0000
—337584000	44829102998.50	423990245394572.0000
44829102.9985	1190323012287.03	50536397070042.0000
—285584000	—130347308800.60	43749226491787.7500
—240754897.0015	1059975613486.43	6787170578254.2500
—85113600	—161064838800.6000	6773044690061.3440
—325868497.0015	898910774685.8300	13225888192.9060
— 76793600	— 23926244850.0750	8427769782.58374
—402662097.0015	874984529835.7550	4798118410.32226
— 68473600	— 24289204850.0750	4213848172.06165
—471135697.0015	850695324985.6800	584270238.26061
— 7389200	— 33952238728.0120	505660113.21862
—478524897.0015	846743086257.6680	78610125.04199
— 7259200	— 3961144456.0120	75848986.16508
—485784097.0015	842781941801.6560	2761138.87691
— 7129200	— 4962543.2821	2528299.40037
—492913297.0015	842776978258.3739	232839.47654
— 1116544	— 4962557.114	168553.29304
—494029841.0015	842772015701.059	64286.18350
— 1113216	— 2481288.030	58993.65256
— 49543057.0015	842769634412.32|9	5292.53094
— 1109888	— 2481292.39	5056.59879
—496252945.0015	842767153119.93	235.93215
— 1383.208	— 297755.58	168.55329
—496254428.21	842766855364.3|8	67.37886
— 1383.2	— 297755.6	58.99365
—496255711.4	842766559608.7	8.38521
— 1383.2	— 44663.3	7.58490
—496257094.6	842766512945.|4	.80031
— 691.6	— 44663.	75849
—496257786	84276646828|2.	418[illegible]
— 692	— 149	3371
—49625847|8	84276646679	811
— 69	— 149	759
—49625917	84276646530	52
— 8	— 10	52
—4|9|6|2|5|9|2|5	8|4|2|7|6|6|4|6|5|2|0	

B. Given $x^5 = 125000$, to find x, or to extract the sursolid of 125000.

				Root.
0	0	0	0	125000(10.456395526
10	100	1000	10000	100000
10	100	1000	10000	25000
10	200	3000	40000	21665.29024
20	300	4000	50000	3334.70976
10	300	6000	4163.2256	2952.90353
30	600	10000	54163.2256	381.80623
10	400	408.064	4329.7024	358.16663
40	1000	10408.064	58492.8280	23.63960
10	20.16	416.192	565.1425	19.92992
50.4	1020.16	10824.256	59058.9705	5.70968
4	20.32	424.384	567.860	5.37937
50.8	1040.48	11248.360	59625.930	33031
4	20.48	54.210	68.509	29886
51.2	1060.96	11302.850	59694.43|9	3145
4	20.64	54.34	68.54	2989
51.6	1081.60	11356.1|9	59762.98	156
4	2.60	54.5	3.43	120
52.0	1084.2|0	11411.6	59766.4|1	36
	2.6	6.5	3.4	35
	1086.|8	11418.|1	59769.8	1
	1|0|9|2	1|1|4|2|4	1	
			59770.8	

42. Given $x^3 - 16\frac{1}{3}x = 19\frac{11}{27}$, to find x, or $x^3 - 16.\dot{3}x = 19.\dot{4}0\dot{7}$.

		0.
	$-16.\dot{3}$	
−1....	1	$19.\dot{4}0\dot{7}$(−1.333333
	$-15.\dot{3}$	$15.\dot{3}$ or $-1.\dot{3}$
	1	$4.07\dot{4}$
−3.3....	.99	3.703
	$-12.34\dot{3}$	$.37107\dot{4}$
	9	334363
−3.93....	1179	36711
	$-11.1454\dot{3}$	33044
	9	3667
	0.120	3301
	−11.014	366
	−1|1.|0|0|2	330
		36
		33
		3
		3

Given $x^4 - 34{,}62x^3 + 300{,}6357x^2 - 34{,}62x + 299{,}6357 = 0$; to find one of the values x. Ans. $x =$

22 .Given $y^7-3x^6-2.5x^5+10x^4-x^3-9x^2+2x=2$, to find x to about 9 or 10 places. (7364

<table>
<tr><td>−3</td><td>−2.5</td><td>10</td><td>1</td><td>−9</td><td>2</td><td></td><td>2(2.62599-</td></tr>
<tr><td>2</td><td>−2</td><td>−9</td><td>2</td><td>6</td><td>−6</td><td>−8</td><td>Root</td></tr>
<tr><td>−1</td><td>−4.5</td><td>1</td><td>3</td><td>−3</td><td>−4</td><td>10</td><td></td></tr>
<tr><td>2</td><td>2</td><td>−5</td><td>−8</td><td>−10</td><td>−26</td><td></td><td>6.3112896</td></tr>
<tr><td>1</td><td>−2.5</td><td>−4</td><td>−5</td><td>−13</td><td>−30</td><td></td><td>3.6887104</td></tr>
<tr><td>2</td><td>6</td><td>7</td><td>6</td><td>2</td><td>40.518816</td><td></td><td>2.7866270</td></tr>
<tr><td>3</td><td>3.5</td><td>3</td><td>+1</td><td>−11</td><td>10.518816</td><td></td><td>.9020834</td></tr>
<tr><td>2</td><td>10</td><td>27</td><td>60</td><td>78.53136</td><td>120.543936</td><td></td><td>.7498790</td></tr>
<tr><td>5</td><td>13.5</td><td>30</td><td>61</td><td>67.53136</td><td>131.06275|2</td><td></td><td>1522044</td></tr>
<tr><td>2</td><td>14</td><td>55</td><td>69.8856</td><td>133.37520</td><td>8.26860</td><td></td><td>1373076</td></tr>
<tr><td>7</td><td>27.5</td><td>85</td><td>130.8856</td><td>200.90656</td><td>139.33135</td><td></td><td>148968</td></tr>
<tr><td>2</td><td>18</td><td>31.476</td><td>91.4064</td><td>202.79040</td><td>8.46560</td><td></td><td>137700</td></tr>
<tr><td>9</td><td>45.5</td><td>116.476</td><td>222.2920</td><td>403.69696</td><td>147.7969|5</td><td></td><td>11268</td></tr>
<tr><td>2</td><td>6.96</td><td>35.868</td><td>115.6920</td><td>9.73325</td><td>2.1789</td><td></td><td>10710</td></tr>
<tr><td>11.6</td><td>52.46</td><td>152.344</td><td>337.9840</td><td>413.43021</td><td>149.9758</td><td></td><td>558</td></tr>
<tr><td>6</td><td>7.32</td><td>40.476</td><td>142.8720</td><td>9.850</td><td>2.1915</td><td></td><td>459</td></tr>
<tr><td>12.2</td><td>59.78</td><td>192.820</td><td>480.8560</td><td>423.280</td><td>152.167|3</td><td></td><td>99</td></tr>
<tr><td>6</td><td>7.68</td><td>45.300</td><td>5.8064</td><td>9.968</td><td>397</td><td></td><td>92</td></tr>
<tr><td>12.8</td><td>67.46</td><td>238.120</td><td>486.662|4</td><td>433.248</td><td>152.564</td><td></td><td>7</td></tr>
<tr><td>6</td><td>8.04</td><td>50.340</td><td>5.843</td><td>2.529</td><td>397</td><td></td><td>6</td></tr>
<tr><td>13.4</td><td>75.50</td><td>288.460</td><td>492.5|06</td><td>435.7|7|7</td><td>152.96|1</td><td></td><td>1</td></tr>
<tr><td>6</td><td>8.40</td><td>1.859</td><td>5.9</td><td>2.5</td><td>4</td><td></td><td></td></tr>
<tr><td>14.0</td><td>83.90</td><td>290.319</td><td>498.4</td><td>438.3</td><td>1|5|3.|0|0</td><td></td><td></td></tr>
<tr><td>6</td><td>8.96</td><td>1.86</td><td>504.3</td><td>440.8</td><td></td><td></td><td></td></tr>
<tr><td>14.6</td><td>92.66</td><td>292.17</td><td>1.5</td><td>.5</td><td></td><td></td><td></td></tr>
<tr><td>6</td><td>30</td><td>1.9</td><td>505.|8</td><td>4|4|1.|3</td><td>296</td><td></td><td></td></tr>
<tr><td>1|5.|2</td><td>92.9|6</td><td>294.0</td><td>5|0|7</td><td></td><td>2|9|8</td><td></td><td></td></tr>
<tr><td></td><td>9|3.|2</td><td>2</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

23. Given $\sqrt[3]{(7x^3+4x^2)}=28-\sqrt{(20x^2-10x)}$ to find x.

Cubing $7x^3+4x^2=21952-2352\sqrt{(20x^2-10x)}+1680x^2-840x+(10x-20x^2)\sqrt{(20x^2-10x)}$. Transfer $7x^3-1676x^2+840x=21952+\sqrt{(20x^2-10x)}(10x-20x^2-2352)$. Or $7x^3-1676x^2+840x-21952=\sqrt{(20x^2-10x)}(10x-20x^2-2352)$. Again, $(7x^3-1676x^2+840x-21952)^2=(20x^2-10x)(10x-20x^2-2352)^2$. By actual involution and multiplication,

$$49x^6-23464x^5+2820736x^4-3123008x^3+74288704x^2-36879360x+481890304=8000x^6-12000x^5-1887600x^4-1882600x^3+111108480x^2-55319040x;$$

Therefore, by transposition we shall have the following equation of the 6th degree, or order: $7951x^6+11464x^5-933136x^4+1240408x^3+36819776x^2-18439680x=481890304$.

24. Given $7951x^6+11464x^5-933136x^4+1240408x^3+36819776x^2-1843968x=481890304$ to find x to 9 or 10 places.

<table>
<tr><td>7951</td><td></td><td></td><td></td><td></td><td>Root, 4.510662488</td></tr>
<tr><td>11464</td><td>—933136</td><td>1240408</td><td>36819776</td><td>—18439680</td><td>481890304</td></tr>
<tr><td>31804</td><td>173072</td><td>—3040256</td><td>—7199392</td><td>118481536</td><td>400167425</td></tr>
<tr><td>43268</td><td>—760064</td><td>—1799848</td><td>29620384</td><td>100041856</td><td>81722880</td></tr>
<tr><td>31804</td><td>300288</td><td>—1839104</td><td>—14555808</td><td>60258304</td><td>80020123</td></tr>
<tr><td>75072</td><td>—459776</td><td>—3638952</td><td>15064576</td><td>160300160</td><td>1702757</td></tr>
<tr><td>31804</td><td>427504</td><td>—129088</td><td>—15072160</td><td>—259915</td><td>1596963</td></tr>
<tr><td>106876</td><td>—32272</td><td>—3768040</td><td>—7584</td><td>16004024|5</td><td>105794</td></tr>
<tr><td>31804</td><td>554720</td><td>2089792</td><td>—512245</td><td>—33946</td><td>95815</td></tr>
<tr><td>138680</td><td>522448</td><td>—1678248</td><td>—51982|9</td><td>15970078</td><td>9979</td></tr>
<tr><td>31804</td><td>681936</td><td>653758</td><td>—15909</td><td>—445</td><td>9582</td></tr>
<tr><td>170484</td><td>1204384</td><td>—102449|0</td><td>—6789|1</td><td>1596963|3</td><td>397</td></tr>
<tr><td>31804</td><td>103132</td><td>78632</td><td>2210</td><td>— 43</td><td>319</td></tr>
<tr><td>202288</td><td>130751|6</td><td>—3181|7</td><td>—457|9</td><td>1|5|9|6|9|20</td><td>78</td></tr>
<tr><td>3976</td><td>10512</td><td>7600</td><td>12</td><td></td><td>64</td></tr>
<tr><td>20626|4</td><td>14126|4</td><td>441|8</td><td>— 44|5</td><td></td><td>14</td></tr>
<tr><td>398</td><td>1071</td><td>813</td><td>1</td><td></td><td>13</td></tr>
<tr><td>2102|4</td><td>1519|7</td><td>125|6</td><td>—4|3</td><td></td><td>1</td></tr>
<tr><td>40</td><td>109</td><td>2</td><td></td><td></td><td></td></tr>
<tr><td>214|2</td><td>162|9</td><td>1|2|7</td><td colspan="3">{4·510662488, root as required.}</td></tr>
</table>

25. Given $x^3 - 17x^4 + 54 = 350$, or $x^3 - 6x = 6$, to find x.

<table>
<tr><td>—17</td><td></td><td>—17</td><td>Ans.</td><td></td><td></td><td>+0</td><td>Ans.</td></tr>
<tr><td>14</td><td>54</td><td>350(14.95406861</td><td></td><td></td><td>—6</td><td>6(2847322</td><td></td></tr>
<tr><td>—3×14</td><td>—42</td><td>168</td><td>Root</td><td>2 ···</td><td>4</td><td>—4</td><td>Root</td></tr>
<tr><td>Divisor.</td><td>12</td><td>182</td><td></td><td>Divisor.</td><td>—2</td><td>10</td><td></td></tr>
<tr><td></td><td>196</td><td>170.379</td><td></td><td></td><td>4</td><td>9.152</td><td></td></tr>
<tr><td>25.9 ····</td><td>23.31</td><td>11.621</td><td></td><td>68 ····</td><td>5·44</td><td>848</td><td></td></tr>
<tr><td>Divisor</td><td>189.31</td><td>10.740875</td><td></td><td>Divisor .</td><td>11.44</td><td>714304</td><td></td></tr>
<tr><td></td><td>81</td><td>.880125</td><td></td><td></td><td>64</td><td>133696</td><td></td></tr>
<tr><td>27.75 ····</td><td>1·3875</td><td>·865276</td><td></td><td>8·44 ····</td><td>3376</td><td>127796</td><td></td></tr>
<tr><td>Divisor</td><td>214.8175</td><td>14849</td><td></td><td>Divisor.</td><td>17.8576</td><td>5900</td><td></td></tr>
<tr><td></td><td>25</td><td>12986</td><td></td><td></td><td>16</td><td>5496</td><td></td></tr>
<tr><td>2|7·8|54. ····</td><td>1114</td><td>1863</td><td></td><td>8·5|27 ······</td><td>597</td><td>404</td><td></td></tr>
<tr><td>Divisor.</td><td>216.318|9</td><td>1731</td><td></td><td>Divisor.</td><td>18·2565</td><td>366</td><td></td></tr>
<tr><td></td><td>2</td><td>132</td><td></td><td></td><td>3</td><td>38</td><td></td></tr>
<tr><td>Divisor.</td><td>2|1|6|·4|3|1</td><td>130</td><td></td><td>Divisor.</td><td>1|8|3|1|9</td><td>37</td><td></td></tr>
</table>

25. Given $x^3+\frac{8}{5}x^2-5x=-\frac{8}{5}$, to find x.

Given $x^3-1.8\dot{3}x^2+.3x=-.36$, to find x. ·1·8$\dot{3}$ Root

```
          ·8                     .3  .    —·36(·760059217570146
 —5    —2·6(1.351029|  1.13—·793    —·322
  1·8  —3·1    8502|     —·46    —  446
 —3·1    .4              ·49   .—  446239
  1      ·3103        .326  1959         —426
4.18 1·256  ·1341        —·74473        —36025549875
  1·034   ·1311527          36          —66411167916
    9      29583      .446716  2233583  —64843617393
4.838  24194   28724398   — ·7205109975  —1567550523
  2·62305       858934            25    —1440961054
       25       575496    .44|6|8|2   20214   —126589469
4.9|3|98 49398'  283438     —·7204846377     —72048043
  2·8724398     258973            89      —54541426
         1       24465     —·720480527       —50433631
       10        23020     —·|7|2|0|4|8|0|4|3|8  —4107795
2.|8|7|7|4|8      1445                        —3602402
                  1438                   337   —505393
                     7                   288   —504336
                     6                    49     —1057
                                          43      —720
```

4· A, B and C put in \$65, and A's money was in trade 4, B's 5, and C's 6 months. When they divided their stock and profit, A' stock and gain was \$42, B's stock and gain \$30, and C's gain \$ 9. What did each put in for his share?

By taking the ratio of each man's gain, divided by his time, and then reducing to a common denominator,

$\left\{\frac{42-x}{4}\right)+\left(\frac{30-y}{5}\right)+\frac{9}{6}\right\}$ will give the following Rule viz.:

Give each man's gain or loss and time, and the whole stock. to find each mans particular stock

Rule: Multiply each man's gain or loss into all the times except his own; then, as the sum of these products is to the whole stock, so is each man's product to his stock.

Let x and y denote the stock A and B put in. Then

$$\left.\begin{array}{l} 42-x=\text{A's gain}=(42-x)5.6=1260-30x \\ 30-y=\text{B's gain}=(30-y)4.6=\ 720-24y \\ 9\ \ =\text{C's gain}\ \ \ 9.4.5\ =\ 180 \end{array}\right\}\text{A, B and C's Prod.}$$

$$2160-30x-24y:65::\left\{\begin{array}{l}1260-30x:x\\ 720-24y:y\end{array}\right.$$

$$\text{Or}\left\{\begin{array}{l}2160x-30x^2-24xy=65(1260-30x)=81900-1950x\\ 2160y-30xy-24y^2=65(\ 720-24y)=46800-1460y\end{array}\right\}$$

$$\left\{\begin{array}{l}4110x-30x^2-24xy=81900\\3720y-30xy-24y^2-46800\end{array}\right\}\text{ or }\left\{\begin{array}{l}685x+5x^2-4xy=13650\\620y-5xy-4y^2=7800\end{array}\right\}$$

Then $685x-5x^2-13650=4xy$, and $y=\dfrac{685v-5x^2-13650}{4x}$, and by substituting this value of y in the following equation, $620y-5xy=7800+4y^2$, and reducing by ex. 6, pa. 15, we get $x^3-443x^2+31500x=573300$, and $x=30=$ A's stock.

26 Given $3x^5-9x^4-2x^3+13x^2+2x=10$, to find x,

$x^5-3x^4-.6x^3+4.3x^2+.6x=3.3$. or $3.\dot{3}\left\{\begin{array}{l}[80046\\2.6734374\end{array}\right\}$

-3
2
-1
2
1
2
3
2
5
2
7.6
6
8.2
.6
8.8
6
9.4
6
10.07
7
10.14
7
10.21
7
10.28
7
10.3_5

$-.\dot{6}$
$-2.$
-2.6
$-2.$
$-.6$
6.
5.3
10.
15.3
4.56.
19.893
4.92.
24.813
5.28.
30.093
5.64.
35.733
7049.
36.43823
7098.
37.1480_3
7147.
37.86273
7196
38.5823
310.
38.613|3
31
38.744
3
38·67

$4.\dot{3}$
$-5.\dot{3}$
$-1.$
1.3
$-2.\dot{3}$
$10.\dot{6}$
$8.\dot{3}$
11.936.
20.2693
14.888.
35.1573
18·056.
53.2133
2.5506763
55.7640096
2.6003623
58.364372.
2.6503913
61.014763
115840
61.120603
11593
61.23653
11602
61.35255
154
61.3680
15
6|1.3|8|3

$.\dot{6}$
$-2.$
-1.3
$-4.\dot{6}$
-6
12.1616
6.1616
21.0944
27.2560
3.903480675
31.159480676
4.08550604.
35.244986716
18336181
35.42834852
18370960
35.61205812
245472
35.6366053
245532
35.661158|5
1841
35.663000
1841
35.66484|1
4
3|5.6|6|4|8|8

2·6
6·
$3·6969_6$
2·30304
2·18116364736
12187635263
10628504556
1559130707
1425464212
133666495
106989000
26677495
24965416
1712079
1426595
285484
285319
165
143
22
21
1

I. Given $7x^6 - 26x^5 - 26x = -7$, to find x to about 6 or 8 places of decimals.

						Root.
—26	0.	0	0	—26		—7 (3.7320
21	—15	—45	—135	—405		1293 [50808.
—5	—15	—45	—135	—431		1286
21	48	99	162	+81		1127.456046
16	33	54	27	—350		158.543954
21	111	432	1458	1960.65149		147.985357
37	144	486	1485	1610.65149		10.558827
21	174	954	1315.9307	3123.71745		10.296445
58	318	1440	2800.9307	4734.36894		.262272
21	237	439.901	1661.3228	198.47206		258102
79	555	1879.901	4462.4535	4932.8410|0		4170
21	73.43	—493.703	2046.4570	201.7029		4130
100	628.43	2373.604	6508.9105	5134.5439	40	40
4.9	76.86	549.906	106.8249	13.6786	36	40
104.9	705.29	2923.510	6615.7354	5148.222|5	4	
4.9	80.29	608.510	107.696	13.679	4	
109.8	785.58	3532.020	6723.43|1	5161.90|0		
4.9	83.72	28.810	108.57	34		
114.7	869.30	3560.83|0	6832.0|0	5162.2|4		
4.9	87.15	28.933	7.3	34		
119.6	956.45	3589.8|60	6839.3	5|1|6|2.|6		
4.9	3.89	29.0	7.3			
124.5	960.3|4	3618.9	6846·|6			
4.9	3.9	29	7			
129.4	964.|2	3648	6|8|5|4			
2	4	2				
12|9|.6	9|6|8	3|6|5|0				

Ans. $x = 3.7320508076$.

1. A and B put in 165 dollars in trade. A's money was in trade 12, and B's 8 months; when they shared stock and gain, A received 67 dollars and B 126 dollars. What was each man's stock? Let $x =$ A's dollars and $165-x=$ B's dollars; then $12x$ and $8(165-x)=1320-8x$ will represent A and B's stock and time of each; hence their sum is $4x + 1320$, and also their gain $= \{(67+126)-165\} = 28$. But I have $4x + 1320 : 28 :: 12x : \frac{336x}{1320+4x}=$ A's gain, and A's stock and gain $= (x+\frac{336x}{1320+4x}) = 67$, by the question, or $1320x+4x^2+336x=88440+268x$, or $x^2+347x=22110$. (By art. 70, case 1,) I have $x=55=$ A's stock, and B's stock $=$ 110 dollars.

K. Given $x^5 - 117x^3 - 2340x^2 = -40500$, to find x.

<table>
<tr><td>0</td><td>—117</td><td>—2340</td><td>0</td><td>—40500 (3.84929542</td></tr>
<tr><td>3</td><td>9</td><td>—324</td><td>—7992</td><td>—23976 Root.</td></tr>
<tr><td>3</td><td>—108</td><td>—2664</td><td>—7992</td><td>—16524</td></tr>
<tr><td>3</td><td>18</td><td>—270</td><td>—8802</td><td>—15441.27232</td></tr>
<tr><td>6</td><td>—90</td><td>—2934</td><td>—16794</td><td>—1082.72768</td></tr>
<tr><td>3</td><td>27</td><td>—189</td><td>—2507.5904</td><td>—877.40141</td></tr>
<tr><td>9</td><td>—63</td><td>—3123</td><td>—19301.5904</td><td>—205.32627</td></tr>
<tr><td>3</td><td>36</td><td>—11.488</td><td>—2508.2816</td><td>—198.79248</td></tr>
<tr><td>12</td><td>—27</td><td>—3134.488</td><td>—21810.8720</td><td>—6.53379</td></tr>
<tr><td>3</td><td>12.64</td><td>—864</td><td>—124.9232</td><td>—4.42336</td></tr>
<tr><td>15.8</td><td>—14.36</td><td>—3135.352</td><td>—21935.035|2</td><td>—2.11043</td></tr>
<tr><td>8</td><td>13.28</td><td>+10.272</td><td>—124.922</td><td>—1.99054</td></tr>
<tr><td>16.6</td><td>—1.08</td><td>—3125.080</td><td>—22059.957</td><td>—11989</td></tr>
<tr><td>8</td><td>13.92</td><td>999</td><td>—28.096</td><td>—11058</td></tr>
<tr><td>17.4</td><td>12.84</td><td>—3124.081</td><td>—22088.05|3</td><td>—931</td></tr>
<tr><td>8</td><td>11.36</td><td>1.028</td><td>—28.09</td><td>—885</td></tr>
<tr><td>18.2</td><td>24.20</td><td>—3123.05|3</td><td>—22116.15</td><td>—46</td></tr>
<tr><td>8</td><td>76</td><td>1.06</td><td>—63</td><td>—44</td></tr>
<tr><td>1|9.|0</td><td>24.96</td><td>—3121.99</td><td>—22116.78</td><td>—2</td></tr>
<tr><td></td><td>* 25.7</td><td>25</td><td>—28</td><td></td></tr>
<tr><td></td><td>8</td><td>—3121.7|4</td><td>—2|2|1|1|7.|1</td><td></td></tr>
<tr><td></td><td>* 26.5</td><td>3</td><td></td><td></td></tr>
<tr><td></td><td>2|7.|3</td><td>—3121.4</td><td></td><td></td></tr>
</table>

2. Two notes, one of 126 dollars, payable in 6 months, and the other of \$150, payable in 9 months, were discounted for \$8,50. What rate of interest were they discounted at; Let x denote the interest of one dollar for 12 months; then the amount of \$1 in 6 months being $1+\frac{1}{2}x$, and in 9 months $1+\frac{3}{4}x$ the present value of the bill due at the end of 6 mos. will, $\therefore$ be $\frac{120}{1+\frac{1}{2}x}$; and that of the bill, due at the end of 9 mos, $\frac{150}{1+\frac{3}{4}x}$; $\therefore \frac{120}{1+\frac{1}{2}x}+\frac{150}{1+\frac{3}{4}x} = (120+150-8,50) = 261.50$, by the question, and reduction $120+90x+150+75x = 261.5(1+\frac{5}{4}x+\frac{3}{8}x^2)$, or $270+165x = \frac{1}{8}\{261.5\times(8+10x+3x^2)\}$; $\therefore \frac{2160}{261,5}+\frac{1320}{261,5}x = (3x^2+10x+8)$, $\therefore x^2+\frac{2590}{1569}x = \frac{136}{1569}$; hence, by art. 70, case 1, we find $x = \sqrt{\{\frac{136}{1569}+(\frac{1295}{1569})^2\}}-\frac{1295}{1569} = \frac{\sqrt{(136\times1569)+(1295\times1295)}-1295}{1569}$ $=0.05093$, and $\times 100 = \$5.093$ cts. Ans.

Given $x^4 - 12x^3 + 12x = 3$; or, $x^2 - 210x = 880$, to find x.

	−210	0.				
16^2....	256	880(16.25252920)				
Div.	26	736 nearly.				
	256	144				
4.82....	964	113.528				
Div.	567.64	30.472				
	4	28.987625				
48.65 ...	2.4325	1.484375				
Div.	597.7525	1.162570				
	25	319805				
48.7,52....	975	291155				
Div.	582.285|0	28650				
	24	23293				
Div.	5,8,2.3,3	5357				
		5241				

1	0	−12	12	+3(2.858
2	4	16	−8	06
2	−8	4	+11	
2	8	0	8.9856	
4	0	4	2.0144	
2	12	15.232	1.7194	
6	12	11.232	2960	
2	7.04	21.376	2919	
8.8	19.04	32.608	31	
.8	7.68	1.780	29	
9.6	26.72	34.38|8	2	
.8	8.32	1.81		
10.4	35.04	36.20		
.8	.56	.29		
1|1|.2	35.6	3|6|.4|9		
	.6			
	3|6.|2			

Q. Given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$, to find x.

2	3	4	5	54321 (8.414454749
8	80	664	5344	42792
10	83	668	5349	11529
8	144	1816	19872	11088.97344
18	227	2484	25221 D	440.02656
8	208	3480	2501.4336	304.11051
26	435	5964 D	27722.4336	135.91605
8	272	289.584	2620.0064	122.02900
34	707 D	6253.584	30342.4400D	13.88705
8 D	16.96	296.432	68.611	12.21500
42.4	726.96	6550.016	30411.051	1.67205
4	17.12	303.344	68.689	1.52700
42.8	741.08	6853.360	30479.74|0	14505
4	17.28	7.762	27.51	12216
43.2	758.36	6861.122	30507.25	2289
.4	17.44	7.76	27.51	2138
43.6	775.80	6868.8|8	3053.47 6	151
.4	44	77	27	122
44.0	776.2	6876.6 6	30537.5	29
			3|0|5|4|0|2	27

Given $x^4 - 36x^2 + 71x = 36$; and $2x^4 - 16x^3 + 40x^2 - 30x = -1$, to find x, and $-x$. root.

0	—36	72	36(1.26	—16	40	—30	—1(1.284
1	1	—35	37 796	2	—14	26	—4 ·72
1	—35	37	—1	—14	26	—4	3
1	·2	—33	—.3664	2	—12	14	2.0992
2	—33	4	—.6336	—12	14	10	.9008
1	3	—5.832	—.5467	2	—10	.496	·8511
3	—30	—1.832	—869	—10	4	10.496	497
1	.84	—5.656	—764	2	—1.52	.208	421
4.2	—29.16	—7.488	—105	—8	2.48	10.704	76
.2	88	—1.624	—98	.4	—1.44	—66	74
4.4	—28.28	—9.112	—7	—7.6	1.04	10.638	2
.2	.92	—1.60	—7	.4	—1.36	—11	2
4.6	—27.36	—10.7|1		—7.2	—32	1|0|.5|2	
.2	.29	—2		.4	—51		
4.8	—27.0|7	—1|0|.9		—6.8	—.83		
	3			4	—51		
	26.7			—6.4	—1|.3|4		

Q. Given $x^3 + 7x^2 + 3x = 57$, to find x; and $x^4 - 12x^2 + 12 = 3$, to find x.

		+7		−12 +6		
	+3	57(2.317	10	—3	+12	3)—3.9007
9....	18	42	—3	18	9	—63
Div.	21	15	—3	15	21	+66
	4	14.097	—3	27	—45	65·0241
13|3....	3.99	903	—6	42	—24	+·9759
Div.	46.99	512091	—3	11·61	—48·249	·925624
	9	390909	—9	53·61	—72·249	+50276
13.91....	1391	360121	—3	12.42	— 59·427	39843
Div.	51.2091	30788	—12.9	66·03	—131·676	+10433
	1	25776	—.9	13·23	— ·556	9297
13.937	976	5012	—13.8	79·26	—132·232	1136
Div.	51.445|9	4640	—9	11	—556	1063
	7	372	—14.7	79.37	—132.788	73
	5|1.|5|5|1	361	—9	10	—2	66
		11	—15.6	79.4|8	—1|3|2.81	7

32. Given the sum of (n) terms of the series $1+2+3+4+$&c. $=A$, to find the sum of $1^2+2^2+3^2+4^2+$&c. to (n) terms.

Ans. $\frac{A}{3}\sqrt{(8A+1)}$.

6. Given $x^5-4.962x^2=1$, to find x to ten or twelve places.

0	0	−4.962	0	1.(1.74256154615
1	1 1	1	−3.962	−3.962 [Root.
1	2 3	−3.962	−3.962	4.962
1	3 6 4	3	−.962	3.82039
2	10.	−.962	−4.924	1.14161
1	3.99	6	10.3817	1.0681282024
3	13.99	5.038	5.4577	734817376
1	4.48	9.793	19.4320	573192130
4	18.47	14.831	24.8897	161625246
1	4.97	12.929	1.81350656	143896180
5.7	23.44	27.760	26.70320656	17729066
.7	5.46	16.408	1.86084224	17283636
6.4	28.90	44.168	28.56404880	445430
7.1	3416	1.169664	955577	288089
.7	29.2416	45.337664	28.6596065	157341
7.8	3432	1.183392	956790	144045
.7	29.5848	46.521156	28.7552855	13296
8.54	3448	1.197184	23950	11524
4	29.9296	47.7183240	28.779236	1772
8.58	3464	6059	24950	1728
4	30.2760	47.7786	28.803186	44
8.62	17	606	287	29
4	30.293	47.8395	28.80606	15
8.66	2	61	287	14
8.710	30.31	47.900	28.80893	1

A gentleman has a circular bowling-green, containing 2750 square yards, which lying low, is incommoded by water in the winter, and finds it necessary to raise it a yard and a half higher, by the earth to be digged from the outer side of it, making thereby a trench or ditch quite around the remaining part of it, whose breadth at bottom shall be 3 yards, and the inner side or circumference of which is perpendicular, but the outer one to slope at an angle 45°. He desires to know the depth to make the ditch, and the expense of building a wall 2 feet thick quite around the inner side of the ditch, from the bottom up to the raised surface of the bowling-green, at 16 cts. the solid yard.

Let $\sqrt{\frac{2750}{.7854}}=59\cdot1727=d$ the diam. of the whole green. Put $a=.7854$, and $x=$ the required depth of the trench: then $x+3$ = its breadth at top (because the slope is half a right angle), and

$d-2\cdot(x+3) = d - 2x - 6 =$ the diam. of the raised green; and $\therefore\ (d-2x-6)^2 \times \frac{3}{2}a =$ the solidity of the new part raised.

But the content of the ditch (being equal to the difference between a cylinder whose diam. is $d-2x-6$, and the conic frustum whose diameters are d and $d-2x$, (their common height being x) will be $\frac{1}{3}\{d^2+d\cdot(d-2x)+(d-2x)^2\}\times ax-(d-2x-6)^2\times ax$, which must be $= (d-2x-6)^2 \times \frac{3}{2}a$, the content of the raised part. Hence $x^3 - \frac{1}{4}(d-15)\cdot 3x^2 - \frac{1}{4}(d-4).27x = 9\{.\frac{1}{4}(d-6)^2\}$, or $x^3 - 33.1295x^2 - 372.4157x = -1590.3345$; the root of which is $x = 3.365$yards, the depth of the trench. $\therefore$ the height of the wall $(x+1\frac{1}{2})$ is 4.865, and the diam. $(d-6-2x)$ of the raised part is 46.4427; to which adding $1\frac{1}{3}$ (double the thickness of the wall) we have 47.776 for the diam. of the green and wall together: then (by Mensuration, p. ,) the area of the ring or top of the wall is $(47\cdot776+46\cdot4427)\times(47.776-46.4427)\times\cdot7854=94.2187 \times 1\frac{1}{3}\times.7854 = 98.6656$; which multiplied by 4.865, the height, we have 480 yards for the content; which, at 16 cents each, amounts to \$ 76,80 the expense of the wall.

Given $x^4 - 38x^3 + 210x^2 + 538x = -289$, or, $x^4 + 24x^3 - 114x^2 - 24x = -1$, find x, in each of the equations.

−38	+210	538	−289	(30.5356	24	−114	−24	−1	(4.236
30	−240	−900	−10860	5375	4	112	−8	−128	0679
−8	−30	−362	1057¼		28	−2	−32	127	
30	660	18900		9826.8125	4	128	504		105.5216
22	630	18538		744.1875	32	126	472		21.4784
30	1560	1115.625		625.7853	4	144	55.61		17.8109
52	2190	19653.625		118.4022	36	270	527.61		3.6675
30	41.25	1136.375		104.7035	4	8	57.22		3.6263
82.5	2231.25	20790.000		13.6987	4|0	278	584.83		412
.5	41.50	69.511		12.5739		8	8.64		364
83.0	2272.75	20859.51|1		1.1268		286	593.|98		48
.5	41.75	69.59		1.0477		8	8.9		42
83.5	2314.50	20929.10		791		294	603.		6
.5	2.52	11.60		630		1	2		5
84|0	2317.0|2	20940.7	14	161		29|5	604		
	2.5	20952.	13	147					
	2319.5	2|0|9|5|3		14					

Ans. $x = 4\cdot2360679$

$x = 30\cdot5356537529$

249. Divide the number 50 into two such parts, that $\frac{3}{4}$ of one part, added to $\frac{5}{6}$ of the other may make 40.

Let x be the first part, then will $50 - x$ be the other part, and $\frac{3}{4}x+\frac{5}{6}(50-x)=40$, by the question, whence $(\frac{3}{4}-\frac{5}{6})x+\frac{250}{6}=40$, or, by reduction, $x=250\times2-40\times12=20$, and $50-x=30$.

Given $x^3 - 6x = 2$, to find x

```
  —6              0     Root.
2....4            2(2.601679135
D. —2          —4
    4           6
6.6...3.96      5.976
D.   9.96          24
     36         14287801
7.801.  7801     9712199
    14.287801    8580171
            1    1132028
7·80|36··· 4682  1001351
    14.300285 500  130677
           55 429  128746
    14.3050|2  71    1931
    1|4.3|0|5|1 71   1431
                      500
```

Given $x^3 + \cdot 75x^2 + \cdot 875x = .5625$, to find x.

```
           .875        +.75
1.15....46         .5625(.4143705
D.     1.335       .534        04
          16         285
1.96......196      19746
D.     1.9746       8754
           1       8008944
1.984......7936     745056
D.    2.002236      603236
            16      141820
1.|99|23....598     140806
      2.010786        1014
            14        1006
      2.011|52           8
      2.0|1|17           8
```

Given $x^3 - 6x = 5.6$, to find x.

```
                       0.   Root.
     —6           5.5(2.8252638
2....  4        —4
Div.  —2          9.6
       4          9.152
6.8 . . . 5.44     .448
Div.   11.44     .353768
         64          94232
8.42 . . . . 1684    89498
Div.  17.6884         4734
            4   54    3589
8.4|6|5 . . . 423  14  1145
       17.8995  14  1077
            2           68
      1|7.9|4|4
```

Given $x^3 - 12x = 15$, to find x.

```
                       0.   Root.
      —12            15 (3.9719607
3×3= 9               —9          [68
Div.  —3              24
       9             21.519
9.9 . . . 8.91        2.481
Div.    23.91         2.411773
          81            69227
11.77 . . . 8239        35295
Div.   34.4539          33932
            49    24    31786
11.911 . . . 119   3    2146
       35.2946     2    2119
            11     1      27
       3|5.3|1|7
```

A trader maintained himself for three years, at the expense of \$50 a year; and in each of those years, augmented that part of his stock which was not so expended by one third thereof. At the end of the third year, his original stock was doubled. What was that stock?

Let x = the number of dollars required; then $x - 50$ = the sum not expended; and with this he traded; ∴ $\frac{1}{3}(x-50)$ = his gain the first year, and $\frac{4}{3}(x-50)$ = the sum he had at the end of the first year; ∴ $\frac{1}{3}(4x-200)-50 = \frac{1}{3}(4x-350)$ = the sum he

traded with the 2d year; ∴ $\frac{4}{3}\cdot\{\frac{1}{1}(4x-350)\}=\frac{1}{3}(16x-1400)=$ the sum he had at the end of the second year; and $\frac{1}{3}(16x-1400)-50=\frac{1}{3}(16x-1850)=$ the sum he traded with at the end of the third year. And $\frac{4}{3}\cdot\{\frac{1}{3}(16x-1850)\}=$ the sum he had at the end of the third year; whence $\frac{4}{3}\cdot\{\frac{1}{3}(16x-1850)\}=2x$, and $32x-3700=27x$; by transposition, $5x=3700$, and $x=740$, Ans.

Given $x^3+9x=6$, or $x^3-3x=3$, find x, in each equation.

9	0	*root.*
6 36	6 (.6378342	
Div. 9.36	5.616	
36	.384	
1.83. . . . 549	.304047	
Div. 10.1349	79953	
9	71428	
1.897. . . .133	8525	
Div. 10.2040	8175	
2	350	
10.219	307	
10.2|2	2	43
	2	41

−3	0	Root.
2. . . .4	3 (2.1038034026	
Div. 1	2	
4	1.	
6.1 . . .61	.961	
Div. 9.61	39	
1	30746727	
6.303 . . 18909	8253273	
Div. 10.248909	8218300	
9	27	34973
6.30|98. . 5043	20	30834
10.273506	7	4139
10.2|7|2|875	6	4112

Given $x^4-3x^3+75x=10000$, to find x.

9	240*	−3	75	10000 (9.88700270095
9	243	81	702	6993 [Root.
18	283	78	777	3007
9	29.44	162	2160	2677.5616
27	512.44	240*	2937	329.4384
9	30.08		409.952	306.16628736
36.8	542.52		3346.952	23.27211264
8	30.72		434.016	23.26163178
37.6	573.24		3780.968	1048086
8	3.1424		46.110592	776088
38.4	576.3824		3827.078592	271998
8	3.1488		46.362496	271630
39.28	579.5312		3873.44108|8	368
8	3.1552		3.49754	349
39.36	582.6864		3876.93863	19
8	2371		3.49896	19
39.44	582.9235		3880.43|759	
3|9.5|2	5|8|3.16		3880.44	

Given the sum of three numbers in harmonical proportion $=26$, and their continued product $=576$; find the numbers.

Ans. 6, 8, and 12.

. Given $x^3-3x^2=5$; and Given $x^3+9x=30$, to find x.

0 −3
....0 5(3.425988757362
0 . 0
9 5
6.4..2.56 4.624
Div. 11.56 376
16 288488
7.22...1444 87512
Div. 14.4244 73027625
4 14484375
7.265.. 3633 13183581
14.60553 1300794
7|2|759...655 1172444
14.6484|2 841 128350
57 733 117250
14.655|5 108 11100
1|4.6|5|6|1 103 10259

Given $x^3-12x=-12$, to find x.

−12 −12(1.11574951
1.... 1 −11 676
−11 −1
1 − 869
3.1.... .31 −131
Div. −8.69 −83369
1 −47631
3.31 331 −41435125
Div. −8.3369 −6195875
1 −5787588
3.335.... 16675 −408287
Div. −8.287025 −330625
25 −78662
3.3457.... 2342 −74390
−8.267983 56−4272
13 50−4133
−8.26563 6 −139
−8.2|6|5|6 − 83

9 0
2 .... 4 30(2.1808498
Div. 13 26
4 4
6.1 .... 61 2.161
Div. 21.61 1.839
1 1.819232
6.38....6104 19768
Div. 22.7404 18610
64 1158
6.54 .... 52 931
23.26|24 227
2|3.|2|7 209
18
Root, 18
$x=2.1808498$

Given $x^3+x^2+x=100$ find x.

1 1
5.... 20 100(4.2644
Div. 21 84
16 16
1.32 .. 2.64 11.9 28
Div. 59.64 4.072
04 3.788376
16.66...8196 283624
Div. 63.1396 256072
36 27552
13.784... 55136 25631
Div. 64.017936 1921
16 1282
13.792.... 5517 639
64.078605 577
276 62
64.078881 58
1 4
6|4.0|7|9 4

31. Two travellers, A and B, set out together from the same place; A travels 8 miles the first day, 12 the second, 16 the third, &c.; B goes 1 mile the first day, 4 the second, 9 the third, &c.; how many days must they travel before B overtakes A. 7 days.

Of Irrational Quantities, or Surds.

Irrational Quantities, or Surds, are those of which the values cannot be accurately expressed in numbers, or are such as have no exact root, and are usually expressed by means of the radical sign $\sqrt{\ }$, or by fractional indices; in which latter case the numerator shows the power the quantity is to be raised to, and the denominator its root. Thus, $\sqrt{2}$, or $2^{\frac{1}{2}}$, denotes the square root of 2; $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$, the cube of the square of a, and $a^{\frac{m}{n}}$, is the nth root of the nth power of a.

Case I. *To reduce a rational quantity to the form of a Surd.*

Rule. Raise the quantity to a power corresponding with that denoted by the index of the surd to which it is to be reduced; and over this new quantity place the radical sign, or proper index, and it will be of the form required.

1. Let 3 and 5 be reduced to the form of the square root.
   Here $3\times3=3^2=9$; whence $\sqrt{9}$. Ans.
   Here $(5)^2=25$; therefore $\sqrt{25}$, Ans.

2. Reduce $2x^2$ and $-3x$ to the form of the cube root.
   Here $(2x^2)^3=8x^6$; whence $\sqrt[3]{8x^6}$, or $(8x^6)^{\frac{1}{3}}$.
   Here $(-3x)^3=-27x^3$; therefore $\sqrt[3]{-27x^3}$, Ans.

3. Let $-2a$ be reduced to the form of the fourth root.
   Here $(-2a)^4=16a^4$; therefore $-\sqrt[4]{16a^4}$, Ans.

4. Let a^2 be reduced to the form of the fifth root, and $\sqrt{a}+\sqrt{b}$, $\frac{\sqrt{a}}{2a}$ and $\frac{a}{b\sqrt{a}}$ to the form of the square root.
   Here $(a^2)^5=a^{10}$; therefore $\sqrt[5]{a^{10}}$, Ans.

And $(\sqrt{a}+\sqrt{b})^2=a+b+2\sqrt{ab}$; $\therefore \sqrt{(a+b+2\sqrt{ab})}$. Again, $(\frac{\sqrt{a}}{2a})^2=\frac{a}{4a^2}=\frac{1}{4a}$; $\therefore\sqrt{(\frac{1}{4}a)}$. Also, $(\frac{a}{b\sqrt{a}})^2=\frac{a^2}{b^2a}=\frac{a}{b^2}$; $\sqrt{\frac{a}{b^2}}$.

* A quantity of the kind here mentioned, as for instance $\sqrt{2}$, is called an irrational number, or a surd, because no number, either whole or fractional, can be found which, when multiplied by itself, will produce 2. But its approximate value may be determined to any degree of exactness by the common rule for extracting the square root, being 1 and certain non-periodic decimals, which never terminate.

Note. Any rational quantity may be reduced by the above rule to the form of the surd to which it is joined, and their product be then placed under the same index or radical sign.

Thus $2\sqrt{2}=\sqrt{4}\times\sqrt{2}=\sqrt{(4\times2)}=\sqrt{8}$,
And $2\sqrt[3]{4}=\sqrt[3]{8}\times\sqrt[3]{4}=\sqrt[3]{(8\times4)}=\sqrt[3]{32}$.
Also $3\sqrt{a}=\sqrt{9}\times\sqrt{a}=\sqrt{(9\times a)}=\sqrt{9a}$.
And $\frac{1}{2}\sqrt[3]{4a}=\sqrt[3]{\frac{1}{8}}\times\sqrt[3]{4a}=\sqrt[3]{(\frac{1}{8}\times4a)}=\sqrt[3]{\frac{a}{2}}$.

Here $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^2}} = \sqrt[3]{\frac{8a^3}{27}} \times \sqrt[3]{\frac{9}{4a^2}} = \sqrt[3]{\frac{72a^3}{108a^2}} = \sqrt[3]{\frac{72a}{108}} = \sqrt[3]{\frac{2a}{3}}$.

1. Reduce $5\sqrt{6}$ and $\frac{1}{5}\sqrt{5a}$ each to a simple radical form.

Here $5\sqrt{6} = \sqrt{25} \times \sqrt{6} = \sqrt{150}$, Ans.

Here $\frac{1}{5}\sqrt{5a} = \sqrt{\frac{1}{25}} \times \sqrt{5a} = \sqrt{(\frac{a}{5})}$, Ans.

CASE II. *To reduce quantities of different indices to others that shall have a given index.*

RULE.—Divide the indices of the proposed quantities by the given index, and the quotients will be the new indices for those quantities. Then, over the said quantities with their new indices, place the given index, and they will be the equivalent quantities required.

1. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$, or $5^{\frac{1}{2}}$ and $6^{\frac{1}{3}}$, or $a^{\frac{1}{2}}$ and $b^{\frac{2}{3}}$ each to quantities that shall have the index $\frac{1}{6}$.

Here $\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \times \frac{6}{1} = \frac{6}{2} = 3$, the first index;

And $\frac{1}{3} \div \frac{1}{6} = \frac{1}{3} \times \frac{6}{1} = \frac{6}{3} = 2$, the 2d index. Whence $(3^3)^{\frac{1}{6}}$ and $(2^2)^{\frac{1}{6}}$, or $27^{\frac{1}{6}}$, and $4^{\frac{1}{6}}$, are the quantities required.

Here $\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \times \frac{6}{1} = 3$, 1st index, $\frac{1}{3} \div \frac{1}{6} = \frac{1}{3} \times \frac{6}{1} = 2$, 2d index; hence $(5^3)^{\frac{1}{6}}$ and $(6^2)^{\frac{1}{6}}$, or $125^{\frac{1}{6}}$ and $36^{\frac{1}{6}}$, are the answers sought:

Here $\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \times \frac{6}{1} = 3$, 1st index; $\frac{2}{3} \div \frac{1}{6} = \frac{2}{3} \times \frac{6}{1} = 4$, 2d index.

Therefore $(a^3)^{\frac{1}{6}}$ and $(b^4)^{\frac{1}{6}}$ are the answers.

3. Reduce $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$, to quantities that shall have the common index $\frac{1}{8}$.

Here $\frac{1}{2} \div \frac{1}{8} = \frac{1}{2} \times \frac{8}{1} = 4$, 1st index, $\frac{1}{4} \div \frac{1}{8} = \frac{1}{4} \times \frac{8}{1} = 2$, 2d index; therefore $(2^4)^{\frac{1}{8}}$ and $(4^2)^{\frac{1}{8}} = 16^{\frac{1}{8}}$ and $16^{\frac{1}{8}}$. Ans. $16^{\frac{1}{8}}$ and $16^{\frac{1}{8}}$.

4. Reduce a^2 and $a^{\frac{1}{2}}$ to quantities that shall have the common index $\frac{1}{4}$. Here $\frac{2}{1} \div \frac{1}{4} = \frac{2}{1} \times \frac{4}{1} = 8$, 1st index; $\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = 2$. 2d.

Therefore $(a^8)^{\frac{1}{4}}$ and $(a^2)^{\frac{1}{4}}$ are the quantities sought.

Note. Surds may also be brought to a common index by reducing the indices of the quantities to a common denominator, and then involving each of them to the power denoted by its numerator.

1. Reduce $3^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$, or $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$, or $1^{\frac{1}{2}}$ and $a^{\frac{1}{5}}$, or $a^{\frac{4}{3}}$ and $b^{\frac{1}{4}}$ each to quantities having a common index.

Here $3^{\frac{1}{2}} = 3^{\frac{3}{6}} = (3^3)^{\frac{1}{6}} = (27)^{\frac{1}{6}}$, and $4^{\frac{1}{3}} = 4^{\frac{2}{6}} = (4^2)^{\frac{1}{6}} = (16)^{\frac{1}{6}}$.

Whence $(27)^{\frac{1}{6}}$ and $(16)^{\frac{1}{6}}$, Ans.

Here $\frac{1}{3}$ and $\frac{1}{4} = \frac{4}{12}$ and $\frac{3}{12}$; hence $(4^4)^{\frac{1}{12}}$ and $(5^3)^{\frac{1}{12}}$, Ans.

Here $\frac{1}{2}=\frac{3}{6}$ and $\frac{1}{3}=\frac{2}{6}$; therefore $(a^3)^{\frac{1}{6}}$ and $(a^2)^{\frac{1}{6}}$, Ans.

Here again $\frac{1}{3}=\frac{4}{12}$ and $\frac{1}{4}=\frac{3}{12}$; therefore $(a^4)^{\frac{1}{12}}$ and $(b^3)^{\frac{1}{12}}$.

2. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to quantities that shall have a common index.

Here $\frac{1}{n}, \frac{1}{m}$ reduces to the common denominators $\frac{m}{mn}$ and $\frac{n}{mn}$; therefore $(a^m)^{\frac{1}{mn}}$ and $(b^n)^{\frac{1}{mn}}$, Ans.

CASE III. *To reduce surds to their most simple forms.*

RULE. Resolve the given number, or quantity, into two factors, one of which shall be the greatest power contained in it, and set the root of this power before the remaining part, with the proper radical sign between them.*

1. Let $\sqrt{48}$ and $\sqrt{108}$ each be reduced to its most simple form. Here $\sqrt{48}=\sqrt{(16\times3)}=4\sqrt{3}$, and $\sqrt{108}=\sqrt{(27\times4)}=3\sqrt{4}$.

Note 1. When any number, or quantity, is prefixed to the surd, that quantity must be multiplied by the root of the factor abovementioned, and the product then joined to the other part, as before.

1. Let $2\sqrt{32}$ and $5\sqrt[3]{24}$ each be reduced to its most simple form. Here $2\sqrt{32}=2\sqrt{(16\times2)}=8\sqrt{2}$, & $5\sqrt[3]{24}=5\sqrt[3]{(8\times3)}=10\sqrt[3]{3}$.

Note 2. A fractional surd may also be reduced to a more convenient form, by multiplying both the numerator and denominator by such a number, or quantity, as will make the denominator a complete power of the kind required; and then joining its root, with 1 put over it, as a numerator to the other part of the surd.

1. Let $\sqrt{\frac{2}{7}}$, $\sqrt{\frac{2}{5}}$, $\sqrt{125}$, $\sqrt{294}$, $\sqrt[3]{56}$, $\sqrt[3]{192}$, $7\sqrt{80}$, $9\sqrt[3]{81}$, and $\frac{3}{121}\sqrt{\frac{5}{6}}$, each be reduced to its most simple form.

Here $\sqrt{\frac{2}{7}}=\sqrt{\frac{14}{49}}=\sqrt{(\frac{1}{49}\times14)}=\frac{1}{7}\sqrt{14}$, Ans. $3\sqrt[3]{\frac{2}{5}}=3\sqrt[3]{\frac{50}{125}}=3\sqrt[3]{(\frac{1}{125}\times50)}=\frac{3}{5}\sqrt[3]{50}$, Ans. $\sqrt{125}=\sqrt{(25\times5)}=5\sqrt{5}$, Ans. $\sqrt{294}=\sqrt{(49\times6)}=7\sqrt{5}$, Ans. $\sqrt[3]{56}=\sqrt[3]{(8\times7)}=2\sqrt[3]{7}$, Ans. $\sqrt[3]{192}=\sqrt[3]{(64\times3)}=4\sqrt[3]{3}$, A. $7\sqrt{80}=7\sqrt{(16\times5)}=28\sqrt{5}$, A. $9\sqrt[3]{81}=8\sqrt[3]{(27\times3)}=27\sqrt[3]{3}$, Ans.

Here, reducing the radical, we have $\sqrt{\frac{5}{6}}=\sqrt{\frac{5\times6}{36}}=\frac{1}{6}\sqrt{30}$; therefore $\frac{3}{121}\sqrt{\frac{5}{6}}=\frac{3}{121}\times\frac{1}{6}\sqrt{30}=\frac{1}{242}\sqrt{30}$, Ans.

10. Reduce $\frac{4}{7}\sqrt[3]{\frac{3}{16}}$, $\sqrt{98a^2x}$, and $\sqrt{(x^3-a^2x^2)}$, each to the most simple form.

Here $\sqrt{\frac{3}{16}}=\sqrt{\frac{3\times4}{64}}=\frac{1}{4}\sqrt[3]{12}$;; hence $\frac{4}{7}\sqrt[3]{\frac{3}{16}}=\frac{1}{7}\sqrt[3]{12}$, Ans.

$\sqrt{98a^2x}=(49a^2\times2x)=7a\sqrt{2x}$, Ans.

$\sqrt{(x^3-a^2x^2)}=\sqrt{\{x^2(x-a^2)\}}=x\sqrt{(x-a^2)}$, Ans.

*When the given surd contains no factor that is an exact power of the kind required, it is already in its most simple form. Thus, $\sqrt{15}$ cannot be reduced lower, because neither of its factors, 5 nor 3, is a square.

CASE IV. *To add Surd Quantities together.*

RULE. When the surds are of the same kind, reduce them to their simplest forms, as in the last case; then, if the surd part be the same in each of them, annex it to the sum of the rational parts, and it will give the whole sum required.

But if the quantities have different indices, or the surd part in each be not the same, they can only be added together by the signs $+$ and $-$.

1. It is required to find the sum of $\sqrt{27}$ and $\sqrt{48}$.

$$\left.\begin{array}{l}\text{Here } \sqrt{27}=\sqrt{(9\times 3)}=3\sqrt{3}\\ \text{And } \sqrt{48}=\sqrt{(16\times 3)}=4\sqrt{3}\end{array}\right\}=7\sqrt{3} \text{ the sum.}$$

2. It is required to find the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

$$\left.\begin{array}{l}\text{Here } \sqrt[3]{500}=\sqrt[3]{(125\times 4)}=5\sqrt[3]{4}\\ \text{And } \sqrt[3]{108}=\sqrt[3]{(27\times 4)}=3\sqrt[3]{4}\end{array}\right\}=8\sqrt[3]{4} \text{ the sum.}$$

3. Required to find the sum of $4\sqrt{147}$ and $3\sqrt{75}$.

$$\left.\begin{array}{l}\text{Here } 4\sqrt{147}=4\sqrt{(49\times 3)}=28\sqrt{3}\\ \text{And } 3\sqrt{75}=3\sqrt{(25\times 3)}=15\sqrt{3}\end{array}\right\}=43\sqrt{3} \text{ the sum.}$$

4. Required to find the sum of $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$.

$$\left.\begin{array}{l}\text{Here } 3\sqrt{\frac{2}{5}}=3\sqrt{\frac{10}{25}}=\frac{3}{5}\sqrt{10}\\ \text{And } 2\sqrt{\frac{1}{10}}=2\sqrt{\frac{10}{100}}=\frac{2}{10}\sqrt{10}\end{array}\right\}=\frac{4}{5}\sqrt{10} \text{ the sum.}$$

5. Required to find the sum of $\sqrt{72}$ and $\sqrt{128}$.

$$\left.\begin{array}{l}\text{First } \sqrt{72}=\sqrt{(36\times 2)}=6\sqrt{2}\\ \text{And } \sqrt{128}=\sqrt{(64\times 2)}=8\sqrt{2}\end{array}\right\}=14\sqrt{2}, \text{ Ans.}$$

6. Required to find the sum of $\sqrt{180}$ and $\sqrt{405}$.

$$\left.\begin{array}{l}\text{Here } \sqrt{180}=\sqrt{(36\times 5)}=6\sqrt{5}\\ \text{Also } \sqrt{405}=\sqrt{(81\times 5)}=9\sqrt{5}\end{array}\right\}=15\sqrt{5}, \text{ Ans.}$$

7. Required to find the sum of $3\sqrt[3]{40}$ and $\sqrt[3]{135}$.

$$\left.\begin{array}{l}\text{First } 3\sqrt[3]{40}=3\sqrt[3]{(8\times 5)}=6\sqrt[3]{5}\\ \text{And } \sqrt[3]{135}=\sqrt[3]{(27\times 5)}=3\sqrt[3]{5}\end{array}\right\}=9\sqrt[3]{5}, \text{ Ans.}$$

8. Find the sum of $4\sqrt[3]{54}$ and $5\sqrt[3]{128}$.

$$\left.\begin{array}{l}\text{Here } 4\sqrt[3]{54}=4\sqrt[3]{(27\times 2)}=12\sqrt[3]{2}\\ \text{And } 5\sqrt[3]{128}=5\sqrt[3]{(64\times 2)}=20\sqrt[3]{2}\end{array}\right\}\therefore 32\sqrt[3]{2}, \text{ Ans.}$$

9. Find the sum of $9\sqrt{243}$ and $10\sqrt{363}$.

$$\left.\begin{array}{l}\text{Here } 9\sqrt{243}=9\sqrt{(81\times 3)}=81\sqrt{3}\\ \text{And } 10\sqrt{363}=10\sqrt{(121\times 3)}=110\sqrt{3}\end{array}\right\}=191\sqrt{3}, \text{ Ans.}$$

10. Find the sum of $3\sqrt{\frac{2}{3}}$ and $7\sqrt{\frac{27}{50}}$.

By first reducing the fractional surds, we have

$$\sqrt{\tfrac{2}{3}}=\sqrt{\tfrac{6}{9}}=\tfrac{1}{3}\sqrt{6}, \text{ and } \sqrt{\tfrac{27}{50}}=\sqrt{\tfrac{54}{100}}=\tfrac{1}{10}\sqrt{(9\times 6)}=\tfrac{3}{10}\sqrt{6};$$

$$\left.\begin{array}{l}\text{Hence } 3\sqrt{\frac{2}{3}}=3\times\frac{1}{3}\sqrt{6}=\sqrt{6}\\ \text{And } 7\sqrt{\frac{27}{50}}=7\times\frac{3}{10}\sqrt{6}=\frac{21}{10}\sqrt{6}\end{array}\right\}=\frac{31}{10}\sqrt{6}, \text{ Ans.}$$

11. Find the sum of $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$.

Here $\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{2}{8}}=\frac{1}{2}\sqrt[3]{2}$, and $\sqrt[3]{\frac{1}{32}}=\sqrt[3]{\frac{2}{64}}=\frac{1}{4}\sqrt[3]{2}$;

Hence $12\sqrt[3]{\frac{1}{4}}+3\sqrt[3]{\frac{1}{32}}=6\sqrt[3]{2}+\frac{3}{4}\sqrt[3]{2}=6\frac{3}{4}\sqrt[3]{2}$, Ans.

12. Find the sum of $\frac{1}{2}\sqrt{a^2b}$ and $\frac{1}{3}\sqrt{4bx^4}$.

Here $\frac{1}{2}\sqrt{a^2b}=\frac{1}{2}\sqrt{(a^2\times b)}=\frac{1}{2}a\sqrt{b}$
And $\frac{1}{3}\sqrt{4bx^4}=\frac{1}{3}\sqrt{(4x^4\times b)}=\frac{2}{3}x^2\sqrt{b}$ } $=(\frac{1}{2}a+\frac{2}{3}x^2)\sqrt{b}$, Ans.

13. Find the sum of $\sqrt{50}$ and $\sqrt{72}$, or $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{1}{15}}$. or

$$\left.\begin{array}{l}\sqrt{50}=\sqrt{(25\times2)}=5\sqrt{2}\\ \sqrt{72}=\sqrt{(36\times2)}=6\sqrt{2}\end{array}\right\}=11\sqrt{2},\text{ sum, or}$$

$$\left.\begin{array}{l}\sqrt{\frac{3}{5}}=\sqrt{\frac{15}{25}}=\frac{1}{5}\sqrt{15}\\ \sqrt{\frac{1}{15}}=\sqrt{\frac{15}{225}}=\frac{1}{15}\sqrt{15}\end{array}\right\}=\frac{4}{15}\sqrt{15},\text{ the sum.}$$

14. Find the sum of $\sqrt[3]{56}$, and $\sqrt[3]{189}$, or $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$.

$$\left.\begin{array}{l}\sqrt[3]{56}=\sqrt[3]{(8\times7)}=2\sqrt[3]{7}\\ \text{and } \sqrt[3]{189}=\sqrt[3]{(27\times7)}=3\sqrt[3]{7}\end{array}\right\}=5\sqrt[3]{7},\text{ sum, or}$$

$$\left.\begin{array}{l}\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{2}{8}}=\frac{1}{2}\sqrt[3]{2}\\ \sqrt[3]{\frac{1}{32}}=\sqrt[3]{\frac{2}{64}}=\frac{1}{4}\sqrt[3]{2}\end{array}\right\}=\frac{3}{4}\sqrt[3]{2},\text{ Ans.}$$

15. Find the sum of $3\sqrt{a^2b}$ and $5\sqrt{16a^4b}$.

$$\left.\begin{array}{l}3\sqrt{a^2b}=3\sqrt{(a^2\times b)}=3a\sqrt{b}\\ 5\sqrt{16a^4b}=5\sqrt{(16a^4\times b)}=20a^2\sqrt{b}\end{array}\right\}=\text{sum}(20a^2+3a)\sqrt{b},\text{ Ans.}$$

Case V. *To find the difference of Surd Quantities.*

Rule. When the surds are of the same kind, prepare the quantities as in the last rule; then the difference of the rational parts annexed to the common surd will give the whole difference required.

But if the quantities have different indices, or the surd part be not the same in each of them, they can only be subtracted by means of the sign —.

1. It is required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

$$\left.\begin{array}{l}\text{Here } \sqrt{448}=\sqrt{(64\times7)}=8\sqrt{7}\\ \text{And } \sqrt{112}=\sqrt{(16\times7)}=4\sqrt{7}\end{array}\right\}=4\sqrt{7}\text{ the difference.}$$

2. Required to find the difference of $\sqrt[3]{192}$ and $\sqrt[3]{24}$.

$$\left.\begin{array}{l}\text{Here } \sqrt[3]{192}=\sqrt[3]{(64\times3)}=4\sqrt[3]{3}\\ \text{And } \sqrt[3]{24}=\sqrt[3]{(8\times3)}=2\sqrt[3]{3}\end{array}\right\}=2\sqrt[3]{3},\text{ the diff.}$$

3. Required to find the difference of $5\sqrt{20}$ and $3\sqrt{45}$.

$$\left.\begin{array}{l}\text{Here } 5\sqrt{20}=5\sqrt{(4\times5)}=10\sqrt{5}\\ \text{And } 3\sqrt{45}=3\sqrt{(9\times5)}=9\sqrt{5}\end{array}\right\}=\sqrt{5},\text{ diff.}$$

4. Find the difference of $\frac{3}{4}\sqrt{\frac{2}{3}}$, and $\frac{2}{5}\sqrt{\frac{1}{6}}$.

$$\left.\begin{array}{l}\text{Here } \frac{3}{4}\sqrt{\frac{2}{3}}=\frac{3}{4}\sqrt{\frac{6}{9}}=\frac{3}{12}\sqrt{6}=\frac{1}{4}\sqrt{6}\\ \text{And } \frac{2}{5}\sqrt{\frac{1}{6}}=\frac{2}{5}\sqrt{\frac{6}{36}}=\frac{2}{30}\sqrt{6}=\frac{1}{15}\sqrt{6}\end{array}\right\}=\frac{11}{60}\sqrt{6},\text{ the diff.}$$

5. Required to find the difference of $2\sqrt{50}$ and $\sqrt{18}$.

$$\left.\begin{array}{l}\text{Here } 2\sqrt{50}=2\sqrt{(25\times2)}=10\sqrt{2}\\ \text{And } \sqrt{18}=\sqrt{(9\times2)}=3\sqrt{2}\end{array}\right\}=7\sqrt{2},\text{ the diff.}$$

6. Required to find the difference of $\sqrt[3]{320}$ and $\sqrt[3]{40}$.

$$\left.\begin{array}{l}\text{Here } \sqrt[3]{320}=\sqrt[3]{(64\times5)}=4\sqrt[3]{5}\\ \text{And } \sqrt[3]{40}=\sqrt[3]{(8\times5)}=2\sqrt[3]{5}\end{array}\right\}=2\sqrt[3]{5},\text{ diff.}$$

7. Required to find the difference of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{1}{27}}$.

$$\left.\begin{array}{l}\text{Here } \sqrt{\tfrac{3}{5}}=\sqrt{\tfrac{15}{25}}=\tfrac{1}{5}\sqrt{15}\\ \text{And } \sqrt{\tfrac{5}{27}}=\sqrt{\tfrac{15}{81}}=\tfrac{1}{9}\sqrt{15}\end{array}\right\}=\tfrac{4}{45}\sqrt{15}, \text{ difference.}$$

8. Find the difference of $2\sqrt{\frac{1}{2}}$ and $\sqrt{8}$.

$$\left.\begin{array}{l}\text{Here } 2\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{2}{4}} = 2\sqrt{2}\\ \text{And } \sqrt{8}= (4\times2)=2\sqrt{2}\end{array}\right\}=\sqrt{2}, \text{ the diff.}$$

9. Find the difference of $3\sqrt[3]{\frac{1}{3}}$ and $\sqrt[3]{72}$.

$$\left.\begin{array}{l}\text{Here } 3\sqrt[3]{\tfrac{1}{3}}=3\sqrt[3]{\tfrac{9}{27}}=\sqrt[3]{9}\\ \text{And } \sqrt[3]{72}=\sqrt{(8\times9)}=2\sqrt[3]{9}\end{array}\right\}=\sqrt[3]{9}, \text{ the diff.}$$

10. Required to find the difference of $\sqrt[3]{\frac{2}{3}}$ and $\sqrt[3]{\frac{9}{32}}$.

$$\left.\begin{array}{l}\text{Here } \sqrt[3]{\tfrac{2}{3}} =\sqrt[3]{\tfrac{18}{27}}=\tfrac{1}{3}\sqrt[3]{18}\\ \sqrt[3]{\tfrac{9}{32}}=\sqrt{\tfrac{18}{64}}=\tfrac{1}{4}\sqrt[3]{18}\end{array}\right\}=\tfrac{1}{12}\sqrt[3]{18}, \text{ the diff.}$$

11. Find the difference of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$.

$$\left.\begin{array}{l}\text{Here } \sqrt{80a^4x}=\sqrt{(16a^4\times5x)}=4a^2\sqrt{5x}\\ \text{And } \sqrt{20a^2x^3}=\sqrt{(4a^2x^2\times5x)}=2ax\sqrt{5x}\end{array}\right\}=(4a^2-2ax)\sqrt{5x}.$$

12. Find the difference of $8\sqrt[3]{a^3b}$ and $2\sqrt[3]{a^6b}$.

$$\left.\begin{array}{l}\text{Here } 8\sqrt[3]{a^3b}=8\sqrt[3]{(a^3\times b)}=8a\sqrt[3]{b}\\ \text{And } 2\sqrt[3]{a^6b}=2\sqrt[3]{(a^6\times b)}=2a^2\sqrt[3]{b}\end{array}\right\}=(8a-2a^2)\sqrt[3]{b}, \text{ diff.}$$

13. Find the difference of $\sqrt{75}$ and $\sqrt{48}$, or $\sqrt[3]{256}$ and $\sqrt[3]{32}$.

$$\left.\begin{array}{l}\sqrt{75}=\sqrt{(25\times3)}=5\sqrt{3}\\ \sqrt{48}=\sqrt{(16\times3)}=4\sqrt{3}\end{array}\right\}=\sqrt{3}= \text{difference, as required.}$$

$$\text{or } \left.\begin{array}{l}\sqrt[3]{256}=\sqrt[3]{(64\times4)}=4\sqrt[3]{4}\\ \sqrt[3]{32}=\sqrt[3]{(8\times4)}=2\sqrt[3]{4}\end{array}\right\}=2\sqrt[3]{4} \text{ difference.}$$

14. Find the difference of $\sqrt{\frac{3}{4}}$ and $\sqrt{\frac{1}{3}}$, or $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{3}{8}}$.

$\sqrt{\frac{3}{4}}=\frac{1}{2}\sqrt{3}$, and $\sqrt{\frac{1}{3}}=\sqrt{(\frac{1}{3}\times\frac{3}{3})}=\frac{1}{3}\sqrt{3}$; then $(\frac{1}{2}-\frac{1}{3})\sqrt{3}=\frac{1}{6}\sqrt{3}$,
$\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\frac{1}{3}\sqrt{6}$, and $\sqrt{\frac{3}{8}}=\sqrt{\frac{6}{16}}=\frac{1}{4}\sqrt{6}$; $(\frac{1}{3}-\frac{1}{4})\sqrt{6}=\frac{1}{12}\sqrt{6}$,

15. Find the difference of $\sqrt[3]{\frac{3}{5}}$ and $\sqrt[3]{\frac{25}{9}}$, or $\sqrt{24a^2b^2}$ and $\sqrt{54b^4}$.

$$\left.\begin{array}{l}\sqrt[3]{\tfrac{25}{9}}=\sqrt[3]{\tfrac{75}{27}}=\tfrac{1}{3}\sqrt[3]{75}\\ \sqrt[3]{\tfrac{3}{5}}=\sqrt[3]{\tfrac{75}{125}}=\tfrac{1}{5}\sqrt[3]{75}\end{array}\right\}=(\tfrac{1}{3}-\tfrac{1}{5})\sqrt[3]{75}=\tfrac{2}{15}\sqrt[3]{75}, \text{ Ans.}$$

$$\text{or } \left.\begin{array}{l}\sqrt{24a^2b^2}=\sqrt{(4a^2b^2\times6)}=2ab\sqrt{6}\\ \sqrt{54b^4} = \sqrt{(9b^4\times6)} = 3b^2\sqrt{6}\end{array}\right\}=(3b^2-2ab)\sqrt{6}, \text{ Ans.}$$

Case VI. *To multiply Surd Quantities together.*

Rule. When the surds are of the same kind, find the product of the rational parts, and the product of the surds, and the two joined together, with the common radical sign between them, will give the whole product required; which may be reduced to its most simple form by Case III.

But if the surds are of different kinds, they must be reduced to a common index, and then multiplied together as usual.

It is also to be observed, as before mentioned, that the product of different powers, or roots, of the same quantity, is found by adding their indices.

1. Required the product of $\frac{1}{2}\sqrt{\frac{2}{3}}$ and $\frac{2}{3}\sqrt[3]{\frac{1}{3}}$.

$\frac{1}{2}\sqrt[3]{\frac{2}{3}} \times \frac{3}{4}\sqrt[3]{\frac{5}{6}} = \frac{3}{8}\sqrt[3]{\frac{10}{18}} = \frac{3}{8}\sqrt[3]{\frac{5}{9}} = \frac{3}{8}\sqrt[3]{\frac{15}{27}} = \frac{1}{8}\sqrt[3]{15}$.

2. Find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

$3\sqrt{8} \times 2\sqrt{6} = 6\sqrt{48} = 6\sqrt{(16\times 3)} = 24\sqrt{3}$, Ans.

3. Find the products of $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$.

$$\left.\begin{array}{l}\text{Here } 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = (2^3)^{\frac{1}{6}} = 8^{\frac{1}{6}} \\ \text{And } 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = (3^2)^{\frac{1}{6}} = 9^{\frac{1}{6}}\end{array}\right\} = (8^{\frac{1}{6}} \times 9^{\frac{1}{6}}) = (72)^{\frac{1}{6}}, \text{ Ans.}$$

4. Required to find the product of $5\sqrt{a}$ and $3\sqrt[3]{a}$.

$$\left.\begin{array}{l}\text{Here } 5\sqrt{a} = 5a^{\frac{1}{2}} = 5a^{\frac{3}{6}} \\ \text{And } 3\sqrt[3]{a} = 3a^{\frac{1}{3}} = 3a^{\frac{2}{6}}\end{array}\right\} = 15a^{\frac{5}{6}} = 15(a^5)^{\frac{1}{6}}, \text{ or } 15\sqrt[6]{a^5}, \text{ Ans.}$$

5. Required to find the product of $5\sqrt{8}$ and $3\sqrt{5}$.

$5\sqrt{8} \times 3\sqrt{5} = 15\sqrt{40} = 15\sqrt{(4\times 10)} = 30\sqrt{10}$, Ans.

6. Find the product of $\sqrt[3]{18}$ and $5\sqrt[3]{4}$.

$\sqrt[3]{18} \times 5\sqrt[3]{4} = 5\sqrt[3]{72} = 5\sqrt[3]{(8\times 9)} = 10\sqrt[3]{9}$.

7. Required the product of $\frac{1}{4}\sqrt{6}$ and $\frac{2}{15}\sqrt{9}$.

$\frac{1}{4}\sqrt{6} \times \frac{2}{15}\sqrt{9} = \frac{1}{30}\sqrt{54} = \frac{1}{30}\sqrt{(9\times 6)} = \frac{1}{10}\sqrt{6}$.

8. Required the product of $\frac{1}{2}\sqrt{18}$ and $5\sqrt{20}$.

$\frac{1}{2}\sqrt{18} = 5\sqrt{20} = \frac{5}{2}\sqrt{360} = \frac{5}{2}\sqrt{(36\times 10)} = 15\sqrt{10}$.

9. Required the product of $2\sqrt{3}$ and $13\frac{1}{2}\sqrt{5}$.

$2\sqrt{3} \times 13\frac{1}{2}\sqrt{5} = 27\sqrt{15}$, Ans.

10. Required the product of $72\frac{1}{4}a^{\frac{2}{3}}$ and $120\frac{1}{2}a^{\frac{1}{4}}$.

$$72\tfrac{1}{4}a^{\frac{2}{3}} \times 120\tfrac{1}{2}a^{\frac{1}{4}} = \tfrac{289}{4} \times \tfrac{241}{2} \times a^{\frac{2}{3}} \times a^{\frac{1}{4}} = \tfrac{69649}{8}a^{\frac{8}{12}} \times a^{\frac{3}{12}}$$
$$= 8706\tfrac{1}{8}a^{\frac{11}{12}}.$$

11. Required the product of $4+2\sqrt{2}$ and $2-\sqrt{2}$.

12. Required the product of $(a+b)^{\frac{1}{n}}$ and $(a+b)^{\frac{1}{m}}$.

11. Mult. $4+2\sqrt{2}$

By $\quad 2-\sqrt{2}$

$$\begin{array}{r} 8+4\sqrt{2} \\ -4\sqrt{2}-4 \\ \hline \text{Ans. } 8-4=4 \end{array}$$

12. Mult. $(a+b)^{\frac{1}{n}} = (a+b)^{\frac{m}{mn}}$

By $\quad (a+b)^{\frac{1}{m}} = (a+b)^{\frac{n}{mn}}$

Product $\quad = (a+b)^{\frac{m+n}{mn}}$.

Multiply $3\sqrt{2}$ by $2\sqrt{8}$, or $\frac{1}{3}\sqrt[3]{4}$ by $\frac{3}{4}\sqrt[3]{12}$, or $\frac{5}{2}\sqrt{\frac{3}{8}}$ by $\frac{9}{10}\sqrt{\frac{2}{5}}$.

$3\times 2\times\sqrt{2}\times\sqrt{8} = 6\sqrt{16} = 24$, Ans.

$\frac{1}{3}\times\frac{3}{4}\times\sqrt[3]{4}\times\sqrt[3]{12} = \frac{1}{4}\sqrt[3]{(8\times 6)} = \frac{1}{2}\sqrt[3]{6}$,

Or $\frac{5}{2}\times\frac{9}{10}\times\sqrt{\frac{3}{8}}\times\sqrt{\frac{2}{5}} = \frac{3}{2}\sqrt{\left(\frac{3\times 5}{20\times 5}\right)} = \frac{3}{20}\sqrt{15}$, Ans.

Multiply $2\sqrt[3]{14}$ by $3\sqrt[3]{4}$, or $2a^{\frac{2}{3}}$ by $a^{\frac{4}{3}}$, or $(a+b)^{\frac{1}{3}}$ by $(a+b)^{\frac{3}{4}}$

$2\times 3\sqrt[3]{14}\times\sqrt[3]{4} = 6\sqrt[3]{56} = 6\sqrt[3]{(8\times 7)} = 12\sqrt[3]{7}$, Ans.

$(a+b)^{\frac{1}{3}}\times(a+b)^{\frac{3}{4}} = (a+b)^{\frac{4}{12}}\times(a+b)^{\frac{9}{12}} = (a+b)^{\frac{4+9}{12}=\frac{13}{12}}$, Ans.

Multiply $2x+\sqrt{b}$ and $2x-\sqrt{b}$, or $2x^{\frac{1}{2}}$ and $3x^{\frac{1}{3}}$.

$$\text{Or } 2\times1\times a^{\frac{2}{3}}\times a^{\frac{4}{3}}=2a^{\frac{6}{3}}=2a^{2}, \text{ Ans.}$$

Multiply $(a+2\sqrt{b})^{\frac{1}{2}}$ by $(a-2\sqrt{b})^{\frac{1}{2}}$, or $4x^{\frac{1}{n}}$ by $2y^{\frac{1}{n}}$.

$$\begin{array}{lll}
2x+\sqrt{b} & (a+2\sqrt{b})^{\frac{1}{2}} & 2x^{\frac{1}{n}} \\
2x-\sqrt{b} & (a-2\sqrt{b})^{\frac{1}{2}} & 3x^{\frac{1}{n}} \\
\hline
4x^2+2x\sqrt{b} & a^2+2a\sqrt{b} & 6(x^2)^{\frac{1}{n}}=6\sqrt[n]{x^2}, \text{Ans.} \\
\quad -2\sqrt{xb}-b & \quad -2a\sqrt{b}-4b & \\
\hline
4x^2-\qquad b, \text{ Ans.} & (a^2\qquad -4b)^{\frac{1}{2}}, \text{ Ans.} &
\end{array}$$

$$\text{Here } 4x^{\frac{1}{n}}\times2y^{\frac{1}{n}}=8(xy)^{\frac{1}{n}}, \text{ or }=8\sqrt[n]{xy}, \text{ Ans.}$$

Case VII. *To divide one Surd Quantity by another.*

Rule. When the surds are of the same kind, find the quotient of the rational parts, and the quotient of the surds, and the two joined together, with the common radical sign between them, will give the whole quotient required.

But if the surds are of different kinds, they must be reduced to a common index, and then be divided as before.

It is also to be observed, that the quotients of different powers or roots of the same quantity, are found by subtracting their indices.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

Here $\dfrac{8\sqrt{108}}{2\sqrt{6}}=4\sqrt{18}=4\sqrt{(9\times2)}=12\sqrt{2}$, Ans.

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

Here $\dfrac{8\sqrt[3]{512}}{4\sqrt[3]{2}}=2\sqrt[3]{256}=2\sqrt[3]{(64\times4)}=8\sqrt[3]{4}$, Ans.

3. It is required to divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$, and $\sqrt{7}$ by $\sqrt[3]{7}$.

$$\text{Here } \frac{\frac{1}{2}\sqrt{5}}{\frac{1}{3}\sqrt{2}}=\tfrac{3}{2}\sqrt{\tfrac{5}{2}}=\tfrac{3}{2}\sqrt{\tfrac{10}{4}}=\tfrac{3}{4}\sqrt{10}, \text{ Ans.}$$

$$\text{And } \frac{\sqrt{7}}{\sqrt[3]{7}}=\frac{7^{\frac{1}{2}}}{7^{\frac{1}{3}}}=\frac{7^{\frac{3}{6}}}{7^{\frac{2}{6}}}=7^{\frac{3}{6}-\frac{2}{6}}=7^{\frac{1}{6}}, \text{ Ans.}$$

4. Required to divide $6\sqrt{54}$ by $3\sqrt{2}$, and $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$.

Here $\dfrac{6\sqrt{54}}{3\sqrt{2}}=2\sqrt{27}=2\sqrt{(9\times3)}=6\sqrt{3}$, Ans.

And $\dfrac{4\sqrt[3]{72}}{2\sqrt[3]{18}}=2\sqrt[3]{4}$, which will not reduce lower.

5. Required to divide $5\frac{3}{4}\sqrt{\frac{1}{135}}$ by $\frac{2}{3}\sqrt{\frac{1}{5}}$.

Here $5\frac{3}{4}\div\frac{2}{3}=\frac{23}{4}\times\frac{3}{2}=\frac{69}{8}$, and $\sqrt{\frac{1}{135}}\div\sqrt{\frac{1}{5}}=\sqrt{\frac{1}{27}}=\sqrt{\frac{3}{81}}$ $=\frac{1}{9}\sqrt{3}$; $\therefore\frac{69}{8}\times\frac{1}{9}\sqrt{3}=\frac{69}{72}\sqrt{3}=\frac{23}{24}\sqrt{3}$, Ans.

6. Divide $4\frac{5}{7}\sqrt{\frac{2}{3}}$ by $2\frac{2}{5}\sqrt{\frac{3}{4}}$, and $4\frac{1}{2}\sqrt[2]{a}$ by $2\frac{2}{3}\sqrt[3]{ab}$.

First $4\frac{5}{7}\div2\frac{2}{5}=\frac{33}{7}\times\frac{5}{12}=\frac{165}{84}$, and $\sqrt{\frac{2}{3}}\div\sqrt{\frac{3}{4}}=\sqrt{(\frac{2}{3}\times\frac{4}{3})}=\sqrt{\frac{8}{9}}$ $=\sqrt{\frac{4\times2}{9}}=\frac{2}{3}\sqrt{2}$; hence $\frac{165}{84}\times\frac{2}{3}\sqrt{2}=\frac{55}{42}\sqrt{2}$. Ans.

Here $4\frac{1}{2}\div2\frac{2}{3}=\frac{9}{2}\times\frac{3}{8}=\frac{27}{16}$, and $\sqrt{a}\div\sqrt[3]{ab}=a^{\frac{1}{2}}\div a^{\frac{1}{3}}b^{\frac{1}{3}}=$ $\frac{a^{\frac{1}{6}}}{b^{\frac{1}{3}}}=\frac{a^{\frac{1}{6}}}{b^{\frac{2}{6}}}=(\frac{a}{b^2})^{\frac{1}{6}}$; therefore $\frac{27}{16}(\frac{a}{b^2})^{\frac{1}{6}}$, the answer.

7. Required to divide $32\frac{2}{5}\sqrt{a}$ by $13\frac{3}{4}\sqrt[3]{a}$.

Here $32\frac{2}{5}\div13\frac{3}{4}=\frac{162}{5}\times\frac{4}{55}=\frac{648}{275}$, and $\sqrt{a}\div\sqrt[3]{a}=a^{\frac{1}{2}}\div a^{\frac{1}{3}}=$ $a^{\frac{3}{6}-\frac{2}{6}}=a^{\frac{1}{6}}$. Therefore $\frac{648}{275}a^{\frac{1}{6}}$, the answer.

8. Required to divide $9\frac{3}{8}a^{\frac{1}{n}}$ by $4\frac{9}{11}a^{\frac{1}{m}}$.

Here $9\frac{3}{8}\div4\frac{9}{11}=\frac{75}{8}\times\frac{11}{53}=\frac{825}{424}$,

And $a^{\frac{1}{n}}\div a^{\frac{1}{m}}=a^{\frac{m}{mn}}\div a^{\frac{n}{mn}}=a^{\frac{m-n}{mn}}$; consequently, $\frac{825}{424}a^{\frac{m-n}{mn}}$, Ans.

9. Divide $\sqrt{20}+\sqrt{12}$ by $\sqrt{5}+\sqrt{3}$

Here $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}+\sqrt{3}}=\sqrt{4}$, or 2, Ans.

Note 1. Since the division of surds is performed by subtracting their indices, it is evident that the denominator of any fraction may be taken into the numerator, or the numerator into the denominator, by changing the sign of its index, or exponent.

2. It likewise appears, from the same rule, that

$$\frac{a^m}{a^m}=a^{m-m}=a^0;\text{ but we have also }\frac{a^m}{a^m}=1.$$

Whence it follows, that a^0 is a symbol equivalent to unity, and may, consequently, be always replaced by 1, whenever it occurs.*

Case VIII. *To involve or raise Surd Quantities to any power.*

Rule. When the surd is a simple quantity, multiply its index by 2 for the square, 3 for the cube, &c. and it will give the power of the surd part; which, being annexed to the proper power of the rational parts, will give the whole power required.

And if it be a compound quantity, multiply it by itself, the proper number of times, according to the usual rule.†

1. Thus $\frac{1}{a}=\frac{a^{-1}}{1}$ or a^{-1}; and $\frac{1}{a^n}=\frac{a^{-n}}{1}$, or a^{-n}.

2. Also $\frac{b}{a^2}=\frac{ba^{-2}}{1}$, or ba^{-2}; and $\frac{a^{-n}}{b^{-m}}=\frac{1}{a^nb^{-m}}$, or $\frac{b^m}{a^n}$.

* This expression, properly speaking, is only the result of an

3. Let $\frac{1}{a^2}$ be expressed with a negative index.

4. Let $a^{-\frac{1}{2}}$ be expressed with a negative index.

Here $\frac{1}{a^2}=a^{-2}=a^{-2}$ the answer. Here $a^{-\frac{1}{2}}=\frac{1}{a^{\frac{1}{2}}}$, Ans.

5. Let $\frac{1}{a+x}$ be expressed with a negative index.

Here $\frac{1}{a+x}=\frac{1}{(a+x)^1}=(a+x)^{-1}$, Ans.

6. Let $a(a^2-x^2)^{-\frac{1}{3}}$ be expressed with a negative index.

Here $a(a^2-x^2)^{-\frac{1}{3}}=\frac{a}{(a^2-x^2)^{\frac{1}{3}}}$, Ans. or $\frac{1}{a(a^2-y^2)^{\frac{1}{3}}}$

7. Required to find the square of $\frac{2}{3}a^{\frac{1}{3}}$, and the cube of $\frac{2}{3}\sqrt{3}$.

Here $\left\{\frac{2}{3}a^{\frac{1}{3}}\right\}2=\frac{4}{9}a^{\frac{1}{3}\times2}=\frac{4}{9}a^{\frac{2}{3}}=\frac{4}{9}\sqrt[3]{a^2}$, Ans.

And $\frac{8}{27}=3^{\frac{3}{2}}=\frac{8}{27}\sqrt{27}=\frac{8}{27}\sqrt{9\times3}=\frac{8}{9}\sqrt{3}$, Ans.

8. Required to find the square of $3\sqrt[3]{3}$, and the cube of $17\sqrt{21}$.

Here $(3\sqrt{2})^2=(3\times3^{\frac{1}{3}})^2=9.3^{\frac{2}{3}}=9\sqrt[3]{9}$, Ans.

$(17\sqrt{21})^3=17^3\sqrt{(21^2+21)}=17^3\times21\sqrt{21}=103173\sqrt{21}$, Ans.

9. Required to find the 4th power of $\frac{1}{6}\sqrt{6}$. Ans. $\frac{1}{36}$.

Here $(\frac{1}{6}\sqrt{6})^4=(\frac{1}{6}\times6^{\frac{1}{2}})^2=\frac{1}{6^4}\times6^2=\frac{1}{6^2}$; therefore $\frac{1}{36}$, Ans.

10. Find the square of $3+2\sqrt{5}$, the cube of $\sqrt{x}+3\sqrt{y}$, and the 4th power of $\sqrt{3}-\sqrt{2}$.

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operation, to which we are led by the common practice of representing the powers of quantities by numeral indices; which, in the present case, gives the quotient a^0, instead of 1, as it would be according to the usual method of division.

† Any power, or root, of a fraction, as before observed, is equal to the same power, or root, of the numerator, divided by the like power, or root, of the denominator. Thus,

$$\left(\frac{a}{b}\right)^2=\frac{a^2}{b^2},\ \text{and}\ \left(\frac{a}{b}\right)^m=\frac{a^m}{b^m};\ \text{also}\ \sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{ab}}{b},\ \text{and}\ \left(\frac{a}{b}\right)^{\frac{1}{n}}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}.$$

And if any quantity that is affected with the sign of the square root, is to be raised to the second power, it is done by suppressing the sign. Thus, $(\sqrt{a})^2$, or $\sqrt{a}\times\sqrt{a}=a$, and $\{\sqrt{(a+b)}\}^2$, or $\sqrt{(a+b)}\times\sqrt{(a+b)}=a+b$.

$$\begin{array}{l}
\sqrt{3}-\sqrt{2}\\
\sqrt{3}-\sqrt{2}\\
\hline
3-\sqrt{6}\\
\quad -\sqrt{6}+2\\
\hline
5-\ 2\sqrt{6}\\
5-\ 2\sqrt{6}\\
\hline
25-\ 10\sqrt{6}\\
\quad -\ 10\sqrt{6}+24\\
\hline
49-\ 20\sqrt{6},\ \text{Ans.}
\end{array}$$

$$\begin{array}{ll}
\text{Mult.} & \sqrt{x}+3\sqrt{y}\\
\text{By} & \sqrt{x}+3\sqrt{y}\\
\hline
& x+3\sqrt{xy}\\
& \quad +3\sqrt{xy}+9y\\
\hline
& x+6\sqrt{xy}+9y\\
& \sqrt{x}+3\sqrt{y}\\
\hline
& x\sqrt{x}+6x\sqrt{y}+\ 9y\sqrt{x}\\
& \quad +3y\sqrt{y}+18y\sqrt{x}+27y\sqrt{y}\\
\hline
& x\sqrt{x}+9x\sqrt{y}+27y\sqrt{x}+27y\sqrt{y}=\text{Cube.}
\end{array}$$

$$\begin{array}{l}
3+2\sqrt{5}\\
3+2\sqrt{5}\\
\hline
9+6\sqrt{5}\\
+9\sqrt{5}+4\sqrt{25}\\
\hline
9+12\sqrt{5}+20\\
\text{or } 29+12\sqrt{5}
\end{array}$$

CASE IX. *To find the roots of Surd Quantities.*

RULE. When the surd is a simple quantity, multiply its index by $\frac{1}{2}$ for the square root, by $\frac{1}{3}$ for the cube root, &c. and it will give the root of the surd part; which being annexed to the root of the rational part, will give the whole root required. And if it be a compound quantity, find its root by the usual rule.

1. Find the square root of $9\sqrt[3]{3}$, and the cube root of $\frac{1}{8}\sqrt{2}$.

Here $(9\sqrt[2]{3})^{\frac{1}{2}}=9^{\frac{1}{2}}\times2^{\frac{1}{3}\times\frac{1}{2}}=9^{\frac{1}{2}}\times3^{\frac{1}{6}}=\sqrt[6]{3}$, Ans.

And $(\frac{1}{8}\sqrt{2})^{\frac{1}{3}}=(\frac{1}{8})^{\frac{1}{3}}\times(2^{\frac{1}{2}\times\frac{1}{3}})=\frac{1}{2}(2^{\frac{1}{6}})=\frac{1}{2}\sqrt[6]{2}$, Ans.

2. It is required to find the square root of 10^3.

Here $\sqrt{10^3}=\sqrt{(10^2+10)}=10\sqrt{10}$, Ans. $10\sqrt{(10)}$.

3. Find the cube root of $\frac{8}{27}a^4$, and the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$.

$$\sqrt{\frac{8}{27}a^4}=\sqrt{(\frac{8}{27}a^3\times a)}=\frac{2}{3}a\sqrt[3]{a},\ \text{and}\ \frac{a}{3}\sqrt{\frac{a}{3}}=(\frac{a}{3})^1\times(\frac{a}{3})^{\frac{1}{2}}=(\frac{a}{3})^{\frac{3}{2}}$$

Consequently $\sqrt[3]{(\frac{a}{3}\sqrt{\frac{a}{3}})}=\sqrt[3]{(\frac{a}{3})}^{\frac{3}{2}}=(\frac{a}{3})^{\frac{3}{6}}=(\frac{3a}{9})^{\frac{1}{2}}=\frac{1}{3}\sqrt{3a}$.

4. It is required to find the 4th root of $\frac{16}{81}a^{\frac{2}{3}}$.

Here $\sqrt{\frac{16}{81}a^{\frac{2}{3}}}=\frac{2}{3}a^{\frac{2}{3}\times\frac{1}{4}}=\frac{2}{3}a^{\frac{1}{6}}$, Ans.

5. Find the square roots of $x^2-4x\sqrt{a}+4a$, and $a+2\sqrt{ab}+b$.

$$\begin{array}{l}
\text{Here } a+2\sqrt{ab}+b(\sqrt{a}+\sqrt{b}\\
\quad a\\
\hline
2\sqrt{a}+\sqrt{b})2\sqrt{ab}+b\\
\qquad\qquad 2\sqrt{ab}+b
\end{array}
\qquad
\begin{array}{l}
x^2-4x\sqrt{a}+4a(x-2\sqrt{a}\\
x^2 \qquad\qquad\qquad\qquad \text{Ans.}\\
\hline
2x-2\sqrt{})-4x\sqrt{a}+4a\\
\qquad\qquad -4x\sqrt{a}+4a
\end{array}$$

CASE X. *To transform a Binomial, or a Residual Surd, into a General Surd.*

RULE. Involve the given binomial, or residual, to a power corresponding with that denoted by the surd; then set the radical sign of the same root over it, and it will be the general surd required.

1. It is required to reduce $2+\sqrt{3}$ to a general surd.

Here $(2+\sqrt{3})^2=4+3+4\sqrt{3}=7+4\sqrt{3}$; therefore $2+\sqrt{3}=\sqrt{(7+4\sqrt{3})}$, the answer.

2. It is required to reduce $\sqrt{2}+\sqrt{3}$ to a general surd.

Here $(\sqrt{2}+\sqrt{3})^2=2+3+2\sqrt{6}=5+2\sqrt{6}$; therefore

$$\sqrt{2}+\sqrt{3}=\sqrt{(5+2\sqrt{6})}, \text{ the answer.}$$

3. It is required to reduce $\sqrt[3]{2}+\sqrt[3]{4}$ to a general surd.

Here $(\sqrt[3]{2}+\sqrt[3]{4})^3=6+6\sqrt[3]{2}+6\sqrt[3]{4}$; therefore

$$\sqrt[3]{2}+\sqrt[3]{4}=\sqrt[3]{\{6(1+\sqrt[3]{2}+\sqrt[3]{4})\}}, \text{ the answer.}$$

4. Reduce $3-\sqrt{5}$, and $\sqrt{2}-2\sqrt{6}$ to a general surd.

$(3-\sqrt{5})^2=9-6\sqrt{5}+5=14-6\sqrt{5}$, and $\sqrt{(14-6\sqrt{5})}$, Ans.

6. Reduce $2\sqrt[3]{3}-3\sqrt[3]{9}$, and $7\sqrt[3]{3}-3\sqrt[3]{9}$ to a general surd.

In examples involving cube root radicals, it is useful to know the following form of the cube of a binomial: viz.

$$(a\pm b)^3=a^3\pm b^3+3ab(a\pm b)$$

Hence $(2\sqrt[3]{3}-3\sqrt[3]{9})^3=24-243-18\sqrt[3]{27}(2\sqrt[3]{3}-3\sqrt{9})=-219-54(2\sqrt[3]{3}-3\sqrt[3]{9})$; consequently

$$\sqrt[3]{(-219-54(2\sqrt[3]{3}-3\sqrt[3]{9}))} \text{ is the general surd.}$$

or, Ans. $\sqrt[3]{(162\sqrt[3]{9}-108\sqrt[3]{3}-219.)}$

$2\sqrt[3]{3}-3\sqrt[3]{9}$ multiply
$2\sqrt[3]{3}-3\sqrt[3]{9}$ by

$4\sqrt[3]{9}-6\sqrt[3]{27}$
$\quad -6\sqrt[5]{27}+9\sqrt[3]{81}$

$4\sqrt[3]{9}-12\sqrt[3]{27}+9\sqrt[3]{81}$
or $4\sqrt[3]{9}-36+27\sqrt[3]{3}$
$\quad 2\sqrt[3]{3}-3\sqrt[3]{9}$

$8\sqrt[3]{27}-72\sqrt[3]{3}+54\sqrt[3]{9}$
$\quad -12\sqrt[3]{81}+108\sqrt[3]{9}-81\sqrt[3]{27}$

$8\sqrt[3]{27}-108\sqrt[3]{3}+162\sqrt[3]{9}-81\sqrt[3]{27}$,
or $24-108\sqrt[3]{3}+162\sqrt[3]{9}-243$ Hence
I have $\sqrt[3]{\{(-219-54(2\sqrt[3]{3}-3\sqrt[5]{})\}}$
$-3\sqrt[3]{9}))$ Ans.

$7\sqrt[3]{3}-3\sqrt[3]{9}$ multiply
$7\sqrt[3]{3}-3\sqrt[3]{9}$ by

$49\sqrt[3]{9}-(21\sqrt[3]{27}=63)$
$\quad -21\sqrt[3]{27}+9\sqrt[3]{81}$

Prod. $49\sqrt[3]{9}-126+27\sqrt[3]{3}$.

$49\sqrt[3]{9}-126+27\sqrt[3]{3}$ multiply
$7\sqrt[3]{3}-3\sqrt[3]{9}$ by

$343\sqrt[3]{27}-882\sqrt[3]{\ }3+189\sqrt[3]{9}$
$\quad -147\sqrt[3]{81}+378\sqrt[4]{9}-81\sqrt[3]{27}$

$1029-1323\sqrt[3]{3}+567\sqrt[3]{9}-243$ or
$(786-1323\sqrt[3]{3}+567\sqrt[3]{9})^{\frac{1}{3}}$ Ans.

* $12\sqrt[3]{27}=12\times3=36$ and $9\sqrt[3]{81}=(3\times9)\sqrt[3]{3}=27\sqrt[3]{3}$, and $12\sqrt[3]{81}=12\sqrt[3]{(27\times3)}=36\sqrt[3]{3}$. Again $147\sqrt[3]{81}=147\sqrt[3]{(27\times3)}=147(\sqrt[3]{27}\times\sqrt[3]{3})=(147\times3)\times\sqrt[3]{3}=441\sqrt[3]{3}$, and $882\sqrt[3]{3}+441\sqrt[3]{3}=1323\sqrt[3]{3}$.

Case XI. *To extract the square root of a binomial, or residual surd.*

Rule. Substitute the numbers, or letters, of which the given surd is composed, in the place of their equals, in one of the two following formulæ, according as it is a binomial or a residual, and it will give the root required.

$$\sqrt{(a+\sqrt{b})}=\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)})}+\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)})}$$
$$\sqrt{(a-\sqrt{b})}=\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)})}-\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)})}$$

And if the second part of the binomial, or residual, in this case, be an imaginary surd, the same theorems will still hold, by only changing $-b$ into $+b$, as below.

$$\sqrt{(a+\sqrt{-b})}=\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2+b)})}+\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2+b)})}$$
$$\sqrt{(a-\sqrt{-b})}=\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2+b)})}-\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2+b)})}$$

Where it is to be observed, that the only cases that are useful in this extraction, are when a is rational, and a^2-b in the first of these formulæ, or a^2+b in the latter, is a complete square.*

1. Find the square root of $11+6\sqrt{2}$, or $11+\sqrt{72}$.

Here $\sqrt{(\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(\tfrac{11}{2}+\tfrac{1}{2}\sqrt{(121-72)})}=\sqrt{(\tfrac{11}{2}+\tfrac{7}{2})}=3$; and $\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(\tfrac{11}{2}-\tfrac{1}{2}\sqrt{(121-72)})}=\sqrt{(\tfrac{11}{2}-\tfrac{7}{2})}=\sqrt{2}$; whence $\sqrt{(11+6\sqrt{2})}$, or $\sqrt{(11+\sqrt{72})}=3+\sqrt{2}$, Ans. And by a similar substitution in the second formula, we shall have $\sqrt{(11-6\sqrt{2})}=3-\sqrt{2}$.

2. It is required to find the square root of $1+4\sqrt{-3}$, or $1+\sqrt{-48}$.

Here $\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1+48)})}=\sqrt{(\tfrac{1}{2}+\tfrac{7}{2})}=2$; & $\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{(1+48)}}=\sqrt{(\tfrac{1}{2}-\tfrac{7}{2})}=\sqrt{-3}$, whence $\sqrt{(1+4\sqrt{-3})}$, or $\sqrt{(1+\sqrt{-48})}=2+\sqrt{-3}$, Ans.

And by operating in the same manner with the last formula, we shall have $\sqrt{(1-4\sqrt{-3})}=2-\sqrt{-3}$.

3. Required to find the square root of $6\pm2\sqrt{5}$, or $23\pm8\sqrt{7}$, or $36\pm10\sqrt{11}$, or $33\pm12\sqrt{6}$, or $((1+4\sqrt{-3})=(1+\sqrt{-48}))$, or $((3\pm4\sqrt{-1})=(3\pm\sqrt{-16}))$, or $-1+\sqrt{-8}$, or $a^2+2x\sqrt{(a^2-x^2)}$, or $6+2\sqrt{2}-\sqrt{(12)}-\sqrt{(24)}$, each of the 9 foregoing questions.

Here I have $\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(\tfrac{6}{2}+\tfrac{1}{2}\sqrt{(36-20)})}=\sqrt{(3+2)}=5$, and $\sqrt{\tfrac{1}{2}-\tfrac{1}{2}(a^2-b)}=\sqrt{(\tfrac{6}{2}-\tfrac{1}{2}\sqrt{(36-20)})}=\sqrt{(3-2)}=1$, and $\therefore\ \sqrt{5}\pm1=$ Ans. or $\sqrt{(\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{(a^2-b)}}=\sqrt{\tfrac{23}{2}}+$

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* This method of extracting the square root of a binomial surd, was known as early as the time of LUCAS DE BURGO; who, in his Treatise entitled *Summa de Arithmetica, &c.* printed in 1494, has given a rule for that purpose, which, though not expressed in general terms, is similar to the one above used. Rules for extracting the cube roots of certain binomial surds, which are useful in some cases of cubic equations, were also introduced, with the solutions of these equations, by TARTALIA and CARDAN, in their Treatises on this subject; and others, of a different kind, have since been invented by later writers; but all these rules, as well as those for the higher orders, are more or less tentative, and only answer for particular numbers; in which cases, that given in the text will be found as commodious as any that has yet been proposed.

$\frac{1}{2}\sqrt{(529-448)})=\sqrt{(11\frac{1}{2}\pm4\frac{1}{2})}=4$, and $\sqrt{7}$, $\therefore 4\pm\sqrt{7}$ the 2nd Ans., the 3d, $\sqrt{(\frac{1}{2}a\pm\frac{1}{2}\sqrt{(a^2-b)}} = \sqrt{(\frac{1^3}{2}\pm\frac{1}{2}\sqrt{(36^2-1100)})} = \sqrt{(18\pm7)}=5$, and $\sqrt{11}$; $\therefore 5\pm\sqrt{11}$, Ans. ; or I have the 4th, $\sqrt{(\frac{1}{2}a\pm\frac{1}{2}\sqrt{(a^2-b)}}=\sqrt{(\frac{3^3}{2}\pm\frac{1}{2}\sqrt{33^2-864})}) = \sqrt{(16\frac{1}{2}\pm7\frac{1}{2})} = \sqrt{24}$ and 3 ; $\therefore 2\sqrt{6}\pm3$, Ans.

Here $\sqrt{(\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\frac{1}{2}+\frac{1}{2}\sqrt{(1+48)})}=\sqrt{(\frac{1}{2}+\frac{1}{2})} = \sqrt{4}=2$; and $\sqrt{(\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\frac{1}{2}-\frac{1}{2})\sqrt{(1+48)})}=\sqrt{(\frac{1}{2}-\frac{7}{2})}=\sqrt{-3}$. Whence $\sqrt{(1+\sqrt{-48})}=2+\sqrt{-3}$, Ans.

Here $\sqrt{(\frac{1}{2}a+\frac{1}{2}\sqrt{)a^2+b)})}=\sqrt{(\frac{3}{2}+\frac{1}{2}\sqrt{(9+16)})} = \sqrt{(\frac{3}{2}+\frac{5}{2})}= \sqrt{4}=2$; and $(\sqrt{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\frac{3}{2}-\frac{1}{2}\sqrt{(9+16)})} = \sqrt{(\frac{3}{2}-\frac{5}{2})}=\sqrt{-1}$. Whence $\sqrt{(3\pm\sqrt{-16})}=2\pm\sqrt{-1}$, Ans.

Here $\sqrt{(\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(-\frac{1}{2}+\frac{1}{2}\sqrt{(1\mp8)})}=\sqrt{(-\frac{1}{2}+\frac{3}{2})} =\sqrt{1}=1$; and $\sqrt{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2+b)})} = \sqrt{(-\frac{1}{2}-\frac{1}{2}\sqrt{(1+8)})} = \sqrt{(-\frac{1}{2}-\frac{3}{2})}=\sqrt{-2}$. Whence $\sqrt{(-1\pm\sqrt{-8})}=1\pm\sqrt{-2}$, A.

Here $\sqrt{(\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)})}=\sqrt{\{\frac{1}{2}a^2+\frac{1}{2}\sqrt{(a^4-4a^2x^2+4x^4)}\}} = \sqrt{(a^2-x^2)}$;

And $\sqrt{(\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)})}=(\sqrt{\frac{1}{2}a^2-\frac{1}{2}\sqrt{(a^4-4a^2x^2+4x^4)})}=x$.

Whence $\sqrt{(a^2+2x\sqrt{(a^2-x^2)})}=x+\sqrt{(a^2-x^2)}$, Ans.

Let $-\sqrt{12}-\sqrt{24}$ be reduced to a general surd, and it becomes $-\sqrt{(+36+24\sqrt{2})}$. Hence $6+2\sqrt{2}=a$, and $+36+24\sqrt{2}=b$; therefore $\sqrt{(\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(3+\sqrt{2}+\frac{1}{2}\sqrt{(44+24\sqrt{2}-36-24\sqrt{2})})}=\sqrt{(3+\sqrt{2}+\frac{1}{2}\sqrt{8})}=\sqrt{(3+2\sqrt{1})}=1+\sqrt{2}$.

Again, $\sqrt{(\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)})} = \sqrt{(3+\sqrt{2}-\frac{1}{2}\sqrt{(44+24\sqrt{2}-36-24\sqrt{2})})}=\sqrt{(3+\sqrt{2}-\frac{1}{2}\sqrt{8})}=\sqrt{(3+\sqrt{2}-\sqrt{3})}=\sqrt{3}$.

Whence $\sqrt{(6+2\sqrt{2}-\sqrt{12}-\sqrt{24})}=1+\sqrt{2}-\sqrt{3}$, Ans.

CASE XII. *To extract the cube root of a binomial, or residual surd.*

RULE. Let the surd, that is to have its root extracted, be of the form $\sqrt[3]{(a+\sqrt{b})}$, or $\sqrt[3]{(a-\sqrt{b})}$.

Then, if a^2-b be a perfect integral cube, and some whole number, can be found, that when substituted for n, will make $n^3-3\{\sqrt[3]{(a^2-b)}\}n=2a$, the roots of two expressions, in this case, will be $\sqrt[3]{(a+\sqrt{b})}=\frac{1}{2}n+\frac{1}{2}\sqrt{\{n^2-4\sqrt[3]{(a^2-b)}\}}$, and $\sqrt[3]{(a-\sqrt{b})} = \frac{1}{2}n-\frac{1}{2}\sqrt{\{n^2-4\sqrt[3]{(a^2-b)}\}}$. And if the second part of the binomial, or residual, be an imaginary surd, and a^2+b be a perfect integral cube, the extraction may be effected, by finding the integral value of n in the following equation, as before, $n^3-3\{\sqrt[3]{(a^2+b)}n\} = 2a$. In which last case, the roots of the two expressions will be $\left\{\begin{array}{l}\sqrt[3]{(a+\sqrt{-b})}=\frac{1}{2}n+\frac{1}{2}\sqrt{\{n^2-4\sqrt[3]{(a^2+b)}\}}\\ \sqrt[3]{(a-\sqrt{-b})}=\frac{1}{2}n-\frac{1}{2}\sqrt{\{n^2-4\sqrt[3]{(a^2+b)}\}}\end{array}\right\}$, each of which latter formulæ may be obtained, by barely changing the sign of b in the former.

1. Find the cube root of $10\pm6\sqrt{3}$, or $10\pm\sqrt{108}$.

Here $a=10$, and $b=108$; whence $(\sqrt[3]{(a^2-b)}=\sqrt[3]{(100-108)}$ $=-2$, and $n^3-3\{\sqrt[3]{(a^2-b)}n\}=20$, or $n^3+6n=20$, where it readily appears, from inspection, that $n=2$. Whence $\sqrt[3]{(10+\sqrt{108})}$ $=\frac{2}{2}+\frac{1}{2}\sqrt{(4-4\times-2)}=1+\frac{1}{2}\sqrt{12}=1+\sqrt{3}$, and $\sqrt[3]{(10-\sqrt{108})}$ $=\frac{2}{2}-\frac{1}{2}\sqrt{(4-4\times-2)}=1-\frac{1}{2}\sqrt{12}=1-\sqrt{3}$.

2. Fnd the cube root of $2\pm 11\sqrt{-1}$, or $2\pm\sqrt{-121}$.

Here $a=2$, and $b=121$; whence $\sqrt[3]{(a^2+b)}=\sqrt[3]{(4+121)}=5$, $n^3-3\{\sqrt[3]{(a^2+b)}\}n=2a$, or $n^3-15n=4$, where it appears, from inspection, that $n=4$. Whence $\sqrt[3]{(2+\sqrt{-121})}=\frac{4}{2}+\frac{1}{2}\sqrt{(16-4\times 5)}=2+\frac{1}{2}\sqrt{-4}=2+\sqrt{-1}$, and $\sqrt[3]{(2-\sqrt{-121})}=\frac{4}{2}-\frac{1}{2}\sqrt{(16-4\times 5)}=2-\frac{1}{2}\sqrt{-4}=2-\sqrt{-1}$.

3. Required the cube roots of $45\pm29\sqrt{2}$, $9\pm4\sqrt{5}$, $9\pm\sqrt{80}$, $20\pm68\sqrt{-7}$, $35\pm69\sqrt{-6}$, and $81\pm\sqrt{-2700}$.

Answers: $3+\sqrt{2}$ and $3-\sqrt{2}$; $\frac{3}{2}+\frac{1}{2}\sqrt{5}$ and $\frac{3}{2}-\frac{1}{2}\sqrt{5}$; $5+\sqrt{-7}$ and $5-\sqrt{-7}$; $5+\sqrt{-6}$ and $5-\sqrt{-6}$; $-3+2\sqrt{-5}$ and $-3-2\sqrt{3}$.

For Trinomial, Quadrinomial Surds, &c.

RULE. Divide half the product of any two radicals by a third, gives the square of one radical part of the root; this repeated with different quantities, will give the squares of all the parts of the root, to be connected by + and —. But if any quantity occur oftener than once, it must be taken but once. For if $x+y+z$ be any trinomial surd, its square will be $x^2+y^2+z^2+2xy+2xz+2yz$; then if half the product of any two rectangles, as $2xy+2xz$, (or $2x^2yz$) be divided by some third $2yz$, the quotient $\frac{2x^2yz}{2yz}=x^2$, must needs be the square of one of the parts; and the like for the rest.

Extract the □ root of $10+\sqrt{(24)}+\sqrt{(40)}+\sqrt{(60)}$.

Here $\frac{\sqrt{(24}\times\sqrt{(40)}}{2\sqrt{(60)}}=2$, and $\frac{\sqrt{(24)}\times\sqrt{(60)}}{2\sqrt{(40)}}=\sqrt{9}=3$, and $\frac{\sqrt{(40)}\times\sqrt{(60)}}{2\sqrt{(24)}}=\sqrt{(25)}=5$. And the root is $\sqrt{2}+\sqrt{3}+\sqrt{5}$.

Find the square root of $14+\sqrt{(32)}-\sqrt{(48)}+\sqrt{(80)}-\sqrt{(24)}+\sqrt{(40)}-\sqrt{(60)}$. Here $\frac{\sqrt{(32\times48)}}{2\sqrt{(80)}}=\frac{\sqrt{(24)}}{\sqrt{5}}$, this produces nothing. Again $\frac{\sqrt{(32\times48)}}{2\sqrt{(24)}}=\sqrt{(16)}=4$. And $\frac{\sqrt{(40\times60)}}{2\sqrt{(24)}}=\sqrt{(25)}=5$; and $\frac{\sqrt{(32\times40)}}{2\sqrt{(60)}}=\sqrt{4}=2$; and $\frac{\sqrt{(48\times24)}}{2\sqrt{(32)}}=\sqrt{9}=3$; and $\frac{\sqrt{(32\times80)}}{2\sqrt{(40)}}=\sqrt{(16)}=4$, &c., therefore the parts of the root are $\sqrt{4}$, $\sqrt{5}$, $\sqrt{3}$, $\sqrt{2}$, $\sqrt{4}$, &c., and the roots of $2+\sqrt{2}-\sqrt{3}+\sqrt{5}$; for, being squared, it produces the surd.

CASE XIII. *To extract any root* (c) *of a binomial surd* $A \pm B$.

Rule. Let $A^2 - B^2 = D$, take Q such, that $QD = n^3$, the last integer power. Let $\sqrt{\{(A+B) \times \sqrt{Q}\}} = r$, the nearest integer number. Reduce $A\sqrt{Q}$ to the simplest form $p\sqrt{s}$; let the nearest integer $\frac{r + \frac{n}{r}}{2\sqrt{s}} = t$; then the root $= \frac{t\sqrt{s} \pm \sqrt{(t^2 s - n)}}{\sqrt[2c]{Q}}$; if it can be extracted. Note. $+$ is for the binomial $A + B$, and $-$ for the residual $A - B$.

1. What is the cube root of $\sqrt{968} + 25$.

Here $D = 343 = 7 \times 7 \times 7$, and $Q \times 7^3 = n^3$, $Q = 1$, $n = 7$, then $\sqrt[3]{\{(A+B) \times \sqrt{Q}\}} = \sqrt[3]{56} + = r = 4$, and $A\sqrt{Q} = \sqrt{968} = 22\sqrt{2} = p\sqrt{s}$, and $\sqrt{s} = \sqrt{2} \cdot \frac{r + \frac{n}{r}}{2\sqrt{s}} = \frac{4 + \frac{7}{4}}{2\sqrt{2}} = t = 2$, and $t\sqrt{s} = 2\sqrt{2}$; $\sqrt{(t^2 s - n)} = \sqrt{(8-7)} = 1$; $\sqrt[6]{Q} = 1$, and the root $\frac{2\sqrt{2}+1}{1} = 2\sqrt{2}+1$, which I find to succeed.

2. Find the cube root of $68 - \sqrt{4374}$.

Here $D = 250 = 5 \times 5 \times 5 \times 2$, and $5^3 \times 2^3 = 4D = QD = n^3$; and $Q = 4$, $n = 2 \times 5 = 10$, and $\sqrt[3]{\{(A+B) \times \sqrt{Q}\}} = \sqrt{(134 \times 2)} = 6 = r$; $A\sqrt{Q} = 136\sqrt{1} = p\sqrt{s}$, and $\sqrt{s} = 1$; $\frac{r + \frac{n}{r}}{2\sqrt{s}} = \frac{7\frac{2}{3}}{2} = \frac{23}{6} = 4 = t$, and $t\sqrt{s} = 4$, and the root $\frac{t\sqrt{s} - \sqrt{(t^2 s - n)}}{\sqrt[6]{2}} = \frac{\sqrt{(16-10)}}{\sqrt[6]{4}} = \frac{4 - \sqrt{6}}{\sqrt[3]{2}}$; for its cube is $68 - 27\sqrt{6}$.

3. Find the 5th root of $29\sqrt{6} + 41\sqrt{3}$.

Here $D = 3$, $n = 3$, $Q = 81$, $r = 5$, $\sqrt{s} = 6$, $t = 1$, $t\sqrt{s} = 6$, $\sqrt{\{(t^2 s - n)\}} = \sqrt{3}$, $\sqrt{Q} = \sqrt{81} = \sqrt[5]{9}$, and the root to be tried $\frac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$. Scholium. If the quantity be a fraction, or has a common divisor, extract the root of the denominator, or of that common divisor, separately The demonstration may be seen in Sir Isaac Newton's Universal Arithmetic, Gravesend's, or MacLaurin's, or Dr. Waring's Med. Algebra.

CASE XIV. *To find such a multiplier, or multipliers, as will make any binomial surd rational.*

RULE 1. When one, or both of the terms, are any even roots; multiply the given binomial, or residual, by the same expression, with the sign of one of its terms changed, and repeat the operation in the same way, as long as there are surds; when the last result will be rational.

In like manner, a trinomial surd may also be rendered rational, by changing the sign of one of its terms for the multiplier; and a

quadrinomial surd by changing the signs of two of its terms, &c.

2. When the terms of the binomial surds are odd roots, the rule becomes more complicated; but for the sum, or difference, of two cube roots, which is one of the most useful cases, the multiplier will be a trinomial surd consisting of the squares of the two given terms, and their product, with its sign changed.*

Given surd	$5+\sqrt{3}$	$\sqrt{5}+\sqrt{3}$	$\sqrt[4]{5}+\sqrt[4]{3}$	given surd.
Multiplier	$5-\sqrt{3}$	$\sqrt{5}-\sqrt{3}$	$\sqrt[4]{5}-\sqrt[4]{3}$	1st mult.
	$25+5\sqrt{3}$	$5+\sqrt{15}$	$\sqrt{5}-\sqrt{3}$	1st prod.
	$-5\sqrt{3}-3$	$-\sqrt{15}-3$	$\sqrt{5}+\sqrt{3}$	2d mult.
Product	$25-3=22$	$5-3=2$	$5-3=2$	Ans. ratio.

$$\begin{aligned}
&\sqrt{5}+\sqrt{3}-\sqrt{2} \text{ given surd.}\\
&\sqrt{5}+\sqrt{3}+\sqrt{2} \text{ 1st mult.}\\
&5+\sqrt{15}-\sqrt{10}\\
&\quad+\sqrt{15}+3-\sqrt{6}\\
&\qquad+\sqrt{10}+\sqrt{6}-2\\
&6+2\sqrt{15} \text{ 1st product.}\\
&-6+2\sqrt{15} \text{ 2d multiplier.}\\
&-36-12\sqrt{15}\\
&\quad+12\sqrt{15}+60\\
&60-36=24, \text{ Ans.}
\end{aligned}$$

$$\begin{aligned}
&\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25} \text{ mult.}\\
&\sqrt[3]{7}-\sqrt[3]{5} \text{ given surd.}\\
&\sqrt[3]{343}+\sqrt[3]{245}+\sqrt[3]{175}\\
&\qquad-\sqrt[3]{245}-\sqrt[3]{175}-\sqrt{125}\\
&\sqrt[3]{343} \quad * \quad * \quad \sqrt[3]{325}=7-5\\
&\text{Given } \sqrt[3]{7}+\sqrt[3]{3} \text{ surd.}\\
&\text{Mult. } \sqrt[3]{7^2}-\sqrt[3]{21}+\sqrt[3]{3^2}\\
&7-\sqrt[3]{147}+\sqrt[3]{63}\\
&\quad+\sqrt[3]{147}-\sqrt[3]{63}+3\\
&\text{Ans. } 10=7+ \quad * \quad * \quad 3
\end{aligned}$$

1. Let $(\sqrt[8]{a}+\sqrt[8]{b})\times(\sqrt[8]{a}-\sqrt[8]{b})=\sqrt[4]{a}-\sqrt[4]{b}$, and again I have $(\sqrt[4]{a}-\sqrt[4]{b})\times(\sqrt[4]{a}+\sqrt[4]{b})=\sqrt{a}-\sqrt{b}$, and 3d $(\sqrt{a}-\sqrt{b})\times(\sqrt{a}+\sqrt{b})=a-b$ rational.

2. Let $a+\sqrt[4]{b}$ be given; then $(a+\sqrt[4]{b})\times(a-\sqrt[4]{b})=a^2-\sqrt{b}$, and $(a^2-\sqrt{b})\times(a^2+\sqrt{b})=a^4-b$, the 2d product.

3. Let $(a-\sqrt[3]{2})\times(a^2+a\sqrt[3]{2}+\sqrt[3]{4})=a^3-2$, product.

4. Let $(\sqrt[3]{a}+\sqrt[3]{b})\times(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2})=a+b$, Answer.

5. Let $\sqrt[4]{5}+\sqrt{3}$ be given, or reduced $(\sqrt[4]{5}+\sqrt[4]{9})(\sqrt[4]{125}-\sqrt[4]{(5^2\times9)}+\sqrt[4]{(5\times9^2)}-\sqrt[4]{9^3})=5-9=-4$, product.

Or $(\sqrt[4]{9}+\sqrt[4]{5})\times\{\sqrt[4]{9^3}-\sqrt[4]{(9^2\times5)}+\sqrt[4]{(9\times5^2)}-\sqrt[4]{5^3}\}=9-5=4$.

* If a multiplier be required, that shall render any binomial surd, whether it consist of even or odd roots, rational, it may be found by substituting the given numbers, or letters, of which it is composed, in the places of their equals, in the following general formula: Binomial $\sqrt[n]{a}\pm\sqrt[n]{b}$

Multiplier $\sqrt[n]{a^{n-1}}\mp\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}\mp\sqrt[n]{a^{n-4}b^3}+$ &c.

where the upper sign of the multiplier must be taken with the upper sign of the binomial, and the lower with the lower; and the series continued to n terms.

CASE XV. *To reduce a fraction, whose denominator is a surd, to another that shall have a rational denominator.*

RULE 1. When the proposed fraction is a simple one, of the form $\frac{b}{\sqrt{a}}$, multiply each of its terms by $\sqrt{a}$, and the resulting fraction will be $\frac{b\sqrt{a}}{a}$. Or when it is of the form $\frac{b}{\sqrt[3]{a}}$, multiply the terms by $\sqrt[3]{a^2}$, and the result will be $\frac{b\sqrt[3]{a^2}}{a}$. And for the general form $\frac{b}{\sqrt[n]{a}}$, multiply by $\sqrt[n]{a^{n-1}}$, & the result will be $\frac{b\sqrt[n]{a^{n-1}}}{a}$.

2. If it be a compound surd, find such a multiplier, by one of the rules of the last case, as will make the denominator rational; and multiply both the numerator and denominator by it.

1. Reduce the fractions $\frac{2}{\sqrt{3}}$; $\frac{3}{\frac{1}{2}\sqrt[4]{5}}$, and $\frac{3}{\sqrt{5}-\sqrt{2}}$, to a fractions whose denominators shall be rational.

Here $\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$; and $\frac{3}{\frac{1}{2}\sqrt[4]{5}}\times\frac{\sqrt[4]{5^3}}{\sqrt[4]{5^3}}=\frac{3\sqrt[4]{5^3}}{\frac{1}{2}\times5}$ $=\frac{6\sqrt[4]{5^3}}{5}=\frac{6}{5}\sqrt[4]{125}$. Here $\frac{3}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=$ $\frac{3\sqrt{5}+3\sqrt{2}}{5-2}=\frac{3\sqrt{5}+3\sqrt{2}}{3}=\frac{\sqrt{5}+\sqrt{2}}{1}=\sqrt{5}+\sqrt{2}$, Ans.

2. Reduce $\frac{\sqrt{2}}{3-\sqrt{2}}$ to a fraction, whose denominater shall be rational. Here

$\frac{\sqrt{2}}{3-\sqrt{2}}=\frac{\sqrt{2}\times(3+\sqrt{2})}{(3-\sqrt{2})\times(3+\sqrt{2})}=\frac{3\sqrt{2}+2}{9-2}=\frac{2+3\sqrt{2}}{7}=\frac{2}{7}$ $+\frac{3}{7}\sqrt{2}$, Ans.

3. Reduce $\frac{2}{\sqrt{3}}$; $\frac{3}{\frac{1}{2}\sqrt[4]{5}}$; $\frac{3}{\sqrt{5}-\sqrt{2}}$; or $\frac{\sqrt{2}}{3-\sqrt{2}}$; or $\frac{4}{\sqrt[3]{7}}$; $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$; or $\frac{\sqrt{2}}{3+\sqrt{x}}$; or $\frac{a-\sqrt{b}}{a+\sqrt{b}}$; or $\frac{10}{\sqrt[3]{7}-\sqrt[3]{5}}$; or $\frac{\sqrt[3]{3}}{\sqrt[3]{9}+\sqrt[3]{10}}$; or $\frac{4}{\sqrt[4]{4}+\sqrt[4]{5}}$; or $\frac{2}{\sqrt{5}+\sqrt{3}}$; or $\frac{8}{\sqrt{3}+\sqrt{2}+1}$ and $\frac{1}{\sqrt{5}+\sqrt{7}+\sqrt{11}}$, each of the 14 preceding questions to others that shall have rational denominators.

$\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}\times\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}}=\frac{\sqrt{42}-\sqrt{18}}{7-3}=\frac{\sqrt{42}-\sqrt{18}}{4}$, Ans.

$\frac{x}{3+\sqrt{x}}\times\frac{3-\sqrt{x}}{3-\sqrt{x}}=\frac{3x-x\sqrt{x}}{9-x}$, Ans. $\frac{a-\sqrt{b}}{a+\sqrt{b}}\times\frac{a-\sqrt{b}}{a-\sqrt{b}}=$

$\frac{(a-\sqrt{b})^2}{a-\sqrt{b}}=\frac{a^2+b-2a\sqrt{b}}{a^2-b}$; $\frac{10}{3\sqrt[3]{7}-\sqrt[3]{5}}\times\frac{\sqrt[3]{7^2}+\sqrt[3]{35}+\sqrt[3]{5^2}}{\sqrt[3]{7^2}+\sqrt[3]{35}+\sqrt[3]{5^2}}$ $=5\times(\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25})$, Ans.

$\frac{\sqrt[3]{3}}{\sqrt[3]{9}+\sqrt[3]{10}}\times\frac{\sqrt[3]{81}-\sqrt[3]{90}+\sqrt[3]{100}}{\sqrt[3]{9^2}-\sqrt[3]{90}+\sqrt[3]{10^2}}=\frac{3\sqrt[3]{9}-3\sqrt[3]{10}+\sqrt[3]{300}}{19}$. Answer.

$\frac{4}{\sqrt[4]{4}+\sqrt[4]{5}}\times\frac{\sqrt[4]{4}-\sqrt[4]{5}}{\sqrt[4]{4}-\sqrt[4]{5}}\times\frac{\sqrt{4}-\sqrt{5}}{\sqrt{4}-\sqrt{5}}=\sqrt{10}-2\sqrt{2}+(2-\sqrt{5})\sqrt[4]{5}$, Ans.

$\frac{2}{\sqrt[4]{5}+\sqrt[4]{3}}\times\frac{\sqrt[4]{5}-\sqrt[4]{3}}{\sqrt[4]{5}-\sqrt[4]{3}}\times\frac{\sqrt[4]{5}+\sqrt{3}}{\sqrt[4]{5}+\sqrt{3}}=$ $\frac{2(\sqrt{125}-\sqrt{75}+\sqrt{45}-\sqrt{27})}{5-3=2}=\sqrt{125}-\sqrt{75}+\sqrt{45}-\sqrt{27}$, the answer required. Or thus; multiply the terms of the fractions, $\frac{2}{\sqrt[4]{5}+\sqrt[4]{3}}$ by $\sqrt[4]{5}-\sqrt[4]{3}$, and it becomes $\frac{2\sqrt[4]{5}-2\sqrt[4]{3}}{\sqrt{5}-\sqrt{3}}$; again, multiply the terms of the last fractions by $\sqrt{5}+\sqrt{3}$, and it be- becomes $\frac{(5^{\frac{3}{4}}-5^{\frac{1}{2}}3^{\frac{1}{4}}+3^{\frac{1}{2}}5^{\frac{1}{4}}-3^{\frac{3}{4}})2}{5-3=2}$, Ans.

$\frac{8}{\sqrt{3}+\sqrt{2}+1}\times\frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}+\sqrt{2}-1}=\frac{8\sqrt{3}+8\sqrt{2}-8}{5+2\sqrt{6}-1}=$ $\frac{4\sqrt{3}+4\sqrt{2}-4}{2+\sqrt{6}}$; again, multiply by $-2+\sqrt{6}$, and it becomes $\frac{-8\sqrt{3}-8\sqrt{2}+8+4\sqrt{18}+4\sqrt{12}-4\sqrt{6}}{6-4=2}=4+2\sqrt{18}+2\sqrt{12}$ $-2\sqrt{6}-4\sqrt{3}-4\sqrt{2}=4+6\sqrt{2}+4\sqrt{3}-2\sqrt{6}-4\sqrt{3}-4\sqrt{2}$ $=4+2\sqrt{2}-2\sqrt{6}$, Ans.

Cor. A binomial becomes rational after one operation, a trinomial after two, and a quadrinomial after three, &c.

Cor. The number of operations is equal to the power of 2 in the index.

A and B began to trade with equal sums of money. In the first year A gained 40 dollars and B lost 40; but in the second A lost one third of what he then had, and B gained a sum less by 40 dollars than twice the sum that A had lost; when it appeared that B had twice as much money as A. What money did each begin with?

Let $x=$ the number of dollars each had at first; then $x+40=$ the sum A had after the first year, and $x-40=$ the sum B had; also $\frac{2}{3}.(x+40)=$ the sum A had after the second year, and $x-40+\frac{2}{3}(x+40)-40=$ the sum B had; $\therefore$ $\frac{4}{3}.(x+40)=x-40+\frac{2}{3}.(x+40)-40$, and $\frac{2}{3}.(x+40)=x-80$; $\therefore$ $2x+80=3x-240$, and $x=320$.

Of Cubic Equations.

95. A cubic equation is that in which the unknown quantity rises to 3 dimensions; and, like quadratics, or those of the higher orders, is either simple or compound. A simple or pure cubic equation is of the form $ax^3=b$, or $x^3=\frac{b}{a}$; where $x=\sqrt[3]{\frac{b}{a}}$.

A compound cubic equation is of the form

$$x^3+ax=b,\ x^3+ax^2=b,\text{ or }x^3+ax^2+bx=c,$$

in each of which the known quantities a, b, c, may be either $+$ or $-$.

Or either of the two latter of these equations may be reduced to the same form as the first, by taking away its second term; which is done as follows:

RULE. Take some new unknown quantity, and subjoin to it a third part of the coefficient of the second term of the equation with its sign changed; then, if this sum, or difference, as it may happen to be, be substituted for the original unknown quantity and its powers in the proposed equation, there will arise an equation wanting its second term.

Note. The second term of any of the higher orders of equations may also be exterminated in a similar manner, by substituting for the unknown quantity some other unknown quantity, and the 4th, 5th, &c. part of the coefficient of its second term, with the sign changed, according as the equation is of the 4th, 5th, &c. power.

1. Required to exterminate the second term of the equation $x^3+3ax^2=b$, or $x^3+3ax^2-b=0$. Here $x=z-\frac{3a}{3}=z-a$.

$$\text{Then }\left\{\begin{array}{lll} x^3= & z^3-3az^2+3a^2z- & a^3 \\ 3ax^2= & +3az^2-6a^2z+ & 3a^3 \\ -b= & & -b \end{array}\right\};\text{ whence }z^3-3a^2z+2a^3$$

$-b=0$; or $z^3-3a^2z=b-2a^3$, in which equation the second power (z^2) of the unknown quantity is wanting.

2. Let the equation $x^3-12x^2+3x=-16$ be transformed into another that shall want the second term. Here $x=z+4$.

$$\text{Then }\left\{\begin{array}{ll} (z+4)^3= & z^3+12z^2+48z+64 \\ -12(z+4)^2= & -12z^2-96z-192 \\ +3(z+4)= & +3z+12 \end{array}\right.$$

Whence $z^3-45z-116=-16$; or $z^3-45z=100$, which is an equation where z^2, or the second term, is wanting, as before.

3. Let the equation $x^3-6x^2=10$ be transformed into another that shall want the second term. Ans. $z^3-12z=26$.

4. Let $y^3-15y^2+81y=243$ be transformed into an equation that shall want the second term. Ans. $x^3+6x=88$.

5. Let the equation $x^3+\frac{3}{4}x^2+\frac{7}{8}x-\frac{9}{16}=0$ be transformed into another that shall want the second term. Ans. $y^3+\frac{11}{16}x=\frac{3}{4}$.

Here $y=x+5$,

$$\left.\begin{aligned} y^3 &= x^3+15x^2+\ \ 75x+125 \\ -15y^2 &= \ \ -15x^2-150x-375 \\ +81y &= \qquad\qquad\ +\ 81x+405 \\ -243 &= \qquad\qquad\qquad\quad -243 \end{aligned}\right\}$$

$$x^3+6x-88=0.$$

Here $x=y-\frac{1}{4}$,

$$\left\{\begin{aligned} x^3 &= y^3-\tfrac{3}{4}y^2+\tfrac{3}{16}y-\tfrac{1}{64} \\ +\tfrac{3}{4}x^2 &= \ \ +\tfrac{3}{4}y^2-\tfrac{3}{8}\,y+\tfrac{3}{64} \\ +\tfrac{7}{8}x &= \qquad\qquad\ \ \tfrac{7}{8}\,y+\tfrac{7}{32} \\ -\tfrac{9}{16} &= \qquad\qquad\qquad\quad -\tfrac{9}{16} \end{aligned}\right.$$

Hence we have $y^3+\frac{11}{16}y-\frac{3}{4}=0$.

6. Let $x^4+8y^3-5x^2+10x-4=0$ be transformed into another that shall want the second term. Ans. $y^4-29y^2+94y-92=0$.

Here $x=y-2$,

$$\begin{aligned} x^4 &= y^4-8y^3+24y^2-32y+16 \\ 8x^3 &= \qquad 8y^3-48y^2+96y-64 \\ -5x^2 &= \qquad\qquad\quad -5y^2+20x-20 \\ +10x &= \qquad\qquad\qquad\quad\ \ 10y-20 \\ -4 &= \qquad\qquad\qquad\qquad\quad\ -4 \end{aligned}$$

We have $x^4-29y^2+94y-92=0$.

7. Let $x^4-3x^3+3x^2-5x-2=0$ be transformed into another that shall want the third term. Ans. $y^4+y^3-4y-6=0$.

Here $x=y+e$,

$$\begin{aligned} x^4 &= y^4+4ey^3+6e^2y^2+4e^3y+e^4 \\ -3x^3 &= \quad\ -3y^3-9e\,y^2-9e^2y-3e^3 \\ +3x^2 &= \qquad\qquad\ \ +3y^2+6e\,y+3e^2 \\ -5x &= \qquad\qquad\qquad\quad\ -5\ y-5e \\ -2 &= \qquad\qquad\qquad\qquad\quad\ -2 \end{aligned}$$

Now, the value of e, by which the third term is taken away, is had by resolving the quadratic equation $6e^2-9e+3=0$; the roots of which are found $=1$, or $\frac{1}{2}$: hence, by substituting $y+1$ for x, in the given equation, we find $y^4+y^3-4y-6=0$, an equation wanting the third term.

8. Let $3x^3-2x+1=0$ be transformed into another whose roots are the reciprocals of the former. Ans. $y^3-2y^2+3=0$.

Here let $x=\frac{1}{y}$; then $3x^3-2x+1=\frac{3}{y^3}-\frac{2}{y}+1=0$, or $3-2y^2+y^3=0$; hence $y^3-2y^2+3=0$; the roots of which are the reciprocals of the former.

9. Let $x^4-\frac{1}{2}x^3-\frac{2}{3}x^2-\frac{3}{4}x+\frac{1}{18}$ be transformed into another, in which the coefficient of the highest term shall be unity, and the remaining terms integers. Ans. $y^4-3x^3+12y^2-162y+72=0$.

Here let $x=\frac{y}{6}$; then $\frac{y^4}{6^4}-\frac{y^3}{2.6^3}+\frac{y^2}{3.6^2}-\frac{3y}{4.6}+\frac{1}{18}=y^4-3y^3+12y^2-162y+72=0$, the equation required; the roots of which are 6 times those of the former.

Of the solution of Cubic Equations.

97. RULE. Take away the second term of the equation when necessary, as directed in the preceding rule. Then, if the numeral coefficients of the given equation, or of that arising from the reduction above mentioned, be substituted for a and b in either of the following formulæ, the result will give one of the roots, as required.*

$$x^3+ax=b.$$

$$x=\begin{cases}\sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}+\sqrt[3]{\left\{\frac{b}{2}-\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}} \text{ or} \\ \sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}-\dfrac{\frac{1}{3}a}{\sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}}\end{cases}$$

Where, it is to be observed that when the coefficient a, of the second term of the above equation, is negative, $\frac{a^3}{27}$, as also $\frac{a}{3}$, in the formula, will be negative; and if the absolute term b be negative, $\frac{1}{2}b$ in the formula will be negative; but $\frac{1}{4}b^2$ will be positive.

It may likewise be remarked, that when the equation is of the form $x^3-ax=\pm b$, and $\frac{a^3}{27}$ is greater than $\frac{b^2}{4}$, or $4a^3$ greater than $27b^2$, the solution of it cannot be obtained by the above rule; as the question, in this instance, falls under what is usually called the Irreducible Case of cubic equations. See index.

1. Given $2x^3-12x^2+36x=44$, to find the value of x.

Here $x^3-6x^2+18x=22$, by dividing by 2; and, in order to exterminate the second term, put $x=z+\frac{6}{3}=z+2$:

$$\text{Then}\left|\begin{array}{rl}(z+2)^3= & z^3+6z^2+12z+8\\ -6(z+2)^2= & -6z^2-24z-24\\ 18(z+2)= & 18z+36\end{array}\right|=22,$$

* If, instead of the regular method of reducing a cubic equation of the general form $x^3+ax^2+bx+c=0$, to another, wanting the second term, as pointed out in the preceding article, there be put $x=\frac{1}{3}(y-a)$, we shall have, by substitution and reduction, $y^3+(9b-3a^2)y=9ab-27c-2a^3$; where, since the value of y can be determined by either of the formulæ given in this rule, the value of x will also be known, being $x=\frac{1}{3}(y-a)$. And if $b=0$, or the original equation be of the following form, $x^3+ax^2+c=0$, the reduced equation will be $x^3-3a^2y=-27c$, where the value of y being found as above, we shall have, as before, $x=\frac{1}{3}(y-a)$, which formulæ, it may be observed, are more convenient, in some cases, than those resulting from the preceding article; as the coefficients thus obtained are always integers; whereas, by the former method, they are frequently fractions.

Whence $x^3+6x+20=22$, or $x^3+6x=2$; and, consequently, by substituting 6 for a and 2 for b, in the first formula, we shall have

$$x=\sqrt[3]{\{\tfrac{2}{2}+\sqrt{(\tfrac{4}{4}+\tfrac{216}{27})}\}}+\sqrt[3]{\{\tfrac{2}{2}-\sqrt{(\tfrac{4}{4}+\tfrac{216}{27})}\}}=$$
$$\sqrt[3]{\{1+\sqrt{(1+8)}\}}+\sqrt[3]{\{1-\sqrt{(1+8)}\}}=\sqrt[3]{(1+\sqrt{9})}+$$
$$\sqrt[3]{(1-\sqrt{9})}=\sqrt[3]{(1+3)}+\sqrt[3]{(1-3)}=\sqrt[3]{4}-\sqrt[3]{2};$$ therefore $x=z$ $+2=\sqrt[3]{4}-\sqrt[3]{2}+2=2+1.587401-1.259921=2.32748$, Ans.

2. Given $x^3-6x=12$, to find the value of x.

Here a being equal to -6, and b equal to 12, we shall have by the formula $x=\sqrt[3]{\{6+\sqrt{(36-8)}\}}-\dfrac{-2}{\sqrt[3]{\{6+\sqrt{(36-8)}\}}}=$

$$\sqrt[3]{(6+\sqrt{28})}+\frac{2}{\sqrt[3]{(6+\sqrt{28})}}=\sqrt[3]{(6+5.2915)}+\frac{2}{\sqrt[3]{(6+5.2915)}}$$
$$=\sqrt[3]{(11.2915)}+\frac{2}{\sqrt[3]{(11.2915)}}=2.2435+\frac{2}{2.2435}=2.2435+$$
$.8957=1392$; therefore $x=3.1392$, the answer.

3. Given $x^3-2x=-4$, to find the value of x.

Here a being $=-2$, and $b=-4$, we shall have, by the formula,

$x=\sqrt[3]{\{-2+\sqrt{(4-\tfrac{8}{27})}\}}+\sqrt[3]{\{-2-\sqrt{(4-\tfrac{8}{27})}\}}$, or by reduction,
$$\sqrt[3]{(-2+\tfrac{10}{9}\sqrt{3})}-\sqrt[3]{(2+\tfrac{10}{9}\sqrt{3})}=\sqrt[3]{(-2+1.9245)}$$
$$-\sqrt[3]{(2+1.9245)}=\sqrt[3]{(-.0755}-\sqrt[3]{3.9245}=-1226-1.5773=$$
$$-1.9999, \text{ or } -2, \text{ Ans.}$$

Note. When one of the roots of a cubic equation has been found, by the common formula as above, or in any other way, the other two roots may be determined as follows:

Let the known root be denoted by r, and put all the terms of the equation, when brought to the left hand side, $=0$; then, if the equation so formed be divided by $x\pm r$, according as r is positive or negative, there will arise a quadratic equation, the roots of which will be the other two roots of the given cubic.

4. Given $x^3-15x=4$, to find the three roots or values of x.

Here x is readily found, by a few trials, to be equal to 4, and therefore, by division, we get $x-4)x^3-15x-4(x^2+4x+1$;

Whence, according to the note above given, $x^2+4x+1=0$, or $x^2+4x=-1$; the two roots of which quadratic are $-2+\sqrt{3}$, and $-2-\sqrt{3}$; and consequently 4, $-2+\sqrt{3}$, and $-2-\sqrt{3}$, are the three roots of the proposed equation.

1. Given $x^3+3x^2-6x=8$, to find the root of the equation, or the value of x. See index. Ans. $x=2$.

2. Given $x^3-6x=6$, or $x^3+9x=6$, or $x^3+2x^2-23x=70$, or $x^3-17x^2+54x=350$, or $x^3-3x^2=5$, or $x^3+x^2=500$, to find the value of x in each expression. Answers, $\sqrt[3]{4}+\sqrt[3]{2}$, or $\sqrt[3]{9}-\sqrt[3]{3}$, or 5.1346, or 14.954, or 3.426, and 7.61728, as required.

3. Given $x^3-6x=4$, to find the three roots of the equation, or the three values of x. Ans. -2, $1+\sqrt{3}$, and $1-\sqrt{3}$.

4. Given $x^3-5x^2+2x=-12$, to find the three roots of the equation, or values of x. Ans. -3, $1+\sqrt{5}$, and $1-\sqrt{5}$.

6. Find the root of the equation $x^3+x=500$, or $x^3-6x=5.6$.
Ans. $x=7.89500828$, and $x=2.82526384$.

Solution of Cubic Equations by the Tables of Sines and Tangents.

98. Cubic Equations may often be more readily resolved by means of the following trigonometrical formulæ, than by the method before given; particularly when they fall under what is usually called the irreducible case, in which the common method is known to fail.

1. Given $x^3+ax=\pm b$, to find the value of x.

Put $\frac{2ar}{3b}\sqrt{\frac{a}{3}}=\tan. z$, and $\sqrt[3]{(r^2 \tan. \frac{1}{2}z)}=\tan. u$,

Then $x=\pm\frac{2}{r}\sqrt{\frac{a}{3}}\times \cot. 2u$; where the upper or under sign is to be taken according as b is positive or negative.

2. Given $x^3-ax=\pm b$, to find the value of x.

This form has two cases, according as $\frac{2ar}{3b}\sqrt{\frac{a}{3}}$ is less or greater than radius r, or $4a^3$ less or greater than $27b^2$.

In the first of these cases, or when $4a^3>27b^2$,

Put $\frac{2ar}{3b}\sqrt{\frac{a}{3}}=\sin. z$, and $\sqrt[3]{(r^2 \tan. \frac{1}{2}z)}=\tan. u$;

Then $x=\pm\frac{2}{r}\sqrt{\frac{a}{3}}\times \text{cosec. } 2u$, where the upper or under sign is to be taken, according as b is positive or negative.

In the second case, or when $4a^3<27b^2$,

Put $\frac{3br}{2a}\sqrt{\frac{3}{a}}=\cos. z$, then x will have the three following values;

$$x=\pm\frac{2}{r}\sqrt{\frac{a}{3}}\times\cos.\frac{z}{3}\,;\quad x=\mp\frac{2}{r}\sqrt{\frac{a}{3}}\times\cos.\left(60^\circ-\frac{z}{3}\right.$$

$$\text{and}\quad x=\mp\frac{2}{r}\sqrt{\frac{a}{3}}\times\cos.\left(60^\circ+\frac{z}{3}\right)$$

where the upper sign is to be used when b is positive, and the under sign when it is negative, as in the other cases.

1. Given $x^3+18x=6$, to find the only real value of x.

Here $a=18$, and $b=6$; whence $\frac{2ar}{3b}\sqrt{\frac{a}{3}}=2r\sqrt{6}=\tan. z$; and, consequently, by formula 1, we shall have

Log. 6	0.7781513
½ log. 6	0.3890756
Log. 2	0.3010300
Log. r	10.0000000
Log. tan. z	10.6901056

Therefore $z=78^\circ\ 27'\ 47''$, and $\frac{1}{2}z=39^\circ\ 13'\ 53\frac{1}{2}''$.

Secondly, $\sqrt[3]{(r^2 \tan. \frac{1}{2}z)}=\tan. u$, whence

Log. r^2	20.0000000
Log. tan. $\frac{1}{2}z$	9.9119523
Sum	3)29.9119523
Log. tan. u	9.9706507

Therefore $u=43^\circ\ 4'\ 55''$, and $2u=86^\circ\ 7'\ 50''$.

Thirdly, $\frac{2}{r}\sqrt{\frac{a}{3}} \times \cot. 2u = \frac{2}{r}\sqrt{6} \times \cot. 2u$, whence

½ log. 6	0.3890756
Log. 2	0.3010300
Log. cot. $2u$	8.8301834
Sum	9.5202890
Log. r	10.0000000
Log. x	1.5202890

Consequently $x = .3313139$, the positive root of the equation, as required.

2. Given $x^3-3x=1$, which is an equation falling under the irreducible case, to find its three roots.

Here, a being $=3$, and $b=1$, we shall have, by taking the radius $r=1$, $\cos. z=\frac{3b}{2a}\sqrt{\frac{3}{a}}=\frac{1}{2}=.5=\cos. 60^\circ$, whence

$$x=2\sqrt{\frac{a}{3}}\times \cos. \frac{z}{3}=2 \cos. 20^\circ=+1.8793852,$$

$$x=-2\sqrt{\frac{a}{3}}\times \cos. (60^\circ-\frac{z}{3})=-2 \cos. 40^\circ=-1.5320888,$$

$$x=-2\sqrt{\frac{a}{3}}\times \cos. (60^\circ+\frac{z}{3})=-2 \cos. 80^\circ=-.3472964.$$

Therefore the three roots are 1.8793852, —1.5320888, and —.3472964.

And if the equation be $x^3-3x=-1$, the three roots or values of x are —1.8793852, 1.5320888, and .3472964.

Which are the negatives of the roots of the former equation.

1. Given $x^3-2x=-2$, to find the root of the equation, or the value of x. See art. 94, p. 214. Ans. $x=-1.7693$.

2. Given $x^3-300x=-1000$, to find the root of the equation, or the value of x. See index. Ans. $x=3.472964$.

3. Given $x^3-9x=9$, to find all the three roots of the equation, or values of x. Ans. 3.411474, —2.226682, and —1.184792.

4. Given $x^3-x^2-2x+1=0$, to find all the three roots of the equation, or the three values of x. Ans. 2 cos. $\frac{1}{7}n$, —2 cos. $\frac{2}{7}n$, and 2 cos. $\frac{3}{7}n$, where $n=180^\circ$, and rad. $=$ 1.

98. This method of cubic equations by converging series, which in some cases will be found more convenient in practice than either of the former, consists in substituting the numeral parts of the given equations in the place of the literal one, in one of the following general formulæ to which it belongs; and then collecting as many terms of the series as are sufficient for determining the value of the unknown quantity to the degree of exactness required. 1. Given $x^3+ax=b$.

$$1.,\ x=\frac{2b}{\sqrt{\{2(27b^2+4a^3)\}}}\{1+\frac{2.5}{6.9}(\frac{27b^2}{27b^2+4a^3})+\frac{2.5.8.11}{6.9.12.15}(\frac{27b^2}{27b^2+4a^3})^2+\frac{2.5.8.11.14.17}{6.9.12.15.18.21}(\frac{27b^2}{27b^2+4a^3})^3+\&c.\}$$

$$\text{or } x=\frac{2b}{\sqrt{\{2(27b^2+4a^3)\}}}\{1+\frac{2.5}{6.9}(\frac{27b^2}{27b^2+4x^3})A+\frac{8.11}{12.15}(\frac{27b^2}{27b^2+4a^3})B+\frac{14.17}{18.21}(\frac{27b^2}{27b^2+4a^3}C+\frac{20.23}{24.27}(\frac{27b^2}{27b^2+4a^3})D$$

In which case, as well as in the following ones, A, B, C, &c. denote the terms immediately preceding those in which they are first found. 2 Given $x^3-ax=\pm b$, where $\frac{1}{4}b^2$ is supposed to be greater than $\frac{1}{27}a^3$ or $27b^2-4a^3$.

$$2.\ x=\pm 2\sqrt[3]{\frac{b}{2}}\{1-\frac{2}{3.6}(\frac{27b^2-4a^3}{27b^2})-\frac{2.5.8}{3.6.9.12}(\frac{27b^2-4a^3}{27b^2})^2-\frac{2.5.8.11.14}{3.6.9.12.15.18}(\frac{27b^2-4a^3}{27b^2})^3-\&c.\},\text{ or}$$

$$x=\pm 2\sqrt[3]{\frac{b}{2}}\{(-\frac{2}{3.6}(\frac{27b^2-4a^3}{27b^2})A-\frac{5.8}{9.12}(\frac{27b^2-4a^3}{27b^2})B-\frac{11.14}{15.18}(\frac{27b^2-4a^3}{27b^2})C-\frac{17.20}{21.24}(\frac{27b^2-4a^3}{27b^2})D-\&c.$$

In which case the upper sign must be taken when b is positive, and the under sign when it is negative; and the same for the first root in the two following cases.

3. Given $x^3-ax=\pm b$, where $\frac{1}{4}b^2$ is supposed to be less than $\frac{1}{27}a^3$, or $27b^2\times 4a^3$.

$$3.\ x=\pm 2\sqrt[3]{\frac{b}{2}}\{1+\frac{2}{3.6}(\frac{4a^3-27b^2}{27b^2})-\frac{2.5.8}{3.6.9.12}(\frac{4a^3-27b^2}{27b^2})^2+\frac{2.5.8.11.14}{3.6.9.12.15.18}(\frac{4a^3-27b^2}{27b^2})^3-\&c.\},\text{ or}$$

$$x=\pm 2\sqrt[3]{\frac{b}{2}}\{1+\frac{2}{3.6}(\frac{4a^3-27b^2}{27b^2})A-\frac{5.8}{9.12}(\frac{4a^3-27b^2}{27b^2})B+\frac{11.14}{15.18}(\frac{4a^3-27b^2}{27b^2})C-\frac{17.20}{21.24}(\frac{4a^3-27b^2}{27b^2})D+\&c.$$

Which series answers to the irreducible case, and must be used when $4a^3$ is less than $27b^2$. And if the root thus found be put $=r$, the other two roots may be expressed as follows:

$$x=\mp\frac{r}{2}\pm\frac{\sqrt{(4a^3-27b^2)}}{9\sqrt[3]{2b^2}}\{1-\frac{2.5}{6.9}(\frac{4a^3-27b^2}{27b^2})+\frac{2.5.8.11}{6.9.12.15}(\frac{4a^3-27b^2}{27b^2})^2-\frac{2.5.8.11.14.17}{3.6.9.12.15.18.21}(\frac{4a^3-27b^2}{27b^2})^3+\&c,\}$$

$$x=\mp\frac{r}{2}\pm\frac{\sqrt{(4a^3-27b^2)}}{9\sqrt[3]{2b^2}}\{1-\frac{2.5}{6.9}(\frac{4a^3-27b^2}{27b^2})A+\frac{8.11}{12.15}(\frac{4a^3-27b^2}{27b^2})B-\frac{14.17}{18.22}(\frac{4a^3-27b^2}{27b^2})C+\frac{20.23}{25.27}(\frac{4a^3-27b^2}{27b^2})^2D$$

Where $-\frac{1}{2}r$, or $+\frac{1}{2}r$, must be taken according as b is positive or negative; and the double signs $\pm$ must be considered as $+$ in one case and $-$ in the other, as usual. 4. Given $x^3-ax=\pm b$, where $\frac{1}{4}b^2$ is still supposed to be less than $\frac{1}{27}a^3$, or $27b^2=4a^3$.

$$4.\quad x=\pm 2\frac{2b}{\sqrt[3]{\{2(4a^3-27b^2)\}}}\{1-\frac{2.5}{6.9}(\frac{27b^2}{4a^3-27b^2})+\frac{2.5.8.11}{6.9.12.15}(\frac{27b^2}{4a^2-27b^2})^2-\frac{2.5.8.11.14.17}{6.9.12.15.18.21}(\frac{27b^2}{4a^3-27b^2})^3+\&c$$

or}

$$x=\mp\frac{27b^2}{\sqrt[3]{\{2(4a^3-27b^2)\}}}\{1-\frac{2.5}{6.9}(\frac{27b^2}{4a^3-27b^2})A+\frac{8.11}{12.15}(\frac{27b^2}{4a^3-27b^2})B-\frac{14.17}{18.21}(\frac{27b^2}{4a^3-27b^2})C+\frac{20.23}{24.27}(\frac{27b^2}{4a^3-27b^2})$$

D—&c., which series also answers to the irreducible case, and must be used when $4a^2$ is greater than $27b^2$. And if the root thus found be put $=r$, as before, the other two roots may be expressed thus:

$$x=\mp\frac{r}{2}\pm\sqrt[6]{\frac{4a^3-27b^2}{4}}\{1+\frac{2}{3.6}(\frac{27b^2}{4a^3-27b^2})-\frac{2.5.8}{3.6.9.12}(\frac{27b^2}{4a^3-27b^2})^2+\frac{2.5.8.11.14}{3.6.9.12.15.18}(\frac{27b^2}{4a^3-27b^2})^3-\&c.\text{ or}$$

$$x=\mp\frac{r}{2}\pm\sqrt[6]{\frac{4a^3-27b^2}{4}}\{1+\frac{2}{3.6}(\frac{27b^2}{4a^3-27b^2})A-\frac{5.8}{9.12}(\frac{27b^2}{4a^3-27b^2})B+\frac{11.14}{15.18}(\frac{27b^2}{4a^3-27b^2})C-\frac{17.20}{21.24}(\frac{27b^2}{4a^3-27b^2})D+$$

Where the signs are to be taken as in the latter part of the preceding case.

Of Biquadratic Equations.

99. A biquadratic equation, as before observed, is one that rises to the fourth power, or that is of the general form $x^4+ax^3+bx^2+ax+d=0$, the root of which may be determined by means of the following formula, substituting the numbers of the given equation, with their proper signs, in the places of the literal coefficients a, b, c, d. RULE I. Find the value of z in the cubic equation $z^3+(\frac{1}{4}ac-\frac{1}{12}b^2-d)z=\frac{1}{108}b^3+\frac{1}{8}(c^2+da^2)-\frac{1}{24}b(ac+8d)$ by one of the former rules; and let the root, thus determined, be denoted by r

Then find the two values of x in each of the following quadratic equations,

$$x^2+\left(\tfrac{1}{2}a+\sqrt{\left\{\tfrac{1}{4}a^2+2(r-\tfrac{1}{3}b)\right\}}\right)x=-(r+\tfrac{1}{6}b)+\sqrt{\left\{(r+\tfrac{1}{6}b)^2-d\right\}}$$
$$x^2+\left(\tfrac{1}{2}a-\sqrt{\left\{\tfrac{1}{4}a^2+2(r-\tfrac{1}{3}b)\right\}}\right)x=-(r+\tfrac{1}{6}b)-\sqrt{\left\{(r+\tfrac{1}{6}b)^2-d\right\}},$$

and they will be the four roots of the biquadratic equation required. Or, if the equation be of the more commodious form, $x^4+bx^2+cx+d=0$, to which it can always be reduced, by taking away its second term, it may be resolved thus:

RULE II. Find the value of z in the cubic equation $z^3-(\frac{1}{12}b^2+d)z=\frac{1}{108}b^3+\frac{1}{8}c^2-\frac{1}{3}bd$, and let the root thus determined be denoted by r. Then find the two values of x, in each of the two following quadratic equations,

$$x^2+\sqrt{\left\{2(r-\tfrac{1}{3}b)\right\}}\,x=-(r+\tfrac{1}{6}b)+\sqrt{\left\{(r+\tfrac{1}{6}b)^2-d\right\}}$$
$$x^2-\sqrt{\left\{2(r-\tfrac{1}{3}b)\right\}}\,x=-(r+\tfrac{1}{6}b)-\sqrt{\left\{(r+\tfrac{1}{6}b)^2-d\right\}}$$

and they will be the four roots of the biquadratic equation required.

1. Given the equation $x^4-10x^3+35x^2-50x+24=0$, to find its roots. Here $a=-10$, $b=35$, $c=-50$, and $d=24$; whence, by substituting these numbers in the cubic equation

$$z^3+(\tfrac{1}{4}ac-\tfrac{1}{12}b^2-d)z=\tfrac{1}{108}b^3+\tfrac{1}{8}(c^2+da^2)-\tfrac{b}{24}(ac+8d),$$

we shall have the following reduced equation, $z^3-\frac{13}{12}z=\frac{35}{108}$, which being resolved, according to the rule before laid down for that purpose, gives $z=\frac{1}{6}\{\sqrt[3]{(35+18\sqrt{-3})}+\sqrt[3]{(35-18\sqrt{-3})}\}$.

But, by the rule for binomial surds, given in Art. I, Case 2, in the former part of the work, $\sqrt[3]{(35+18\sqrt{-3})}=\frac{7}{2}+\frac{1}{2}\sqrt{-3}$, and $\sqrt[3]{(35-18\sqrt{-3})}=\frac{7}{2}-\frac{1}{2}\sqrt{-3}$;

$$\text{wherefore } z=\tfrac{1}{6}\{\tfrac{7}{2}+\tfrac{1}{2}\sqrt{-3}+\tfrac{7}{2}-\tfrac{1}{2}\sqrt{-3}\}=\tfrac{7}{6}.$$

And if this number be substituted for r, -10 for a, 35 for b, and 24 for d, in the two quadratic equations,

$$x^2+\left[\tfrac{1}{2}a+\sqrt{\left\{\tfrac{1}{4}a^2+2(r-\tfrac{1}{3}b)\right\}}\right]x=-(r+\tfrac{1}{6}b)+\sqrt{\left\{(r+\tfrac{1}{6}b)^2-d\right\}},$$
$$x^2+\left[\tfrac{1}{2}a-\sqrt{\left\{\tfrac{1}{4}a^2+2(r-\tfrac{1}{3}b)\right\}}\right]x=-(r+\tfrac{1}{6}b)-\sqrt{\left\{(r+\tfrac{1}{6}b)^2-d\right\}},$$

they will become, after reducing them to their most simple terms, $x^2-3x=-2$, and $x^2-7x=-12$; from the first of which $x=\frac{3}{2}\pm\sqrt{\frac{1}{4}}=\frac{3}{2}\pm\frac{1}{2}=2$ or 1, and from the second $x=\frac{7}{2}\pm\sqrt{\frac{1}{4}}=\frac{7}{2}\pm\frac{1}{2}=4$ or 3; whence the four roots of the given equation are 1, 2, 3, and 4.

2. Given $x^4+12x-17=0$, to find the four roots of the equation.

Here $a=0$, $b=0$, $c=12$, and $d=-17$:

Whence, by substituting these numbers in the equation $z^3-(\frac{1}{12}b^2+d)z=\frac{1}{108}b^3+\frac{1}{8}c^2-\frac{1}{3}bd$, we shall have, after simplifying the result, $z^3+17z=18$, where it is evident, by inspection, that $z=1$.

And if this number be substituted for r, 0 for b, and -17 for d, in the two quadratic equations

$$x^2+\sqrt{\left\{2(r-\tfrac{1}{3}b)\right\}}\,x=-(r+\tfrac{1}{6}b)+\sqrt{\left\{(r+\tfrac{1}{6}b)^2-d\right\}},$$
$$x^2-\sqrt{\left\{2(r-\tfrac{1}{3}b)\right\}}\,x=-(r+\tfrac{1}{6}b)-\sqrt{\left\{(r+\tfrac{1}{6}b)^2-d\right\}},$$

they will become, after reducing them, in the usual manner, to their most simple forms,

$x^2+2^{\frac{1}{2}}x=-1+3\sqrt{2}$, and $x^2-2^{\frac{1}{2}}x=-1-3\sqrt{2}$, which being resolved, according to the general rule, we shall have,

$$x=-\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}+\sqrt{18})}=-\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}+3\sqrt{2})},$$
$$x=-\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}+\sqrt{18})}=-\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}+3\sqrt{2})},$$
$$x=+\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}-\sqrt{18})}=+\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}-3\sqrt{2})},$$
$$x=+\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}-\sqrt{18})}=+\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}-3\sqrt{2})},$$

which are the four roots of the proposed equation. Where it is to be observed that the first two are real, and the two latter imaginary.

RULE III. The roots of any biquadratic equation of the form $x^4+ax^2+bx+c=0$, may also be determined by the following general formulæ first given by *Euler*, which are remarkable for their elegance and simplicity.

Find the three roots of the cubic equation $z^3+2az^2+(a^2-4c)z=b^2$, by one of the former rules, before given for this purpose; and let them be denoted by r', r'', and r'''. Then we shall have

When b is positive,	When b is negative,
$x=\dfrac{-\sqrt{r'}-\sqrt{r''}-\sqrt{r'''}}{2}$	$x=\dfrac{+\sqrt{r'}+\sqrt{r''}+\sqrt{r'''}}{2}$
$x=\dfrac{-\sqrt{r'}+\sqrt{r''}+\sqrt{r'''}}{2}$	$x=\dfrac{+\sqrt{r'}-\sqrt{r''}-\sqrt{r'''}}{2}$
$x=\dfrac{+\sqrt{r'}-\sqrt{r''}+\sqrt{r'''}}{2}$	$x=\dfrac{-\sqrt{r'}+\sqrt{r''}-\sqrt{r'''}}{2}$
$x=\dfrac{+\sqrt{r'}+\sqrt{r''}-\sqrt{r'''}}{2}$	$x=\dfrac{-\sqrt{r'}-\sqrt{r''}+\sqrt{r'''}}{2}$

101. If the three roots r', r'', r''', of the auxiliary cubic equation be all real and positive, the four roots of the proposed equation will, also, be real; and if one of these roots be positive, and the other two imaginary, or both of them negative, and equal to each other, two of the roots of the given equation will be real, and two imaginary; which are the only cases that produce real results.

3. Given $x^4-25x^2+60x-36=0$, to find the four roots of the equation. Here $a=-25$, $b=60$, and $c=-36$; whence, by substituting these values for their equals, in the cubic equation above given, we shall have $z^3-2\times25z^2+(25^2+4\times36)z=60^2$, or $z-50z^2+769z=3600$; the three roots of which last equation, as found by trial, or by one of the former rules, are 9, 16, and 25, respectively; whence

$$x=\tfrac{1}{2}(-\sqrt{9}-\sqrt{16}-\sqrt{25})=\tfrac{1}{2}(-3-4-5)=-6,$$

$x=\frac{1}{2}(-\sqrt{9}+\sqrt{16}+\sqrt{25})=\frac{1}{2}(-3+4+5)=+3,$
$x=\frac{1}{2}(+\sqrt{9}-\sqrt{16}+\sqrt{25})=\frac{1}{2}(+3-4+5)=+2,$
$x=\frac{1}{2}(+\sqrt{9}+\sqrt{16}-\sqrt{25})=\frac{1}{2}(+3+4-5)=+1,$

And ∴, the four roots of the equation are 1, 2, 3, and -6.

1. Given $x^4-55x^2-30x+504=0$, to find the four roots or values of x. Ans. 3, 7, -4, and -6.

2. Given $x^4+2x^3-7x^2-8x=-12$, to find the four roots or values of x. Ans. 1, 2, -3, and -2.

3. Given $x^4-8x^3+14x^2+4x=8$, $x^4-17x^2-20x-6=0$, $x^4-3x-4x=3$; x^4- *$19x^3+123x^2-302x+200=0$, to find the four roots or values of x.

Ans. $\begin{cases} 3+\sqrt{5}, & 3-\sqrt{5}, \\ 1+\sqrt{3}, & 1-\sqrt{5}. \end{cases}$ Ans. $\begin{cases} 2+\sqrt{7}, & -2-\sqrt{7}. \\ -2+\sqrt{2}, & -2-\sqrt{2}. \end{cases}$

Ans. $\begin{cases} \frac{1}{2}+\frac{1}{2}\sqrt{3}, & \frac{1}{2}-\frac{1}{2}\sqrt{3}, \\ -\frac{1}{2}+\frac{1}{2}\sqrt{3}, & -\frac{1}{2}-\frac{1}{2}\sqrt{3}. \end{cases}$ Ans. $\begin{cases} 1.02804, & 4\cdot00000 \\ 6.57653, & 7.39543 \end{cases}$

4. Given $x^4-27x^3+162x^2+356x-1200=0$, $x^4-12x^2+12x-3=0$, $x^4-36x^2+72x-36=0$, to find the four roots or values of x in each. Ans. $\begin{cases} 2.05608, & -3.0000, \\ 13.15306, & 14.7908. \end{cases}$ $\begin{cases} .60601, & -3.907572 \\ 2.85808, & .443274 \end{cases}$

Ans. 0.8729836, 1.2679494, 4.7320506, -6.8729836

5. Given $x^4-5x^3+13x^2-17x+12=0$, to find the 4 roots of the equation, which are all imaginary

I have $x^2-2x+3=0$, or $x^2-3x+4=0$, Ans.

*To this we may farther add, that when one of the roots of an equation has been found, either by this method or the former, the rest may be determined as follows :

Bring all the terms to the left hand side of the equation, and divide the whole expression so formed by the difference between the unknown quantity (x) and the root first found ; and the resulting equation will then be depressed a degree lower than the given one. Find a root of this new equation, by approximation, as in the first instance, and the number so obtained will be a second root of the original equation. Then, by means of this root, and the unknown quantity, depress the equation a degree lower, and thence find a third root ; and so on, till the equation is reduced to a quadratic ; when the two roots of this, together with the former, will be the roots of the equation required.

Thus, in the equation $x^3-15x^2+63x=50$, the first root is found by approximation to be 1.02804. Hence

$$x-1.02804)x^3-15x^2+63x-50(x^2-13.97196x+48.63627=0.$$

And the two roots of the quadratic equation, $x^2-13.97196x=-48.63627$, found in the usual way, are 6.57653 and 7.39543.

So that the three roots of the given cubic equation $x^3-15x^2+63x=50$, are 1.02804, .6.57653, and 7.39543 ; their sum being $=15$, the coefficient of the second term of the equation, as it ought to be when they are right.

Of the resolution of Equations by approximation, or the method of successive substitutions.

102. Equations of the fifth power, and those of higher dimensions, cannot be resolved by any general rule, or algebraic formula, that has yet been discovered; except in some particular cases, where certain relations subsist between the coefficients of their several terms; or when the roots are rational, and, for that reason, can be easily found by means of a few trials. See p. 113.

In these cases, therefore, recourse must be had to some of the usual methods of approximation, among which that commonly employed is the following, which is universally applicable to all kinds of numeral equations, whatever may be the number of their dimensions; and which, by continuing the process to a sufficient length, will give the value of the root sought to any required degree of exactness. See Index.

RULE I. Find, by trials, a number nearly equal to the root sought, which call r; and let z be made to denote the difference between this assumed root and the true root x.

Then, instead of x, in the given equation, substitute its equal $r+z$, or $r-z$, according as r is too little or too great, and there will arise a new equation, involving only z and known quantities.

Reject all the terms of this equation in which z is of two or more dimensions, and the approximate value of z may then be determined by means of a simple equation.

And if the value thus found be added to r, when it is too little, or subtracted from it when it is too great, it will give a near value of x, or of the root required.

But as this approximation will seldom be sufficiently exact, the operation must be repeated, by substituting the number last found, for r, in the abridged equation exhibiting the value of z; when a second correction of z will be obtained, which being added to, or subtracted from that number, will give a nearer value of x than the former.

And by again substituting this last number for r, in the above mentioned equation, and repeating the same process as often as may be thought necessary, a value of z, and consequently of the root sought, may be found to any degree of accuracy.

Note. The decimal part of the root, as found both by this and the next rule, will in general about double itself at each operation; and therefore it would be useless, as well as troublesome, to use a much greater number of figures than these, in the several substitutions for the values of r.

1. Given $x^3+x^2+x=90$, to find the value of x, by approximation.

Here the root is soon found, by a few trials, to lie between 4 and 5; but nearer to 4 than to 5.

Let therefore $4=r$, and $x=r+z$,

$$\text{Then}\left|\begin{array}{l} x^3=r^3+3r^2z+3rz^2+z^3 \\ x^2=r^2+2rz+z^2 \\ x=r+z \end{array}\right|=90.$$

And, by rejecting the terms z^3, $3rz^2$ and z^2, or, to avoid trouble, omitting them in the operation, as being small in comparison with z, we shall have $r^3+r^2+r+3r^2z+2rz+z=90$;

Whence $z=\dfrac{90-r^3-r^2-r}{3r^2+2r+1}=\dfrac{90-64-16-4}{48+8+1}=\dfrac{6}{57}=.10$, and, consequently, $x=4.1$, nearly. Again, if 4.1 be substituted in the place of r, in the last equation, we shall have.

$$z=\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-68.921-16.81-4.1}{50.43+8.2+1}=.00283,$$

And consequently $x=4.1+.00283=4.10283$, for a second approximation; and if the first four figures, 4.102, of this number, be again substituted for r, in the same equation, an approximate value of x may be obtained, to six or seven places of decimals.

And by proceeding in the same manner, the root may be found still more correctly.

1. Given $x^2+20x=100$, to find the value of x by approximation. Ans. $x=4.1421356237$.

2. Given $x^3-9x=10$, to find the value of x by approximation. Ans. 3.4494897428.

3. Given $x^3+9x^2+4x=80$, to find the value of x by approximation. Ans. $x=2.47213596$.

Here, by a few trials, the root is found to be between 2 and 3, and very nearly equal to 2.5; let, therefore, $x=r+z=2.5+z$; then

$$\begin{array}{rcl} x^3 &=& r^3 + 3r^2z + \&c. \\ 9x^2 &=& 9r^2 + 18rz + \&c. \\ 4x &=& 4r + 4z \\ \hline & & r^3 + 9r^2 + 4r + z(3r^2+18r+4r) \end{array}$$

$=80$, and $z=\dfrac{80-r^3-9r^2-4r}{3r^2+18r+4r}=\dfrac{-1.875}{67.75}=-.0276$. Whence $x=2.5-.0276=2.472$, the root nearly. And if this value be substituted in the above expression for z, we get $x=2.4721359$.

4. Given $x^4-38x^3+210x^2+538x+289=0$, to find the value of x by approximation. Ans. $x=30.5356537529$.

Here x is found, by trial, to be very nearly $=30$; let, therefore, $x=r+z=30+z$; then

$$\begin{aligned}
x^4 &= r^4 + 4r^3z + \&c. \\
-38x^3 &= -38r^3 - 114r^2z - \&c. \\
+210x^2 &= 210r^2 + 420rz + \&c. \\
+538x &= 538r + 538z
\end{aligned}$$

$r^4 - 38r^3 + 210r^2 + 538r + z(4r^3 - 114r^2 + 420r + 538) + 289 = 0$, and

$$z = \frac{r^4 - 38r^3 + 210r^2 + 538r + 289}{114r^2 - 4r^3 - 420r - 538} = \frac{-10571}{-18538} = .56\,;$$

whence $x = 30.56$, nearly. Again, taking 30.5 for the root, and substituting it instead of r in the expression for z, we get $z = 30.5356$.

5. Given $x^5 + 6x^4 - 10x^3 - 112x^2 - 207x - 110 = 0$, to find the value of x by approximation. Ans. 4.46410161.

By a few trials, x is found to be between 4 and 5, and nearly equal to 4.5. Let, therefore, $x = r + z = 4.5 + z$; then

$$\begin{aligned}
x^5 &= r^5 + 5r^4z + \&c. \\
6x^4 &= 6r^4 + 24r^3z + \&c. \\
-10x^3 &= -10r^3 - 30r^2z + \&c. \\
-112x^2 &= -112r^2 - 224rz + \&c. \\
-207x &= -207r - 207z
\end{aligned}$$

$r^5 + 6r^4 - 10r^3 - 112r^2 - 207r + z(5r^4 + 24r^3 - 30r^2 - 227r - 207) = 110$; therefore

$$z = \frac{110 - r^5 - 6r^4 + 10r^3 + 112r^2 + 207r}{5r^4 + 24r^3 - 30r^2 - 227r - 207} = -.0354\,;$$ whence $x = 4.5 - .035 = 4.465$; and, taking this for the root, and substituting again in the value for z, we get $x = 4.464101$, the root nearly.

103. The roots of equations, of all orders, can also be determined, to any degree of exactness, by means of the following easy rule of double position; which, though it has not been generally employed for this purpose, will be found, in certain cases, superior to the former, as it can be applied at once to any unreduced equation, consisting of surds, or compound quanties, as readily as if it had been brought to its usual form.

Rule II. Find, by trial, two numbers nearly equal to the root sought, and substitute them in the given equation instead of the unknown quantity, noting the results that are obtained from each. Then, as the difference of these results is to the difference of the two assumed numbers,

So is the difference between the true result, given by the question, and either of the former, to the correction of the assumed number belonging to the result used,

Which correction being added to that number when it is *too little*, or subtracted from it when it is *too great*, will give the root required, nearly.

And if the number thus determined, and the nearest of the two former, or any other that appears to be more accurate, be now taken as the assumed roots, and the operation be repeated as before, a new value of the unknown quantity will be obtained still more correct than the first; and so on, proceeding in this manner, as far as may be judged necessary.

1 Given $x^3+3x^2+3x=130$, to find a near approximate value of x. Ans. $x=4.078753$.

2. Given $x^3+10x^2+5x=2600$, to find a near approximate value of x. Ans. $x=11.00673$.

It is soon found by trial that x is rather more than 11; let, therefore, 11 and 11.1 be taken for the assumed number; then

$$\begin{array}{rcr}
\text{1st Sup.} & & \text{2d Sup.} \\
1331 & & 1367.631 \\
1210 & & 1232.10 \\
55 & & 55.5 \\
\hline
2596 & \text{Results} & 2655.231
\end{array}$$

$$\therefore \left\{\begin{array}{rrr}
2655.231 & 11.1 & 2600 \\
2596. & 11 & 2596 \\
\hline
59.231 : & .1 :: & 4
\end{array}\right.$$

$: \dfrac{4\times .1}{59.231}=.0067$; whence $x=11+.0067=11.0067$, Ans.

3. Given $2x^4-16x^3+40x^2-30x+1=0$, to find a near value of x.

Dividing by 2, we have $x^4-8x^3+20x-15x=-.5$. By a few trials it is found that x is a little greater than 1; let, therefore, 1.2 and 1.3 be assumed for the numbers; then

$$\begin{array}{rcr}
\text{1st Sup.} & & \text{2d Sup.} \\
2.0736 & & 2.8561 \\
28.8 & & 33.8 \\
\hline
30.8736 & & 36.6561 \\
-31.824 & & -37.076 \\
\hline
-.9504 & \text{results} & -.4199
\end{array}$$

$$\therefore \left\{\begin{array}{rrr}
.9504 & 1.3 & .5 \\
.4199 & 1.2 & .4199 \\
\hline
.5305 : & .1 :: & .0801
\end{array}\right.$$

$: \dfrac{.1\times .0801}{.5305}=.01509$, whence $x=1.3-.0151=1.2849$. Now take 1.284 and 1.285 for the assumed numbers, and repeat the operation. Then x will come out $=1.284724$.

4. Given $x^5+2x^4-3x^3+4x^2+5x=54321$, to find a near value of x.

By a few trials x is found to be nearly $=8\frac{1}{2}$; take, therefore, 8.4 and 8.5 for the assumed numbers; then

$$\begin{array}{lrcr l}
\text{1st Sup.} & 41821.19424 & =\ x^5 = & 44370.53125 & \text{2d Sup.} \\
& 9957.4272 & =\ 2x^4 = & 10440.1250 & \\
& 1778.112 & =\ 3x^3 = & 1842.375 & \\
& 282.24 & =\ 4x^2 = & 289. & \\
& 42. & =\ 5x = & 42.5 & \\
\hline
& 53880.97344 & \text{results} & 56984.53125 &
\end{array}$$

$$\text{Therefore} \left\{\begin{array}{rrr}
56984.53125 & 8.5 & 54321 \\
53880.97344 & 8.4 & 53880.97344 \\
\hline
3103.55781 : & .1 :: & 440.02656 :
\end{array}\right\}$$

: $\frac{.1 \times 440.02656}{3103.55781} = .0141$; whence $x = 8.4 + .0131 = 3.4141$, nearly. Taking 8.414 and 8.415 for the numbers, we have

42170.867905346409824	$= x^5 =$	42195.935727527009375
10023.976207593632	$= 2x^4 =$	10028.74241890125
1787.017385832	$= 4x^3 =$	1787.654620125
283.181584	$= 4x^2 =$	283.2489
42.07	$= 5x =$	42.075
54307.113082772041824		54337.654666553259375
54337.65465655325	8.415	54337.65466655325
53307.11308277204	8.414	54321
30.54158378121 :	.001 ::	16.65466655325 : 000545

and $x = r - z = 8.415 - .000545 = 8.414455$, very near.

5. Given $\sqrt[3]{(7x^3+4x^2)}+\sqrt{(20x^2-10x)}=28$, to find an approximate value of x. Ans. 4.510661.

By trial x is found to be between 4 and 5. Let these two numbe taken for the first value; then

1st Sup. 8	$= \sqrt[3]{(7x^3 + 4x^2)} =$	9.916	2d Sup.
16.73	$= \sqrt{(20x^2 - 10x)} =$	21.213	
24.73	results	31.129	

Therefore
$$\begin{array}{lll} 31.129 & 5 & 28 \\ 24.73 & 4 & 24.73 \\ \hline 6.399 : & 1 :: & 3.27 : \frac{3.27}{6.399} = .51; \text{ whence} \end{array}$$

$x = 4.51$, nearly. Assuming $x = 4.51$ and 4.52, and repeating the operation, we get $x = 4.5106$, the root, nearly.

6. Given $\sqrt{\{144x^2-(x^2-20)^2\}}+\sqrt{\{196x^2-(x^2+24)^2\}}=114$, to find a near value of x.

Here the root is found, by a few trials, to be rather more than 7; let, therefore, 7 and 7.1 be assumed for the numbers; then

1st Sup. 7056 =	$144x^2 =$	7259.04 2d Sup.
4761 =	$(x^2+20)^2 =$	4957.5681
$\sqrt{2295} = 47.906$		$\sqrt{2302.4719} = 47.977$
9604 =	$196x^2 =$	9880.36
5329 =	$(x^2 + 24)^2 =$	5536.8481
$\sqrt{(4275)} = 65.383$		$\sqrt{4343.5119} = 65.905$

$$\left.\begin{array}{lr} 47.906 & 47.977 \\ 65.383 & 65.905 \\ \hline 113.289 \text{ results} & 113.882 \end{array}\right\} \text{ or } \left\{\begin{array}{lll} 113.882 & 7.1 & 114. \\ 113.289 & 7 & 113.882 \\ \hline .593 : & .1 :: & .118 : \end{array}\right.$$

$\frac{.1 \times .118}{.593} = \frac{.0118}{.593} = .2$; whence $x = 7.12$, nearly. By assuming

7.12 and 7.13 for the numbers, we get $x = 7.12209461$, the answer, nearly.

1. Given $x^3+x^2+x=100$, to find an approximate value of x.

Here it will be found, by a few trials, that the value of x lies between 4 and 5.

Hence, by taking these as the two assumed numbers, the operation will stand thus:

First Sup.		*Second Sup.*
4	....x....	5
16	....x^2....	25
64	....x^3....	125
84	Results	155

155....5....100
84....4.... 84
∴ 71 : 1 :: 16 : .225

And consequently $x=4+.225=4.225$, nearly.

Again, if 4.2 and 4.3 be now taken as the two assumed numbers, we shall have

First Sup.		*Second Sup.*
4.2	....x....	4.3
17.64	....x^2....	18.49
74.088	....x^3....	79.507
95.928	Results.	102.297

102.297....4.3..102.297
95.928....4.2..100
∴ 6.369 : .1 :: 2.297 : 036.

And consequently $x=4.3-.036=4.264$, nearly.

Again, let 4.264 and 4.265 be the two assumed numbers. Then

First Sup.		*Second Sup.*
4.264	.... x ..	4.265
18.181696	. x^2 ..	18.190225
77.526752	. x^3 ..	77.581310
99.972448		100.036535

Therefore

100.036535 4.265 100
99.972448 4.264 .99.972448
.064087 : .001 :: .027552 : .0004299, and consequently,

$x = 4.264 + .0004299 = 4.2644299$, very nearly.

2. Given $(\frac{1}{5}x^2 - 15)^2+x\sqrt{x}=90$, to find an approximate value of x.

Here, by a few trials, it will be found that the value of x lies between 10 and 11; which let, therefore, be the two assumed numbers, agreeably to the directions given in the rule.

Then

First Sup.		*Second Sup.*
25	.. $(\frac{1}{5}x^2-15)$ ..	84.64
31.622	.. $x\sqrt{x}$ ..	36.482
56.622	Results	121.122

Hence

121.122 .. 11 .. 121.122
56.622 .. 10 .. 90
64.5 : 1 :: 31.122 : .482

And consequently $x=11-.582=10.518$.

Again, let 10.5 and 10.6 be the two assumed numbers;

Then

First Sup.		*Second Sup.*
49.7025	$(\frac{1}{5}x^2-15)^2$	55.830784
34.0239	....$x\sqrt{x}$..	34.511099
83.7264	.. Results ..	90.341883

Hence

90.341883 .. 10.6 .. 90.341883
83.7264 .. 10.5 .. 90.
6.615483 .. .1 :: .341883 : .0051679, and consequently $x = 10.6 - .0051679 = 10.5948321$, very nearly.

Of Reciprocal Equations.

104. Although no general method has hitherto been discovered for the resolution of equations higher than those of the fourth power, there are, notwithstanding, some particular equations, of all orders under the 10th, which, on account of the relations subsisting between their coefficients, may be solved by the rules that have been already given for the first four orders.

This is particularly the case with what have been usually called reciprocal equations; which are such that the coefficients of their terms, taken from the beginning of the equation, are the same as those of the corresponding terms, taken from the end, with the same signs; or which remain of the same form when the reciprocal of the unknown quantity, or $\frac{1}{x}$, is substituted for x; except that the terms are then reversed. Thus the equations $x^{2m}+px^{2m-1}+qx^{2m-2}+\ldots\ldots+qx^2+px+1=0$, and $x^{2m+1}+px^{2m}+qx^{2m-1}+\ldots\ldots+qx^2+px+1=0$, may always be transformed into others of a degree which is denoted by half the exponent of the highest power of the unknown quantity, if it be an even number, or by half that exponent when it is diminished by 1, if it be an odd number, the method of resolving them, as far as to the 9th order, inclusively, being as follows. To this we may add, that the nature of these equations consists, as abovementioned, in their not being changed by substituting $\frac{1}{x}$ for x; from which it follows, that if a be any one of the roots, its reciprocal $\frac{1}{a}$ will also be a root; and as $+1$, or -1, is always a root of the equation, when the number of its dimensions is odd, it may be readily shown from these circumstances, that every equation of the $2m$th or $2m$th$+1$th order, can be reduced to another of the mth order.

Case I. *When the index of the highest term is an even number.*

Rule I. If the equation be of the fourth power, as $x^4+px^3+qx^2+px+1=0$, find the two values of z in the equation, $z^2+pz+q-2=0$, and let them be denoted by r and r'; then the roots of the two quadratic equations, $x^2-rx+1=0$, and $x^2-r'x+1=0$, will be the four roots of the proposed equation.

2. If it be of the sixth power, as $x^6+px^5+qx^4+rx^3+qx^2+px+1=0$, find the three values of z in the equation, $z^3+pz^2+(q-3)z+r-2p=0$, and let them be denoted by $r.\ r',\ r''$; then the roots of the three quadratics $x^2-rx+1=0$, $x^2-r'x+1=0$, and $x^2-r''x+1=0$, will be the six roots of the proposed equation.

3. And if it be of the eighth power, as $x^8+px^7+qx^6+rx^5+sx^4+rx^3+qx^2+px+1=0$, find the four values of z in the equation, $z^4+pz^3+(q-4)z^2+(r-3p)z+s-2(q-1)=0$, and let them be denoted by $r,\ r',\ r'',\ r'''$.

Then the roots of the four quadratic equations.
$x^2-rx+1=0$, $x^2-r'x+1=0$, $x^2-r''x+1=0$, and $x^2-r'''x+1=0$, will be the eight roots of the proposed equation.

CASE II. *When the index of the highest term is an odd number.*

RULE I. If the equation be of the third power, as $x^3+px^2+px+1=0$, where one of its roots is evidently -1, find the value of z in the simple equation $z+p-1=0$, and let it be denoted by r; then the roots of the quadratic equation $x^2-rx+1=0$, will be the other two roots of the proposed equation.

2. If it be of the fifth power, as $x^5+px^4+qx^3+qx^2+px+1=0$, where one of its roots is also -1, find the two values of z in the equation $z^2+(p-1)z+q-p-1=0$, and then let them be denoted by r, and r'; then the roots of the two quadratics $x^2-rx+1=0$, and $x^2-r'x+1=0$ will be the other four roots of the proposed equation.

3. If it be of the seventh power, as $x^7+px^6+qx^5+rx^4+rx^3+qx^2+px+1=0$, where one of the roots is -1, as before; find the the three values of z in the equation $z^3+(p-1)z^2+(q-p-2)z+r-p-q+1=0$, and let them be denoted by r, r', r''; then the roots of the 3 quadratics $x^2-rx+1=0$, $x^2-r'x+1=0$, and $x^2-r''x+1=0$, will be the other 6 roots of the proposed equation.

4. And if it be of the 9th power, as

$$x^9+px^8+qx^7+rx^6+sx^5+sx^4+rx^3+qx^2+px+1=0,$$

where one of the roots is -1, as in the former cases, find the four values of z in the biquadratic equation

$z^4+(p-1)z^3+(q-p-3)z^2+(r-q-2p+2)z+s-r-q+p+1=0$, and let them be denoted by r, r', r'', r''';

then the roots of the four quadratics

$x^2-rx+1=0$, $x^2-r'x+1=0$, $x^2-r''x+1=0$, and $x^2-r'''x+1=0$, will be the other eight roots of the proposed equation.

Note. If an equation of this kind be of an odd number of dimensions, or if the middle term of one of an even number of dimensions be wanting, the same rules will hold when the signs of the terms, taken from the beginning, are $+$ and $-$ alternately. Thus $x^3-px^2+px-1=0$, and $x^4-px^3+px-1=0$, are reciprocal equations, like those above given, except that $+1$ is now one of the roots, instead of -1.

1. Given $x^4+4x^3-19x^2+4x+1=0$, to find the four roots of the equation. Here $p=4$ and $q=-19$; hence, by Case 1,

$$z^2+pz=2-q \text{ becomes } z^2+4z=2+19=21,$$

Where the two values of z are $+3$ and -7;
Consequently $x^2-3x=-1$, and $x^2+7x=-1$,
From the first of which equations $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$,
And from the second $x=(-\frac{7}{2}\pm\frac{3}{2}\sqrt{5})=(-\frac{7}{2}\pm\frac{1}{2}\sqrt{45})$;

Therefore the four roots of the proposed equation are

$$\tfrac{3}{2}+\tfrac{1}{2}\sqrt{5},\ \tfrac{3}{2}-\tfrac{1}{2}\sqrt{5},\ -\tfrac{7}{2}+\tfrac{3}{2}\sqrt{5}, \text{ and } -\tfrac{7}{2}-\tfrac{3}{2}\sqrt{5}.$$

2. Given $x^5-11x^4+17x^3+17x^2-11x+1=0$, to find the five roots of the equation. Here $p=-11$, and $q=17$; hence, by case 2, $z^2+(p-1)z=1+p-q$ becomes $z^2-12z=-27$, where the two values of z are 9 and 3; $x^2-9x=-1$, and $x^2-3x=-1$, from the first of which equations $x=\frac{9}{2}\pm\frac{1}{2}\sqrt{77}$, and from the second $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$. Therefore the five roots of the proposed equation are -1, $\frac{9}{2}+\frac{1}{2}\sqrt{77}$, $\frac{9}{2}-\frac{1}{2}\sqrt{77}$, $\frac{3}{2}+\frac{1}{2}\sqrt{5}$, and $\frac{3}{2}-\frac{1}{2}\sqrt{5}$.

105. If the coefficients of either of the orders of equations, mentioned in the rule, be in part literal, and so constituted as to render the different terms homogeneous, its roots may be determined by means of the simple substitution of a new unknown quantity, as if they were entirely numeral.

Thus, let there be taken, as an equation of this kind, $x^4+4ax^3-19a^2x^2+4a^3x+a^4=0$, where the indices of the given and unknown quantities, when added together, are the same as those of the first and last terms. Then, by putting $x=az$, we shall have $a^4z^4+4a^4z^3-19a^4z^2+4a^4z+a^4=0$, or, dividing by a^4, $z^4+4z^3-19z^2+4z+1=0$; which equation, like the first of those given above, will have for its roots $\frac{3}{2}+\frac{1}{2}\sqrt{5}$, $\frac{3}{2}-\frac{1}{2}\sqrt{5}$, $-\frac{7}{2}+\frac{3}{2}\sqrt{5}$, and $-\frac{7}{2}-\frac{3}{2}\sqrt{5}$; and the four roots of the proposed equation are $a(\frac{3}{2}+\frac{1}{2}\sqrt{5})$, $a(\frac{3}{2}-\frac{1}{2}\sqrt{5})$, $a(-\frac{7}{2}+\frac{3}{2}\sqrt{5})$, and $a(-\frac{7}{2}-\frac{3}{2}\sqrt{5})$.

An homogeneous equation, containing only two letters, a and x, is of the general form

$$x^n+pax^{n-1}+qa^2x^{n-2}+ra^3x^{n-3}+\dots+ka^{n-1}x+la^n=0,$$

where p, q, r, k, l, are the numeral coefficients.

1. Given $x^4-15x^3+38x^2-15x+1=0$, to find the four roots of the equation. Ans. $\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$, and $6\pm\sqrt{35}$.

Here $p=-15$, $q=38$, we have $z^2-15z=-36$, by case 2; $z=\frac{9}{2}\pm\frac{15}{2}=12$ or -3. Hence we have $x^2-12x=-1$, and $x^2+3x=-1$; $\therefore$ in the first we have $x=6\pm\sqrt{35}$, and in the second case we have $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$.

2. Given $x^4-4ax^3+5a^2x^2-4a^3x+a^4=0$, to find the four roots of the equation. Ans. $a(\frac{3}{2}\pm\frac{1}{2}\sqrt{5})$, and $a(\frac{1}{2}\pm\frac{1}{2}\sqrt{-3})$.

Let $p=-4$, $q=5$; then $z^2-4z=2-5=-3$; $\therefore z=3$ or 1, and we have $x^2-3y=-1$, and $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$.

Again, $x^2-x=-1$; $\therefore x=\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$; consequently we have $a(\frac{3}{2}\pm\frac{1}{2}\sqrt{5})$, and $a(\frac{1}{2}\pm\frac{1}{2}\sqrt{-3})$.

3. Given $x^5-21x^4+37x^3+37x^2-21x+1=0$, to find the five roots of the equation. Ans. -1, $\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$, and $\frac{19}{2}\pm\frac{1}{2}\sqrt{357}$.

Here $p=-21$, and $q=37$; then by case 2, $z^2-22z=-57$; $\therefore z=11\pm8=19$ or 3, and $\therefore y^2-19x=-1$; then $x=\frac{19}{2}\pm\frac{1}{2}\sqrt{357}$. Again $x^2-3x=-1$, we find $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$.

106. Equations of this kind, in which the given and the un-

known quantities can be substituted, alternately, for each other, without producing a new equation, are always capable of being reduced to others of lower dimensions. In order to such a reduction, let the equation, if it be of an even dimension, be first divided by the equal powers of its two quantities in the middle term: then assume a new equation, by putting some quantity (or letter) equal to the sum of the two quotients that arise by dividing those quantities one by the other, alternately; by means of which equation let the said quantities be exterminated; whence a numeral equation will emerge, of half the dimensions with the given literal one.

But, if the equation propounded be of an odd dimension, let it be first divided by the sum of its two quantities, so will it become of an even dimension, and its resolution will therefore depend upon the preceding rule.

Let there be given the equation $x^4-4ax^3+5a^2x^2-4a^3x+a^4=0$.

Here, dividing by a^2x^2, we have $\frac{x^2}{a^2}-\frac{4x}{a}+5-\frac{4a}{x}+\frac{a^2}{x^2}=0$, (or $\frac{x^2}{a^2}+\frac{a^2}{x^2}-4\times\frac{x}{a}+\frac{a}{x}+5=0$, by joining the corresponding terms;) and by making $z=\frac{x}{a}+\frac{a}{x}$, and by squaring both sides, we have also $z^2=\frac{x^2}{a^2}+2+\frac{a^2}{x^2}$, or $z^2-2=\frac{x^2}{a^2}+\frac{a^2}{x^2}$. Therefore, by substituting these values, our equation becomes $z^2-2-4z+5$, or $z^2-4z=-3$; whence $z=3$. But $\frac{x}{a}+\frac{a}{x}$ being $=z$, we have $x^2-zax=-a^2$; and consequently $x=\frac{1}{2}za\pm\surd(\frac{1}{4}a^2z^2-a_2)=\frac{1}{2}a\times\{z\pm\surd(z_2-4)\}$ $=\frac{1}{2}a\times\{3\pm\surd(5)\}$, in the present case.

2. Let there be given $x^5+4ax^4-12a^2x^3-12a^3x^2+4a^4x+a^5=0$.

In this case, we must first divide by $x+a$, and the quotient will come out $x^4+3ax^3-15a^2x^2+3a^3x+a^4-0$: whence, by proceeding as in the former example, we have $\frac{x^2}{a^2}+\frac{a^3}{x^3}+3\times\frac{x}{a}+\frac{a}{x}-15=0$, or $z^2-2+3z-15=0$, and from thence $z=\frac{\surd(77)-3}{2}$.

Given $7x^6-26ax^5-26a^5x+7a^6=0$, which, divided by a^3x^3, becomes $7\times(\frac{x^3}{a^3}+\frac{a^3}{x^3})-26\times(\frac{x^2}{a^2}+\frac{a^2}{x^2})=0$. Now making, as before, $z=\frac{x}{a}+\frac{a}{x}$, we have $z^2-2=\frac{x^2}{a^2}+\frac{a^2}{x^2}$; and mult. again by $z=\frac{x}{a}+\frac{a}{x}$, we likewise have have $z^3-2z=\frac{x^3}{a^3}+\frac{a}{x}+\frac{x}{a}+\frac{a^3}{x^3}=\frac{x^3}{a^3}$

$=\frac{x^3}{a^3}+z+\frac{a^3}{x^3}$; and therefore $z^3-3z=\frac{x^3}{a^3}+\frac{a^3}{x^3}$: which values being substituted above, our equation becomes $7\times(z^3-3z)-26\times(z^2-2)=0$, or $7z^3-26z^2-21z+52=0$. Where, trying the divisors of the last term, which are 1, 2, 4, 13, &c. the third is found to answer; z, consequently, being $=4$.

4. Wherein let there be given $2x^7-13a^2x^5-13a^5x^2+2a^7=0$.

Here dividing first by $x+a$, the quotient will be $2x^6-2ax^5-11a^2x^4+11a^3x^3-11a^4x^2-2a^5x+2a^6=0$; which, divided again by a^3x^3, gives $2\times(\frac{x^3}{a^3}+\frac{a^3}{x^3})-2\times(\frac{x^2}{a^2}+\frac{a^2}{x^2})-11\times(\frac{x}{a}+\frac{a}{x})+11=0$, that is, $2\times(z^3-3z)-2\times(z^2-2)-11z+11=0$, or $2z^3-2z^2-17z+15=0$; whence $z=3$.

107. A literal equation may be made to correspond with a numeral one, by substituting an unit in the room of the given quantity (or letter:) and equations that do not seem, at first, to belong to the preceding class, may sometimes be reduced to such by a proper substitution; that is, by putting the quotient of the first term divided by the last, equal to some new unknown quantity (or letter) raised to the power expressing the dimension of the equation. Thus, if the equation given be $2x^4+24x^3-315x^2+216x+162=0$; by putting $\frac{2x^4}{162}=y^4$, we have $x=3y$; whence, after substitution, the given equation becomes $162y^4+648y^3-2835y^2+648y+162=0$; which now answers to the rule, and may be reduced down to $2y^4+8y^3-35y^2+8y+2=0$.

4. Given $7x^6-26x^5-26x+7=0$, to find the roots of the equation. Ans. The only real roots are $2\pm\sqrt{3}$.

5. Given $2x^7-13a^2x^5-13a^5x^2+2a^7=0$, to find the roots of the equation. Ans. The only reals roots are
$-a$, $a(\frac{3}{2}\pm\frac{1}{2}\sqrt{5})$, and $-\frac{a}{4}\{(2+\sqrt{14})\pm\sqrt{(2+4\sqrt{14})}\}$.

108. *Of Binomial Equations, or such as are of the form* $x^m\pm a^m=0$.

Equations of this kind, which are a peculiar species of reciprocal equations, may be reduced to a more simple form by putting $x=ay$, and then dividing the result by a^m; in which case we shall have $a^my^m\pm a^m=0$, or $y^m\pm1=0$, where the several values of y are the roots of -1, or $+1$; and consequently those of x are the same roots multiplied by a.

Whence, as the first of these forms, $x^m+1=0$, or $x^m=-1$, (using x instead of y,) is a reciprocal equation, wanting all the intermediate terms, its solution may be obtained from the rules be-

fore given for this purpose, by making the coefficients, p, q, r, s, each equal to 0, and finding the result accordingly.

It therefore only remains to determine the several roots of the equation $x^m-1=0$, or $x^m=1$, which, for the first ten orders, may be done as follows.

CASE I. *When the index of the first term is an even number.*

RULE I. If the equation be of the fourth power, as $x^4-1=0$, where two of the roots are, evidently, $+1$ and -1, find the two value of x in the equation $x^2+1=0$, and they will be the other two roots of the proposed equation.

2. If it be of the sixth power, as $x^6-1=0$, where two of the roots are 1 and -1, as before; find the two values of x in each of the equations $x^2+x+1=0$, and $x^2-x+1=0$, and they will be the other four roots of the proposed equation.

3. If it be of the eigth power, as $x^8-1=0$, where two of the roots are also 1 and -1, find the two values of z in the equation $z^2-2=0$, or $z^2=2$, and let them be denoted by r and r'; then the roots of the 3 quadratics $x^2+1=0$, $x^2-rx+1=0$, and $x^2-r'x+1=0$, will be the other six roots of the proposed equation.

4. And if it be of the tenth power, as $x^{10}-1=0$, where two of the roots are 1 and -1, as they are in all even powers, find the four roots of the equation $z^4-3z^2+1=0$, and let them be denoted by r, r' r'', r'''; then the roots of the four quadratics $x^2-rx+1=0$, $x^2-r'x+1=0$, $x^2-r''x+1=0$, & $x^2-r'''x+1=0$, will be the other eight roots of the proposed equation.

CASE II. *When the index of the first term is an odd number.*

RULE I. If the equation be of the third power, as $x^3-1=0$, where one of the roots is evidently 1; find the two values of x in the equation $x^2+x+1=0$, and they will be the other two roots of the proposed equation.

2. If it be of the fifth power, as $x^5-1=0$, where one of its roots is 1, as before; find the two values of z in the equation $z^2+z-1=0$, and let them be denoted by r and r'; then the roots of the two quadratics $x^2-rx+1=0$, and $x^2-r'x+1=0$, will be the other four roots of the proposed equation.

3. If it be of the seventh power, as $x^7-1=0$, where one of its roots is also 1; find the three values of z in the equation $z^3+z^2-2z-1=0$, and let them be denoted by r, r', r''; then the roots of the 3 quadratics, $x^2-rx+1=0$, $x^2-r'x+1=0$, & $x^2-r''x+1=0$, will be the other six roots of the proposed equation.

4. And if it be of the ninth power, as $x^9-1=0$, where one of its roots is likewe 1, as in all the odd powers, find the four values of z in the equation $z^4+z^3-3z^2-2z+1=0$, and let them be denoted by r, r' r'' r'''; then the roots of the four quadratics

$x^2-rx+1=0$, $x^2-r'x+1=0$, $x^2-r''x+1=0$, & $x^2-r'''x+1=0$, will be the other eight roots of the proposed equation.

109 Since two of the values of x, in every case of even powers, are $+1$ and -1, if the first member of an equation of the form $x^{2m}-1=0$, be divided by x^2-1, the quotient will be

$$x^{2m-2}+x^{2m-4}+x^{2m-6}+\dots+x^4+x^2+1=0,$$

And because one of the values of x, in every case of the odd powers, is 1, if the first member of an equation of the form $x^{2m+1}-1=0$, be divided by $x-1$, the quotient will be

$$x^{2m}+x^{2m-1}+x^{2m-2}+x^{2m-3}+\dots+x^2+x+1=0,$$

both of which are reciprocal equations, having all the coefficients of their several terms $=1$; and, consequently, each of them may be resolved by the method employed in the last article, without recurring to any separate considerations.

1. Required to find the four roots of the equations $x^4-1=0$.

Here, the index of x being an even number, two of the roots are $+1$ and -1. by Case I, we have $x^2+1=0$, or $x=\pm\sqrt{-1}$; whence the four roots are 1, -1, $+\sqrt{-1}$, and $-\sqrt{-1}$.

2. It is required to find the three roots of the equation $x^3-1=0$.

Here, the index of x being an odd number, one of the root is 1.

And, by Case II, we shall have $x^2+x+1=0$, or $x^2+x=-1$. Whence, by quadratics, $x=-\frac{1}{2}\pm\sqrt{(\frac{1}{4}-1)}=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

$\therefore$ 1, $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, are the roots sought.

And if it be required to find the three roots of the equation $x^3+1=0$, we shall have, in this case, one of the roots $=-1$, And, by dividing x^3+1 by $x+1$, there will arise $x^2-x+1=0$, or $x^2-x=-1$ Consequently, $x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}-1)}=\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$. that is -1, $\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, are the roots sought.

3. It is required to find the five roots of the equation $x^5-1=0$.

Here 5 being an odd power, 1 root of the given equation is $=1$

Therefore, by the rule, Case II, $z^2+z-1=0$, or $z^2+z=1$;

$z=-\frac{1}{2}+\sqrt{(\frac{1}{4}+1)}=-\frac{1}{2}+\frac{1}{2}\sqrt{5}=r$, and $z=-\frac{1}{2}-\frac{1}{2}\sqrt{5}=r'$,

Whence, also, $x^2-rx+1=0$, or $x^2-(-\frac{1}{2}+\frac{1}{2}\sqrt{5})x=-1$. And,

$$x=\tfrac{1}{4}(-1+\sqrt{5})\pm\sqrt{\{\tfrac{1}{16}(-1+\sqrt{5})^2-1\}}=\tfrac{1}{4}(-1+\sqrt{5})$$
$$\pm\tfrac{1}{4}\sqrt{(-10-2\sqrt{5}}=\tfrac{1}{4}(-1+\sqrt{5})\pm\tfrac{1}{4}(\sqrt{10+2\sqrt{5}})\sqrt{-1}.$$

In like manner, $x^2-r'x+1=0$, or $x^2+(\frac{1}{2}+\frac{1}{2}\sqrt{5})x=-1$, where $x=-\frac{1}{4}(1+\sqrt{5})\pm\sqrt{\{\frac{1}{16}(1+\sqrt{5})^2-1\}}=-\frac{1}{4}(1+\sqrt{5})\pm\frac{1}{4}(\sqrt{10-2\sqrt{5}})\sqrt{-1}$. Therefore the five roots are

$$1,\ \tfrac{1}{4}(-1+\sqrt{5})\pm\tfrac{1}{4}(\sqrt{10+2\sqrt{5}})\sqrt{-1},\ \text{and}$$
$$-\tfrac{1}{4}(1+\sqrt{5})\pm\tfrac{1}{4}(\sqrt{10-2\sqrt{5}})\sqrt{-1}.$$

1. It is required to find the four roots of the equaton $x^4+1=0$.

See Index. Ans. $x=\frac{1}{2}\sqrt{2}\pm\frac{1}{2}\sqrt{-2}$, and $-\frac{1}{2}\sqrt{2}\pm\frac{1}{2}\sqrt{-2}$

2. It is required to find the five roots of the equation $x^5+1=0$.

Ans. -1, $\frac{1}{4}(1+\sqrt{5})\pm\frac{1}{4}(\sqrt{10-2\sqrt{5}})\sqrt{-1}$,

and $\frac{1}{4}(1-\sqrt{5})\pm\frac{1}{4}(\sqrt{10+2\sqrt{5}})\sqrt{-1}$.

3. It is required to find the six roots of the equation $x^6-a^6=0$.
Ans. a, $-a$, $a(-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3})$, and $a(\frac{1}{2}\pm\frac{1}{2}\sqrt{-3})$.

4. It is required to find the eight roots of the equation $x^8-1=0$.
Ans. 1, -1, $\sqrt{-1}$, -1, $(\frac{1}{2}\sqrt{2}\pm\frac{1}{2}\sqrt{-2})$, $(-\frac{1}{2}\sqrt{2}\pm\frac{1}{2}\sqrt{-2})$.

5. Find the roots of the equations $x^{10}-a^{10}=0$, and $x^{16}-1=0$.

Of Equations that have equal Roots.

110. Besides the classes of equations before treated of, there are others of a different kind, that are equally susceptible of being reduced to those of lower dimensions; the most useful of which are such as have two or mose equal roots, with the same or contrary signs; in which case the method of resolving them, as far as the 4th order, inclusively, are as follows:*

RULE I. If a quadratic equation of the general form $x^2\pm ax+b=0$, has two equal roots, with the same sign, they will be each $=-\frac{1}{2}a$, or $+\frac{1}{2}a$, according as the coefficient of the second term is positive or negative. And if the equal roots have contrary signs, the equation must be of the form $x^2-b^2=0$, in which case $x=\pm\sqrt{b^2}$, or $=+b$, and $-b$.

2. If a cubic equation of the general form, $x^3+ax^2+bx+c=0$, has two equal roots, with the same sign, each of them will be a root of the equation $3x^2+2ax+b=0$, and the remaining root will be $=-a-$ twice one of the equal roots. And if the equal roots have contrary signs, we shall have in that case, $ab=c$; and the roots sought will be $\sqrt{-\frac{c}{a}}$, and $-\sqrt{-\frac{c}{a}}$, and the remaining root will be $=a$.

3. If a biquadratic equation of the form

$$\left.\begin{array}{l} x^4+ax^3+bx^2+cx+d=0, \\ -38\ \ 210\ \ 538\ \ 289, \end{array}\right\}$$

has two equal roots, with the same sign, each of them will be a root of the equation

$$(3a^2-8b)x^2+2(ab-6c)x+ac-16d=0;\ x^2-\frac{22416}{2652}x=\frac{25068}{2652}$$

$$x^2-\frac{22416}{2652}x=\frac{25058}{2652},\ \text{or}\ x^2-\frac{1868}{221}x=\frac{2089}{221}.\ \text{By Case II,}$$

$$x^2-\frac{1868}{221}x+(\frac{934}{221})^2=(\frac{2089}{221}+\frac{872356}{48841})=\frac{1334025}{48841};\ x-\frac{934}{221}=$$

* Other methods of resolving equations that have equal roots, have been given by *Maclaurin*, and several of the best foreign writers on Algebra, which, though not so simple as those here laid down, are more commodious for equations of the higher orders, which, if treated in the manner employed in the text, would become very tedious.

$\sqrt{\frac{1334025}{48841}}=\frac{1155}{221}$, or $x=\pm\frac{1155}{221}+\frac{934}{221}=-\frac{221}{221}$, or -1, and the other two roots will be found from the equation

$$x^2+(a+2r)x+b+2ar+3r^2=9, \text{ or } x^2-40x=-289,$$

r being one of the equal roots before obtained.

And if the equal roots have contrary signs, we shall have $a(bc-ad)=c^2$, and $x=\pm\sqrt{-\frac{c}{a}}$ as before; the other two roots being found from the equation $x^2+ax+b+r^2=0$.

4. Also, if the general biquadratic equation $x^4+ax^3+bx^2+cx+d=0$, has three equal roots, with the same sign, each of them will be a root of the equation $6x^2+3ax=-b$. And if two of the three equal roots have contrary signs, we shall have

$$a(bc-ad)x=c^2, \text{ and } x=\pm\sqrt{-\frac{c}{a}}, \text{ as in the last case.*}$$

1. Supposing that two of the roots of the equation $x^3+x^2+33+63=0$, are equal to each other, and have the same sign, it is required to determine them.

Here a being $=1$, $b=-33$, and $c=63$, we shall have, by the rule above laid down, $3x^2+2ax+b=3x^2+2x-33=0$; or $x^2-\frac{2}{3}x=11$; whence $x=-\frac{1}{3}\pm\sqrt{(\frac{1}{9}+11)}=-\frac{1}{3}\pm\sqrt{\frac{100}{9}}=-\frac{1}{3}\pm\frac{10}{3}=3$, or $=-\frac{1}{3}-\frac{10}{3}=-\frac{11}{3}$, the former of which values (3) being substituted for x, in the original equation, is found to succeed.

Whence two of the three roots are each $=3$; and the remaining root $=-7$.

2. Supposing that two of the roots of the equation $x^4+\frac{1}{2}x^3-\frac{3}{2}x^2+\frac{4}{27}x+\frac{2}{9}=0$, are equal to each other, and have the same sign; it is required to determine them.

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* The mode of solution here given might have been easily extended to equations of the 5th, 6th, &c. power; but in those above the 4th degree, the expressions for the equal roots, when they have the same sign, are too complicated to be of any practical use.

Rules might have been also introduced for determining, *a priori*, by meams of the relations of the coefficients alone, whether or not any given equation has equal roots; but, except in the case where they have contrary signs, this may in general be more readily done, by finding the root of the quadratic equation to which it is reducible, and then seeing whether the root, thus obtained, be a root of the original equation or not.

In all cases of this kind, however, it may be of use to know, that an equation cannot have equal roots when the last term, and the coefficient of the last but one, are prime to each other; but the reverse of this is not always true.

Here a being $=\frac{1}{2}$, $b=-\frac{3}{2}$, $c=\frac{4}{27}$, and $d=\frac{2}{9}$, we shall have, by the rule, $(3a^2-8b)x^2+2(ab-6c)x+ac-16d=\frac{51}{4}x^2-\frac{59}{18}x-\frac{94}{27}=0$. Or, by transposition and simplifying the terms, $x^2-\frac{118}{459}x=\frac{376}{1377}$; the two roots of which equation are $\frac{2}{3}$ and $=-\frac{188}{459}$.

And by trial, the first of these, $\frac{2}{3}$, will be found to be a root of the original equation. Whence the 2 roots required are each $=\frac{2}{3}$.

And, since $r=\frac{2}{3}$, we shall have, by the rule,

$x^2+(a+2r)x+b+2ar+3r^2=x^2+(\frac{1}{2}+\frac{4}{3})x-\frac{3}{2}+\frac{2}{3}+\frac{4}{3}=0$, or $x^2+\frac{11}{6}x=-\frac{1}{2}$; the roots of which latter equation are $-\frac{1}{3}$, $-\frac{3}{2}$. Hence the four roots of the given equation are $\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$, and $-\frac{3}{2}$.

3. It is required to determine whether the equation $x^4+11x^3+19x^2-99x-252=0$, has equal roots, and if so, what they are.

Here, since $a=11$, $b=19$, $c=-99$, and $d=-259$, we shall have, by the rule $a(bc-ad)=11(-19\times 99+11\times 252)=11(-1881+2772)=9801=99^2=c^2$; whence the equation has two equal roots, with contrary signs; and, consequently,

$$x=\pm\sqrt{-\frac{c}{a}}=\pm\sqrt{\frac{99}{11}}=\pm\sqrt{9}=3, \text{ and } -3,$$ which two roots will be found to answer the conditions of the question. Also, since $r=\pm 3$, we shall have, by the rule, $x^2+ax+b+r^2+=x^2+11x+28=0$, or $x^2+11x=-28$; the roots of equations are -4 and -7 Whence the four roots of the given equation are $3, -3, -4$, and -7.

1. Supposing that two roots of the equation $x^3-48x-128=0$, are equal, and have the same signs; it is required to determine them. Ans. -4, -4, and 8.

11. A cubic equation of the general form $x^3+px^2+qx+r=0$, has two equal roots with the same signs. Let its 3 roots be denoted by a, b, c; then we shall have the two following equations; $\{a^3+pa^2+qa+r=0\}-\{b^3+pb^2+qb+r=0\}=a^3-b^3+p(a^2-b^2)+q(a-b)=0$; and when divided by $a-b$, will give $a^2+ab+b^2+p(a+b)+q=0$. If we now take $b=0$, the reduced equationin this case will be $3a^2+2pa+q=0$, or $=3x^2+2px+q=0$, where one of the values of x, taken twice with the same sign, will give the two equal roots of the proposed equation.

2. Given the equation $x^3+5\frac{1}{2}x^2-\frac{1}{16}x-\frac{11}{32}=0$, to find whether it has equal roots, and if so, what they are. Ans. $\frac{1}{4}, -\frac{1}{4}$, & $-5\frac{1}{2}$.

3. Supposing the equation $x^4+3a^3-14x^2-12x+40=0$, to have three equal roots, two of which have the same sign; it is required to determine them. Ans. $2, 2, -2$, and -5.

4. Supposing the equation $x^4-\frac{51}{400}x^2-\frac{1}{800}x+\frac{3}{800}=0$, to have two equal roots, with the same sign; it is required to determine all its four roots. Ans. $-\frac{1}{4}, -\frac{1}{4}, \frac{1}{5}$, and $\frac{3}{10}$.

Of Exponential Equations.

111 An exponential quantity is that which is raised to some unknown power, or which has a variable quantity for its index, as a^x, $a^{\frac{1}{x}}$, x^x, or $a^{\frac{1}{x}}$, &c. See Logarithms page 302.

And an exponential equation is that which is formed between any expression of this kind and some other quantity whose value is known; as $a^x = b$, $x^x = a$, &c.; where it is to be observed, that the first of these equations $a^x = b$, when converted into logarithms, is the same as $x \log. a = \log. b$; or $x = \dfrac{\log. b}{\log. a}$; and the second $x^x = a$, when so converted, is the same as $x \log. x = \log. a$; in the last of which cases the approximate value of x may be determined as follows:

Rule. Find, by trial, two numbers nearly equal to the value of x, and substitute them in the given equation, $x \log. x = \log. a$, instead of the unknown quantity, noting the results obtained from each, as in the rule of Double Position, before laid down, in art. 31.

Then, by means of a certain number of successive operations, performed in the same manner as is there described, the value of x may be found to any degree of accuracy required.

1. Given $x^x = 100$, to find an approximate value of x.

Here, by the above formula, we have $x \log. x = \log. 100 = 2$,

And since x is easily found, by a few trials, to be nearly in the middle between 3 and 4, but rather nearer the latter than the former, let 3.5 and 3.6 be taken for the two assumed numbers. Then log. 3.5 = .5440680; which, being multiplied by 3.5, gives 1.904238 = first result; And log. 3.6 = .5563025; which being multiplied by 3.6, gives 2.002689 for the second result. Whence

$$\begin{array}{l}
2.002689\ldots3.6\ldots2.002689\\
1.904238\ldots3.5\ldots2.\\
\hline
.098451 \ :\ .1 \ ::\ .002689 \ :\ .00273
\end{array}$$

for the first correction; which, taken from 3.6, leaves x = 3.59727, nearly. And as this value is found, on trial, to be rather too small, let 3.59727 and 3.59728 be taken as the two assumed numbers. Then log. 3.59727 = .5559731; which, being multiplied by 3.59727, gives 1.9999854 = first result.

And log. 3.59728 = .5559743; which, being multiplied by 3.59728, gives 1.9999953 = second result; whence

$$\begin{array}{l}
1.9999953\ldots3.59728\ldots2\\
1.9999854\ldots3.59727\ldots1.9999053\\
\hline
.0000099 \ :\ .00001 \ ::\ .0000047 : .0000047474
\end{array}$$

for the second correction; which, added to 3.59728, gives x = 3.59728474747, extremely near the truth.

And in the same way may the value of the unknown quantity be determined, in any other equation of this kind.

1. Given $x^x=2000$, to find an approximate value of x.

Here x log. $x=$ log. $2000 = 3.301030$, and x is found, by a few trials, to be rather less than 5; let therefore 4.8 and 4.9 be taken for the assumed numbers; then log. $4.8=0.681241$, which, being multiplied by 4.8, gives $3.269956 =$ the first result; and log. $4.9 = 0.690196$, which being multiplied by 4.9, gives $3.381960 =$ the second result. Whence

3.381960 . . 4.9 . . 3.381960
3.269956 . . 4.8 . . 3.301030

.112004 : .1 :: .080930 : 00722 for the first correction, which, being taken from 4.9, leaves 4.8278, the answer nearly.

2. Given $(6x)^x=96$, to find the approximate value of x.

Here x log. $6x =$ log. 96, and x is easily found, by a few trials, to be rather less than 2; let therefore 1.8 and 1.9 be taken for the two assumed numbers; then log. $6x(=10.8) = 1.033424$, which, multiplied by 1.8, gives $1.860363 =$ the first result; and log. $6x(=11.4) = 1.056905$, which, multiplied by 1.9, gives $2.008119 =$ the second result. Whence

2.008119 . . . 1.9 . . . 2.008119
1.860363 . . . 1.8 . . . 1.982271

.147756 : .1 :: .025848 : .0174 for the first correction, which, taken from 1.9, leaves 1.8826 for the answer,

3. Given $x^x=123456789$, to find the approximate value of x.

Here, after a few trials, or from inspection in a table of powers we find x is between 8 and 9, but nearer the latter than the former. Assume therefore $x=8.6$ and $x=8.7$.

Then log · 123456789 = 8.0915149

Log.8.6=0.9344935	L.8.7=0.9395193	8.17381	8.7	8.09151
Mult. by 8·6	8·7	8·03664	8·6	8·03664
56069610	65766351	·13717 ;	0.1 ::	·05487
7·4759480	7·5161544			: ·04
8·03664410 Res	8·17381791			Whence

$x = 9·6 + ·04 = 8·640$ nearly; And repeating the operation by assuming x equal to 8·64 and 8·641; x is found $= 8·6400268$

4. Given $x^x-x=(2x-x^x)^{\frac{1}{x}}$, to find the approximate value of x. Ans. $x=1.47933$.

5. Given $a^{mx+n}=b^{nx+m}$, to find x. Ans. $x=\dfrac{\text{log. } b^m-\text{log. } a^n}{\text{log. } a^m-\text{log. } b^n}$.

6. Given $\left.\begin{array}{l} a^{xy}=m \\ b^{x+y}=n \end{array}\right\}$ to find the values of x and y. ; See Index.

7. Given $x^x=48$, $(\frac{5}{6})^x=\frac{1}{2}$, and $x^x=100$, to find x.

Ans. $x = 3.26$, $x = 3,8$ and $x = 3.5972$.

Here x is found, by a few trials, to be nearly equal to 1.7; let therefore 1.7 and 1.8 be taken for the assumed numbers:

1st Supposition.		*2d Supposition.*	
By logs. $x^x=$	2.46471	$x^x=$	2.88072
$x^x-x=$	0.76471	$x^x-x=$	1.08072
$2x-x^x=$	.93529	$2x-x^x=$	.71929
log. $(x^x-x)=$	-1.883496	log. $(x^x-x)=$	0.033713
$\frac{1}{x}$ log. $(2x-x^x)=$	-1.982909	$\frac{1}{x}$log. $(2x-x^x)=$	-1.864943

$$\begin{array}{r} 0.033713 \\ -1.864943 \\ \hline .168770 \end{array} \quad \text{results} \quad \begin{array}{r} -1.982909 \\ -1.883496 \\ \hline .098413 \end{array}$$

Therefore .168770 : .1 : : .098413 : .058. Whence $1.8-.058$ $=1.742$, the value fo x, nearly; and, assuming 1.74 and 1.75 for the numbers, and repeating the operation, we get $x=1.74793$.

5. Given $a^x+\frac{1}{a^x}=b$, to find the value of x.

$$a^{2x}-ba^x=-1;\ a^{2x}-ba^x+\frac{b^2}{4}=\frac{b^2-4}{4};\ a^x=\frac{b\pm\sqrt{(b^2-4)}}{2};$$

$$\therefore x\cdot\text{log. } a=\text{log. }\frac{b\pm\sqrt{(b^2-4)}}{2}=\text{log. }\{b\pm\sqrt{(b^2-4)}\}-\text{log. } 2;$$

$$\therefore x=\frac{\text{log. }\{b\pm\sqrt{(b^2-4)}\}-\text{log. } 2}{\text{log. } a}$$

6. Given $a^{mx}+a^{mx-1}=b$, to find the value of x.

$$a^{mx}+a^{mx-1}=b;\ \text{or } a^{mx}+\frac{a^{mx}}{b}=b;\ \therefore a^{mx}=\frac{ab}{a+1}.$$

Take the log. and mx. log. $a=$ log. $(ab)-$log. $(a+1)$;

$$\therefore x=\frac{\text{log.}(ab)-\text{log.}(1+1)}{m\cdot\text{log. } a}.$$

112. In considering the nature of an exponential of the form a^{b^x}, it must be observed that it means a to the power of b^x, and not a^b to the power of x, which would be a^{bx}.

7. Given $x^x=a$, to find the value of x.

Find by trial a number, n, as near to the true value of x as can be done. Let $n+y=x$, then by taking the log. of the given equation,

$$(n+y)\cdot\text{log.}(n+y)=\text{log. } a;\ \text{i. e. } (n+y)\cdot\text{log. } n\cdot(1+\frac{y}{n})=\text{log. } a,$$

$$\text{Or } (n+y)\cdot\text{log. } n+(n+y)\text{M}\left\{\frac{y}{n}-\frac{y^2}{2n^2}+\&\text{c.}\right\}=\text{log. } a;$$

Then by multiplying $(n+y)$ into the series, and rejecting all the terms into which y^2, y^3, &c. enter, we obtain $(n+y)\cdot$log. $n+$

$My = \log. a$. Whence $y = \frac{\log. a - n \log. n}{\log. n + M}$; and $\therefore x = n + y =$ $n + \frac{\log. a - n \cdot \log. n}{\log. n + M} = \frac{\log. a + Mn}{\log. n + M}$, which affords a near approximate value of x. If extreme accuracy be required, we must use this value of x in the place of n, and repeat the operation.

Of the method of Indeterminate Coefficients.

113. This is a species of investigation which is frequently used for obtaining the development of certain fractional and other expressions, without having recourse to the operations of division, or the extraction of roots; the method of performing which is as follows :*

RULE. Assume a series, or other expression, with unknown coefficients, for that which is required to be found; then, having multiplied it by the denominator of the given fraction, or raised it to its proper power, find the value of each of these coefficients, by equating the homologous terms of the two expressions, or putting such of them as have no corresponding terms, equal to 0, as the case may require. Find the development of $\frac{a}{b+cx}$, according to the above method.

$$\text{Assume } \frac{a}{b+cx} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$$

Then, multiplying the right hand side of the equation by $b + cx$, and transposing a, we shall have

$$0 = \begin{array}{l} Ab \\ -a \end{array} + \left.\begin{array}{l} Bb \\ +Ac \end{array}\right| x + \left.\begin{array}{l} Cb \\ +Bc \end{array}\right| x^2 + \left.\begin{array}{l} Db \\ +Cc \end{array}\right| x^3 + \left.\begin{array}{l} Eb \\ +Dc \end{array}\right| x^4 + \left.\begin{array}{l} Fb \\ +Ec \end{array}\right| x^5 + \&c.$$

And by putting the first term, and the coefficients of the several powers of $x = 0$, there will arise the following equations:

$$\left.\begin{array}{l} Ab - a = 0 \\ Bb + Ac = 0 \\ Cb + Bc = 0 \\ Db + Cc = 0, \&c. \\ Eb + Dc = 0, \\ Fb + Ec = 0, \end{array}\right| \text{ or } \left|\begin{array}{ll} A = \frac{a}{b}; & B = -\frac{c}{b}A \\ C = -\frac{c}{b}B, \ \&c. & D = -\frac{c}{b}C; \\ E = \frac{a}{b}D; & F = \frac{a}{b}E \end{array}\right.$$

$$\text{Hence } \frac{a}{b+cx} = \frac{a}{b} - \frac{c}{b}Ax - \frac{c}{b}Bx^2 - \frac{c}{b}Cx^3 - \frac{c}{b}Dx^4 - \&c.$$

* The method here described of assuming certain expressions, with indeterminate coefficients, in the investigation of analytical problems, and then determining their values by means of their relations to other quantities that are known, appears to have been first employed by *Descartes*, who obtained by this means the rule for resolving biquadratic equations, which still goes by his name.

where it is obvious, that each coefficient, after the second is equal to the one that immediately precedes it, multiplied by $-\frac{c}{b}$ which law renders it unnecessary to take a greater number of equations, or to push the calculation farther.

2. Required the development of $\frac{d+ex}{a'+b'+c'x^2}$, according to the [above method,

Assume $\frac{d+ex}{a'+b'x+c'x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\&c.$

Then multiplying the right hand side of the equation by $a+b'x+c'x^2$, and transposing $a+bx$, we shall have

$$\begin{array}{r|r|r|r|l} 0=Aa'+Ba' & x+Ca & x^2+Da' & x^3+Ea & x^4+\&c. \\ -\ d+Ab' & +Bb' & +Cb' & +Db & \\ -e & +Ac' & +Bc' & +Cc & \end{array}$$

And, by putting the first term, and the coefficients of the several powers $=0$, there will arise

$$\left.\begin{array}{r} Aa'-d=0 \\ Ba'+Ab'-e=0 \\ Ca'+Bb'+Ac'=0 \\ Da'+Cb'+Bc'=0, \end{array}\right| \begin{array}{c} \\ \text{or} \\ \\ \&c. \end{array} \left| \begin{array}{l} A=\frac{d}{a'}\,;\ B=-\frac{b'}{a'}A+\frac{e}{a'} \\ C=-\frac{b'}{a'}B-\frac{b'}{a'}A \\ D=-\frac{b'}{a'}C-\frac{c'}{a'}B\ \& \end{array}\right.$$

Lastly $\quad Ea+Db+Cc=0;\ \therefore\ E=-\frac{b}{a}D-\frac{c}{a}C.$

Whence $\frac{d+ex}{a'+b'x+c'x^2}=\frac{a}{a'}-(\frac{'d}{a}A-\frac{e}{a'})x-(\frac{b'}{a'}B-\frac{c'}{a'}A)x^2-(\frac{b'}{a'}C+\frac{c'}{q'})x^3-$&c., where each coefficient, beginning with the third, may be readily deduced from the two that precede it.

So that if if P, Q, R, be any three consecutive coefficients, we shall have $Ra'+Qb'+Pc'=0$; or $R=-\frac{b'}{a'}Q-\frac{c'}{a'}P$.

3. Given $(x^2+p)^2-(qx+r)^2=x^4+ax^2+bx+c$, to find the indefinite coefficients p, q, and r.

Here, by squaring the terms on the left hand side of the equation, and collecting those that are alike, we have

$$x^4+(2p-q^2)x^2-2qrx+p^2-r^2=x^4+ax^2+bx+c.$$

And consequently, by equating the homologous terms,

$$\left.\begin{array}{r} 2p-q^2=a \\ -2qr=b \\ p^2-r^2=c \end{array}\right| \quad \text{or} \quad \left| \begin{array}{l} 2p-a=q^2 \\ -b=2qr \\ p^2-c=r^2 \end{array}\right.$$

where it is plain, that the product of the first and third of these is equal to $\frac{1}{4}$ of the square of the second; or, $2p^3-ap^2-2cp+ac=\frac{1}{4}b^2$.

Whence the value of p may be found by a cubic equation, and then q and r from the former equations.

We shall further observe, that every algebraic function, containing one unknown quantity, or that quantity and its powers, may be assumed of the following form: $A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+$ &c., of which the terms involve only the whole positive powers of the unknown quantity x, and certain coefficients A_0, A_1, A_2, &c. which are wholly independent of the value of that quantity.

114. This may be exemplified by considering the expansions of such functions as usually occur; where the particular series that is to be adopted, in each case, may be determined, by putting $x=0$ and observing the nature of the result.

1. Thus, if the function to be developed is to become $=$ to some particular quantity for $x=0$, the first term of the series must be taken singly, or without an x; and if it is to become equal to zero, for $x=0$, the first term must contain x.

2. Also, if any term be either omitted or introduced into the series that does not belong to it, the comparisons of the homologous terms will always show the impropriety of it, by their leading to absurd equations. Thus taking, for instance, as a partial example of the first of these cases $\frac{1}{1+x}=A_0+A_1x+A_2x^2+A_3x^3+$&c., is plain, that when $x=0$, we shall have $\frac{1}{1}=A_0$, or $A_0=1$; which is therefore the first term of the series.

3. In like manner, if we take the expansion of a fraction of the following kind, $\frac{x}{1+x}=A_1x+A_2x^2+A_3x^3+A_4x^4+$&c. it is equally evident that the first term of the series must contain x, for $x=0$; for, if it were A_0, $\frac{0}{1}$ would be equal to some determinate quantity, which is impossible.

4. Also, if there be taken, as an example of the last of the above cases, $\frac{1}{1-x}=A_0+A_1x \cdots\cdots+A_3x^3+A_4x^4+$ &c., where the term A_2x^2 of the series is omitted, it will be found, by multiplying the right hand member of the identity by $1-x$, and equating the homologous coefficients, that $A_0=1$, and $A_1=1$, as also $A_1=0$; which is absurd.

5. Again, let there be taken, as an additional example, the well known formula $\frac{x^m-v^m}{x-v}=x^{m-1}+A_2x^{m-2}+A_3x^{m-3}+\ldots+v^{m-1}$; where, supposing m to be a whole positive number, it is evident that the series is rightly assumed; because, for $x=0$, the development is reduced to v^{m-1}, and for $v=0$ it is reduced to x^{m-1}, as

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it ought.* Hence, multiplying the right hand member by $x-v$, we shall have

$$x^m - v^m = x^m + \begin{array}{c|c|c|c} A_2 & x^{m-1} + A_3 & x^{m-2} + A_4 & x^{m-3} \ldots\ldots + v^{m-1}x \\ -v & -A_2 v & -A_3 v & \ldots\ldots - v^m \end{array}$$

6. And, consequently, by putting the coefficients of the several terms of the product, except the first and last, equal to 0, there will arise the following relations:

$$\begin{array}{l|c|l} A_2 - \quad v = 0 & & A_2 = v \\ A_3 - A_2 v = 0 & \text{or} & A_3 = v^2 \\ A_4 - A_3 v = 0 & & A_4 = v^3 \\ A_5 - A_4 v = 0, \ \&c. & & A_5 = v^4, \ \&c. \end{array}$$

From which it is evident, that the proposed formula, when expanded, will become

$$\frac{x^m - v^m}{x-v} = x^{m-1} + v x^{m-2} + v^2 x^{m-3} + v^3 x^{m-4} + \ldots\ldots\ldots + v^{m-1}$$

115. In like manner, supposing, when m is a whole number, the following polynomial, or series,

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots\ldots\ldots + Tx + V,$$

to be exactly divisible by $x-a$, we shall have, by assuming a certain series, $x^{m-1} + A'x^{m-2} + B'x^{m-3} + \&c.$ for the quotient

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots + Tx + V =$$
$$(x-a)(x^{m-1} + A'x^{m-2} + B'x^{m-3} + \ldots \quad + S'x + T'),$$

where A, B, C, &c. are supposed to be known coefficients, and A′, B′, C′, &c. other coefficients that are to determined.

Then, by actually performing the multiplication indicated by the second member of the identity, there will arise,

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots Tx + V$$
$$= \begin{array}{|c|c|c|c|c|c} x^m + A' & x^{m-1} + B' & x^{m-2} + C' & x^{m-3} + S' & x^2 + T' & x - aT' \\ -a & -aA' & -aB' & -aR' & +aS' & \end{array}$$

In which case, as the corresponding powers of x in both series have a term, that for the development of the quotient appears to have been rightly assumed. Wherefore, bringing all the terms to the left hand side of the equation, we shall have

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* In the case where the index, in the above expression, is a fractional number, which requires a different investigation, the same formula may be exhibited under the form

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x-v} = x^{\frac{m}{n}-1} \left| \frac{1 + \left(\frac{x}{v}\right) + \left(\frac{x}{v}\right)^2 + \left(\frac{x}{v}\right)^3 + \ldots + \left(\frac{x}{v}\right)^{m-1}}{+1\left(\frac{x}{a}\right)^{\frac{m}{n}} + \left(\frac{x}{v}\right)^{\frac{2m}{n}} + \left(\frac{x}{v}\right)^{\frac{3m}{n}} + \ldots + \left(\frac{x}{v}\right)^{\frac{(m-1)m}{n}}} \right.$$

being, in this state, a similar theorem to that given by *Landen*, in his *Residual Analysis*, from which he attempted to derive the Doctrine of Fluxions, now the method of the Calculus.

$$\begin{array}{l|l|l|l|l|l}
A & x^{m-1}+B & x^{m-2}+C & x^{m-3}+\ldots+R & x^2+T & x+V=0 \\
-A_1 & -B_1 & -C' & -R' & -T' & -aT \\
+a & +aA' & +aB' & +aT' & +aS' &
\end{array}$$

And, consequently, by equating the coefficients of the several powers of x with 0, there will arise the following relations;

$$\begin{array}{l|c|l}
A-A_1+a=0 & & A'=a+A \\
B-B_1+aA'=0 & \text{or} & B'=a^2+aA+B \\
C-C'+aB'=0 & & C'=a^3+a^2A+aB+C \\
D-D'+aC'=0, \text{ \&c.} & & D'=a^4+a^3A+a^2B+aC+D, \text{ \&c.}
\end{array}$$

where the law by which they may be continued being sufficiently evident, we shall have, for the quotient,

$$Q=x^{m-1}\begin{array}{l|l|l|l}
+a & x^{m-2}+a^2 & x^{m-3}+a^3 & x^{m-4}+\ldots+a^{m-1} \\
+A & +aA & +a^2A & +a^{m-2}A \\
 & +B & +aB & +a^{m-3}B \\
 & & +C & +a^{m-4}C \text{ \&c.}
\end{array}$$

And as the process here followed is the same in effect as that which would have taken place from a direct comparison of the corresponding terms of the two members of the identity, it follows, that, when two polynomials are identical, the coefficients of the respective powers of the unknown quantity, in each of them, will be equal.

So that, from these instances, and others of a like kind, that might be given, the form of the series for any expanded algebraic function, and the values of the assumed coefficients may always be determined.

4. Find the expansion of the expression $\frac{a}{b-cx}$. Assume $\frac{a}{c-cx}=A+Bx+Cx^2+Dx^3+Ex^4+\text{\&c.}$ By freeing this equation of fractions

$$a=Ab\begin{array}{l|l|l|l|l}
+Bb & x+Cb & x^2+Db & x^3+Eb & x^4+\text{\&c.} \\
-Ac & -Bc & -Cc & -Dc & +\text{\&c.}
\end{array}$$

By transposition,

$$\left.\begin{array}{l|l|l|l|l}
Ab+Bb & x+Cb & x^2+Db & x^3+Eb & x^4+\text{\&c.} \\
-a-Ac & -Bc & -Cc & -Dc & -\text{\&c.}
\end{array}\right\}=0;$$

$$\text{Therefore } Ab-a=0;\ \therefore Ab=a,\ \therefore A=\frac{a}{b};$$

$$Bb-Ac=0;\ \therefore Bb=Ac,\ \therefore B=\frac{Ac}{b}=\frac{ac}{b^2};$$

$$Cb-Bc=0;\ \therefore Cb=Bc,\ \therefore C=\frac{Bc}{b}=\frac{ac^2}{b^3};$$

$$Db-Cc=0;\ \therefore Db=Cc,\ \therefore D=\frac{Cc}{b}=\frac{ac^3}{b^4};$$

$$\text{Therefore } \frac{a}{b-cx}=\frac{a}{b}+\frac{acx}{b^2}+\frac{ac^2x^2}{b^3}+\frac{ac^3x^3}{b^4}\text{ \&c. Ans.}$$

5. Find the expansion of the expression $\sqrt{(a-x)}$.

Assume $\sqrt{(a-x)}=A+Bx+Cx^2+Dx^3+Ex^4+\&c.$

Clearing the equation of surds by squaring,

$a-x=A^2+2ABx+(B^2+2AC)x^2+(2AD+2BC)x^3+\&c.$

By transposition, $(A^2-a)+(2AB+1)x+(2AC+B^2)x^2+(2AD+2BC)x^3+\&c.=0$. Therefore $A^2-a=0$,

$\therefore A=a^{\frac{1}{2}}$. $2AB+1=0$, $\therefore 2AB=-1$, $\therefore B=-\frac{1}{2A}=-\frac{a^{\frac{1}{2}}}{2a}$.

$$2AC+B^2=0,\quad \therefore 2AC=-B^2,\quad \therefore C=-\frac{B^2}{2A}=-\frac{a^{\frac{1}{2}}}{8a^3};$$

$$2AD+2BC=0,\quad \therefore AD=-BC,\quad \therefore D=-\frac{BC}{A}=-\frac{a^{\frac{1}{2}}}{16a^5};$$

$$\text{Whence } \sqrt{(a-x)}=a^{\frac{1}{2}}\left\{1-\frac{x}{2a}-\frac{x^2}{8a^3}-\frac{x^3}{16a^5},\ \&c.\right\}\text{Ans.}$$

6. Required the development of $\sqrt{(a^2+x^2)}$ by this method.

Assume $\sqrt{(a^2+x^2)}=A+Bx+Cx^2+Dy^3+\&c.$; then, by squaring each side, and transposing, we have

$$\left.\begin{matrix}A^2+2ABx+2AC\\-a^2\qquad\quad +B^2\\ \qquad\qquad +1\end{matrix}\right\}x^2\left.\begin{matrix}+2AD\\+2BC\end{matrix}\right\}x^3\left.\begin{matrix}+2BD\\+C^2\end{matrix}\right\}x^4+\&c.=0;$$

$$\text{Whence}\left\{\begin{array}{ll}A^2-a^2=0, & \text{therefore } A=a\\ 2AB=0, & \dots\dots\dots B=0\\ 2AC-1=0, & \dots\dots\dots C=\frac{1}{2a}\\ 2AD+2BC=0,\ \&c. & \dots\dots\dots D=0,\ \&c.\end{array}\right.$$

$$\text{Therefore } \sqrt{(a^2+x^2)}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\&c$$

7. Required the development of $\frac{x}{1+x+x^2}$ by the same method. Here, since the first term of the series must contain x,

Assume $\frac{x}{1+x+x^2}=Ax+Bx^2+Cx^3+Dx^4+\&c.$ then we have

$$\left.\begin{matrix}A\\-1\end{matrix}\right\}x\left.\begin{matrix}+A\\+B\end{matrix}\right\}x^2\left.\begin{matrix}+A\\+B\\+C\end{matrix}\right\}x^3\left.\begin{matrix}+B\\+C\\+D\end{matrix}\right\}x^4\left.\begin{matrix}+C\\+D\\+E\end{matrix}\right\}x^5+\&c.=0;$$

$$\text{Whence}\left\{\begin{array}{ll}A-1=0, & \text{therefore } A=1\\ A+B=0, & \dots\dots\dots B=-1\\ A+B+C=0, & \dots\dots\dots C=0\\ B+C+D=0, & \dots\dots\dots D=1\\ C+D+E=0, & \dots\dots\dots E=-1;\end{array}\right.$$

$$\text{Therefore } \frac{x}{1+x+x^2}=x-x^3+x^4-x^6+x^7-\&c.$$

1. It is required to convert $\frac{A}{b-ax}$ into a series by the above method. Ans. $\frac{A}{b}(1+\frac{ax}{b}+\frac{a^2x^2}{b^2}+\frac{a^3x^3}{b^3}+\frac{a^4x^4}{b^4}+$&c.

2. It is required to convert $\frac{1+2x}{1-x-x}$ into a series by the above method. Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+29x^6+$&c.

3. It is required to convert $\frac{1-x}{1-2x-3x^2}$ into a series by the same method. Ans. $1+x+5x^2+13x^3+41x^4+121x^5+365x^6+$&c.

Assume $\frac{1-x}{1-2x-3x^2}=A+Bx+Cx^2-Dx^3-Ex^4+Fx^5+Gx^6+$,&c,

This, cleared of fractions, and by transposition, we have

$$\left.\begin{array}{l|l|l|l|l|l|l}
A+ & B\,x+ & C\,x^2+ & D\,x^3+ & E\,x^4+ & F\,x^5+ & G\,x^6+\&c. \\
-1-2A & -2B & -2C & -2D & -2E & -2F & -\&c. \\
+1 & -3A & -3B & -3C & -3D & -3E &
\end{array}\right\}=0.$$

Therefore $A-1=0$; $\therefore A=1$.

$B-2A+1=0 \quad B=2A-1=2-1=1;$

$C-2B-3A=0; \therefore C=2B+3A=2.2+3=5$

$D-2C-3B=0; \therefore D=2C+3B=2.5 \quad +3.1 = \quad 13$

$E-2D-3C=0; \therefore E=2D+3C=2.13 \quad +3.5 =26+15 =41$

$F-2E-3D=0; \therefore F=2E=3D=2.41 \quad +3.13=82+39 =121$

$G-2F-3E=0; \therefore G=2F=3E=2.121+3.41=242+123=365$

$\frac{1-x}{1-2x-3x^2}=1+x+5x^2+13x^3+41x^4+121x^5+365x^6+$&c. Ans.

4. It is required to convert $\sqrt{(1-x)}$ into a series by the same method. Ans. $1-\frac{x}{2}-\frac{x^2}{2.4}-\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}-\frac{3.5.7x^5}{2.4.6.8.10}-$&c.

5. It is required to find, according to the above method, the several roots or values of x, in the equation $x^4-6x^3+13x^2-12x=a$, by means of a quadratic equation.

Ans. $x=\frac{3}{2}\pm\sqrt{\{\frac{1}{4}\pm\sqrt{(a+4)}\}}$.

Put $x=y+z$; then by substitution and ordering the equation, the terms I have $y^4+y^3(4z-6)+y^2(6z^2-18z+13)+y(4z-6)\times(z^2-3z+2)+z^4-6z^3+13z^2-12z=a$. (1.) Let $4z-6=0$; or $z=\frac{3}{2}$; then the odd powers of y disappear from the (1), and it becomes, by substitution for z, its value in the other terms, $y^4-\frac{1}{2}y^2-\frac{63}{16}=a$, or $y^4-\frac{1}{2}y^2=a+\frac{63}{16}$. Hence, by completing the square, I have $y^4-\frac{1}{2}y^2+\frac{1}{16}=a+4 \therefore y^2=\frac{1}{4}\pm\sqrt{(a+4)}$, or $y=\pm\sqrt{\{\frac{1}{4}\pm\sqrt{(a+4)}\}}$; hence $x=y+z=\frac{3}{2}\pm\sqrt{\{\frac{1}{4}\pm\sqrt{(a+4)}}$.

6. Convert $\frac{1-x}{1-3x-2x^2}$, or $\frac{1}{1-2x+x^2}$, or $\frac{1+x}{1-2x+x^2}$, or

$\frac{3-x-6x^2}{1-2x-x^2+2x^3}$ and $\frac{1}{1-2ax+x^2}$, or $\frac{1+5x}{1-x-6x^2}$, or $\frac{2}{1+2x}$, or $\frac{1}{(1+x)^2}$, or $\frac{1+x+2x^2}{(1-x)^2-2x^3}$, and $\frac{1+6x+x^2}{(1-x)^4}$, each of the foregoing expressions by the method of indeterminate coefficients.

Answers: $1+2x+8x^2+28x^3+100x^4+356x^5+$&c., or $1+2x+3x^2+5x^3+8x^4+13x^5+$&c., or $1+3x+5x^2+7x^3+$&c. or $3+5x+7x^2+13x^3+23x^4+$&c., or $1+2ax+(4a^2-1)x^2+(8a^3-4a)x^3+$&c., or $1+6x+12x^2+48x^3+120x^5+$&c., or $2-4x+8x^2-16x^3+32x^4$&c., or $1+2x+3x^2+4x^3+5x^4+6x^5+$&c. or $1+3x+7x^2+13x^3+25x^4+51x^5+103x^6+$&c., or $1+8x+27x^2+64x^3+125x^4+216x^5+343x^6$.

8 Assume $\frac{1-x}{1-3x-2x^2}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5$

This, cleared of fractions, and by transposition, we have

$$\left.\begin{array}{l|l|l|l|l|l|l|l}
A+B & x+C & x^2+D & x^3+E & x^4+F & x^5+G & x^6+H & x^7 \\
+1 & x-3B & x^2-3C & x^3-3D & x^4-3E & x^5-3F & x^6-3G & x^7- \\
-1 & -2A & x^2-2B & x^3-2C & x^4-2D & x^5-2E & x^6-2F & x^7-
\end{array}\right\}=0,$$

$A-1=0$	and $A=1$	1
$B-3A+1=0$,	$\therefore B=3A-1=2$,.	$2x$
$C-3B-2A=0$;	$\therefore C=3B+2A=8$	$8x^2$
$D-3C-2B=0$;	$\therefore D=3C+2B=28$	$28x^3$
$E-3D-2C=0$;	$\therefore E=3D+2C=100$	$100x^4$
$F-3E-2D=0$; :	$\therefore F=3E+2D=356$	$=356x^5$
$G-3F-2E=0$;	$\therefore G=3F+2E=1268$	$1268x^6$

$\frac{1-n}{1-3x-2x}=1+2x+8x^2+28x^3+100x^4+356x^5+1268x^6$ Ans

9 Assume $\frac{a}{a'+b'x}=A+Bx+Cx^2+Dx^3+$&c.;

Then multiplying each side by $a'+b'x$, and transposing, we have $\left.\begin{array}{l}Aa'\\-a\end{array}\right\}x\left.\begin{array}{l}+Aa'\\+Bb'\end{array}\right\}x^2\left.\begin{array}{l}+Da'\\+Cb'\end{array}\right\}x^3+$&c. $=0$;

$$\text{Whence}\left\{\begin{array}{llll}
Aa'-a=0, & \text{therefore} & A= & \frac{a}{a'}. \\
Ba'+Ab'=0 & \ldots & B= & -\frac{b'}{a'}A \\
Ca'+Bb'=0 & \ldots & C= & -\frac{b'}{a'}B \\
Da'+Cb'=0 & \ldots & D= & -\frac{b'}{a'}C \text{ \&c.}
\end{array}\right.$$

Of the Binomial Theorem.

116. The Binomial Theorem is a general algebraical expression, or formula, by which any power, or root of a given quantity, consisting of two terms, is expanded into a series; the form of which, as it was first proposed by *Newton*, being as follows:

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}}[1+\frac{m}{n}Q+\frac{m}{n}(\frac{m-n}{2n})Q^2+\frac{m}{n}(\frac{m-n}{2n})(\frac{m-2n}{3n})Q^3+$$

or

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\frac{m-3n}{4n}DQ+$$

&c. where P is the first term of the binomial, Q the second term divided by the first, $\frac{m}{n}$ the index of the power or root, and A, B, C, &c. the terms immediately preceding those in which they are first found, including their signs $+$ or $-$.

Which theorem may be applied to any particular case, by substituting the numbers or letters in the given example, for P, Q, m and n, in either of the above formulæ, and then finding the result according to the rule.

When the index of the binomial is a whole number, the series will terminate, as observed under the article Involution before treated on; but when it is a negative or fractional number, as in the following examples, the series will proceed on ad infinitum, and will become more convergent the less the second term of the binomial is with respect to the first.*

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* This celebrated theorem, which is of the most extensive use in Algebra, and various other branches of analysis, may be otherwise expressed as follows:

(1) $(a+x)^{\frac{m}{n}} = a^{\frac{m}{n}}[1+\frac{m}{n}(\frac{x}{a})+\frac{m}{n}.\frac{m-n}{2n}(\frac{x}{a})^2+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}(\frac{x}{a})^3+$ &c.

(2) or $(a+x)^{\frac{m}{n}} =$

$$a^{\frac{m}{n}}[1+\frac{m}{n}(\frac{x}{a+x})+\frac{m}{n}.\frac{m+n}{2n}(\frac{x}{a+x})^2+\frac{m}{n}.\frac{m+n}{2n}.\frac{m+2n}{3n}(\frac{x}{a+x})^3+$$

(3) or $(a+x)^{\frac{m}{n}} =$

$$(2a)^{\frac{m}{n}}\{1-\frac{m}{n}(\frac{a-x}{a+x})+\frac{m}{n}.\frac{m+n}{2n}(\frac{a-x}{a+x})^2-\frac{m}{n}.\frac{m+n}{2n}.\frac{m+2n}{3n}(\frac{a-x}{a+x})^3+$$

And if the reciprocals of the same expressions be required, they will be

(4) $\frac{1}{(a+x)^{\frac{m}{n}}} = \frac{1}{a^{\frac{m}{n}}}\{1-\frac{m}{n}(\frac{x}{a})+\frac{m}{n}.\frac{m+n}{2n}(\frac{x}{a})^2-\frac{m}{n}.\frac{m+n}{2n}.\frac{m+2n}{3n}(\frac{x}{a})^3$

1. It is required to convert $(a^2+x)^{\frac{1}{2}}$ into an infinite series.

Here $P=a^2$, $Q=\frac{x}{a^2}$, $\frac{m}{n}=\frac{1}{2}$, or $m=1$ and $n=2$; whence

$$P^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{\frac{1}{2}}=a=A,\quad \frac{m}{n}AQ=\frac{1}{2}\times\frac{a}{1}\times\frac{x}{a^2}=\frac{x}{2a}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{x}{2a}\times\frac{x}{a^2}=-\frac{x^2}{2.4a^3}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times-\frac{x^2}{2.4a^3}\times\frac{x}{a^2}=\frac{3x^3}{2.4.6a^5}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{3x^3}{2.4.6a^5}\times\frac{x}{a^2}=-\frac{3.5x^4}{2.4.6.8a^7}=E,$$

$$\frac{m-4n}{5n}EQ=\frac{1-8}{10}\times-\frac{3.5.x^4}{2.4.6.8a^7}\times\frac{x}{a^2}=\frac{3.5.7x^5}{2.4.6.8.10a^9}=F,\ \&c.$$

$$(a^2+x)^{\frac{1}{2}}=a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}+\frac{3.5.7x^5}{2.4.6.8.10a^9}$$

Where the law of formation of the several terms of the series is sufficiently evident.

2. It is required to convert $\frac{1}{(a+b)^2}$, or its equal $(a+b)^{-2}$, into an infinite series.

Here $P=a$, $Q=\frac{b}{a}$, and $\frac{m}{n}=-\quad$, or $m=-2$ and $n=1$;

$$\text{(5) or } \frac{1}{(a+x)^{\frac{m}{n}}}=\left(\frac{1}{a}\right)^{\frac{m}{n}}\{1-\frac{m}{n}\left(\frac{x}{a+x}\right)+\frac{m}{n}.\frac{m-n}{2n}\left(\frac{x}{a+x}\right)^2-$$

$$\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}\left(\frac{x}{a+x}\right)^3+\&c.\quad (6)\ \frac{1}{(a+x)^{\frac{m}{n}}}=\left(\frac{1}{2a}\right)^{\frac{m}{n}}\{1+\frac{m}{n}\left(\frac{a-x}{a+x}\right)$$

$$+\frac{m}{n}.\frac{m+n}{2n}\left(\frac{a-x}{a+x}\right)^2+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}\left(\frac{a-x}{a+x}\right)^3+\&c.\}$$

It may here also be observed, that if m, taken singly, be made to represent any whole, or fractional number, whether positive or negative, the first of these expressions may be exhibited under the more simple form

$$(a+x)^m=a^m+ma^{m-1}x+\frac{m(m-1)}{1.2}a^{m-2}x^2+\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}a^{m-3}x^3$$

$$+\ldots\ldots\ldots+\frac{m(m-1)(m-2)\ldots\ldots\ldots\{m-(n-1)\}a^{m-n}x^n}{1\,.\,2\,.\,3\,.\,4\ldots\ldots\ldots\ldots\ldots n}$$

where the last term is called the general term of the series, because if 1, 2, 3, 4, &c. be substituted successively for n, it will give all the rest.

whence $P^{\frac{m}{n}}=(a)^{\frac{m}{n}}=a^{-2}=\frac{1}{a^2}=A$, $\frac{m}{n}AQ=-\frac{2}{1}\times\frac{1}{a^2}\times\frac{b}{a}=$
$-\frac{2b}{a^3}=B$; $\frac{m-n}{2n}BQ=\frac{-2-1}{2}\times-\frac{2b}{a^3}\times\frac{b}{a}=\frac{3b^2}{a^4}=C$,

$$\frac{m-2n}{3n}CQ=\frac{-2-2}{3}\times\frac{3b^2}{a^4}\times\frac{b}{a}=-\frac{4b^3}{a^5}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{-2-3}{4}\times-\frac{4b^3}{a^5}\times\frac{b}{a}=\frac{5b^4}{a^6}=E, \text{ \&c. \&c.}$$

Consequently $\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}-$&c.

3. It is required to convert $\frac{a^2}{(a^2-x)^{\frac{1}{2}}}$, or its equal $a^2(a^2-x)^{-\frac{1}{2}}$

into an infinite series. Here $P=a^2$, $Q=-\frac{x}{a^2}$, and $\frac{m}{n}=\frac{-1}{2}$, or $m=-1$ and $n=2$; whence $P^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{-\frac{1}{2}}=\frac{1}{a}=A$, $\frac{m}{n}AQ$
$=-\frac{1}{2}\times\frac{1}{a}\times-\frac{x}{a^2}=\frac{x}{2a^3}=B$,

$\frac{m-n}{2n}BQ=\frac{-1-2}{4}\times\frac{x}{2a^3}\times-\frac{x}{a^2}=\frac{3x^2}{2.4a^5}=C$,

$\frac{m-2n}{3n}CQ=\frac{-1-4}{6}\times\frac{3x^2}{2.4a^5}\times-\frac{x}{a^2}=\frac{3.5x^3}{2.4.6a^7}=D$,

$\frac{m-3n}{4n}DQ=\frac{-1-6}{8}\times\frac{3.5x^3}{2.4.6a^7}\times-\frac{x}{a^2}=\frac{3.5.7x^4}{2.4.6.8a^9}=E$,&c; therefore $\frac{1}{(a^2-x)^{\frac{1}{2}}}=\frac{1}{a}+\frac{1}{2}\{\frac{x}{a^3}\}+\frac{3}{2.4}\{\frac{x^2}{a^5}\}+\frac{3.5}{2.4.6}\{\frac{x^3}{a^7}\}+\frac{3.5.7}{2.4.6.8}\{\frac{x^4}{a^9}\}$
$+$&c; $\frac{a^2}{(a^2-x)^{\frac{1}{2}}}=a+\frac{1}{2}\{\frac{x}{a}\}+\frac{3}{2.4}\{\frac{x^2}{a^3}\}+\frac{3.5}{2.4.6}\{\frac{x^3}{a^5}\}+\frac{3.5.7}{2.4.6.8}\{\frac{x^4}{a^7}\}$

4. It is required to convert $\sqrt[3]{9}$, or its equal $(8+1)^{\frac{1}{3}}$, into an infinite series.

Here $P=8$, $Q=\frac{1}{8}$, and $\frac{m}{n}=\frac{1}{3}$, or $m=1$ and $n=3$; whence
$P^{\frac{m}{n}}=(8)^{\frac{m}{n}}=8^{\frac{1}{3}}=2=A$, $\frac{m}{n}AQ=\frac{1}{3}\times\frac{2}{1}\times\frac{1}{2^3}=\frac{1}{3.2^2}=B$,

$\frac{m-n}{2n}BQ=\frac{1-3}{6}\times\frac{1}{3.2^2}\times\frac{1}{2^3}=-\frac{1}{3.6.2^4}=C$,

$\frac{m-2n}{3n}CQ=\frac{1-6}{9}\times-\frac{1}{3.6.2^4}\times\frac{1}{2^3}=\frac{5}{3.6.9.2^7}=D$,

$$\frac{m-3n}{4n}\text{DQ}=\frac{1-9}{12}\times\frac{5}{3.6.9.2^7}\times\frac{1}{2^3}=-\frac{5.8}{3.6.8.12.2^{10}}=\text{E},$$

$$\frac{m-4n}{5n}\text{EQ}=\frac{1-12}{15}\times-\frac{5.8}{3.6.9.12.2^{10}}\times\frac{1}{2^3}=\frac{5.8.11}{3.6.9.12.15.2^{13}}=\text{F},$$

$$\sqrt[3]{9}=2+\frac{1}{3.2^2}-\frac{1}{3.6.2^4}+\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}+\frac{5.8.11}{3.6.9.12.15.2^{13}}$$

—&c., which series, in this instance, converges very rapidly; and the same will be the case with the example fifth next following. But when the given number falls in, or about the middle, between the two nearest integral powers, this will not take place.

In such cases, therefore, recourse must be had to the use of logarithms, the rule of double position, or some other more ready mode of computation.

5. Required to convert $\sqrt[5]{243}$, or $\sqrt{(1+1)}$, $(a\pm x)^{\frac{1}{2}}$, $(a\pm b)^{\frac{1}{3}}$, $(a-b)^{\frac{1}{4}}$, $(a+x)^{\frac{2}{3}}$, and $(1-x)^{\frac{2}{5}}$, each expression to an infinite series.

6. Convert $\dfrac{1}{(a\pm x)^{\frac{1}{2}}}$, or $\dfrac{a}{(a\pm x)^{\frac{1}{3}}}$, or $\dfrac{1}{(1+x)^{\frac{1}{5}}}$, or $\left(\dfrac{1+x}{a-x}\right)^{\frac{1}{2}}$, or their =s, $(a\pm x)^{-\frac{1}{2}}$, $a(a\pm x)^{-\frac{1}{3}}$, $(1+x)^{-\frac{1}{5}}$, and $(1+x)^{\frac{1}{2}}(a-x)^{-\frac{1}{2}}$ each into an infinite series.

5,. Here $\text{P}=243$, $\text{A}=-\frac{3}{243}=\frac{-1}{3^4}$, $\frac{m}{n}=\frac{1}{5}$, or $m=1$, and $n=5$;

$$\text{P}^{\frac{m}{n}}=243^{\frac{1}{5}}=3=\text{A},\ \frac{m}{n}\text{AQ}=\tfrac{1}{5}\times3\times-\frac{1}{3^4}=\frac{-1}{5.3^3}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{-4}{10}\times\frac{-1}{5.3^3}\times-\frac{1}{3^4}=\frac{-4}{5.10.3^7}=\text{C},$$

$$\frac{m-2n}{3n}\text{CQ}=\frac{-9}{15}\times\frac{-4}{5.10.3^7}\times\frac{-1}{3^4}=\frac{-4.9}{5.10.15.3^{11}}=\text{D};$$

$$\frac{m-3n}{4n}\text{DQ}=\frac{-14}{20}\times\frac{-4.9}{5.10.15.3^{11}}\times\frac{-1}{3^4}=\frac{-4.9.14}{5.10.15.20.3^{15}}=\text{E};$$

$$\sqrt[5]{240}=3-\frac{1}{5\cdot3^3}-\frac{4}{5\cdot10.3^7}-\frac{4.9}{5.10.15.3^{11}}-\frac{4.9.14}{5.10.15.20.3^{15}}-\text{\&c.}$$

5,′. Here $\text{A}=a$, $\text{Q}=\pm\frac{x}{a}$, $m=1$, $n=2$, $\text{P}^{\frac{m}{n}}=a^{\frac{1}{2}}=\text{A}$,

$$\frac{m}{n}\text{AQ}=\tfrac{1}{2}\times a^{\frac{1}{2}}\times\pm\frac{x}{a}=a^{\frac{1}{2}}\times\pm\frac{x}{2a}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{-1}{4}\times a^{\frac{1}{2}}\times\pm\frac{x}{2a}\times\pm\frac{x}{a}=a^{\frac{1}{2}}\times-\frac{x^2}{2.4\,a^2}=\text{C},$$

$\frac{m-2n}{3n}CQ=\frac{-3}{6}\times a^{\frac{1}{2}}\times-\frac{x^2}{2.4a^2}\times\pm\frac{x}{a}=a^{\frac{1}{2}}\times\pm\frac{3x^3}{2.4.6a^3}=D,$

$\frac{m-3n}{4n}DQ=-\frac{5}{8}\times a^{\frac{1}{2}}\times\pm\frac{3x^3}{2.4.6a^3}\times\pm\frac{x}{a}=a^{\frac{1}{2}}\times-\frac{3.5.x^4}{2.4.68a^4}=E.$

$\therefore\sqrt{(a\pm x)}=a^{\frac{1}{2}}\times(1\pm\frac{x}{2a}-\frac{x^2}{2.4a^2}\pm\frac{3x^3}{2.4.6a^3}-\frac{3.5.x^4}{2.4.6.8a^4}\pm\&c.$

(5). Here $P=a$, $q=\pm\frac{b}{a}$, $m=1$, and $n=3$. $P^{\frac{1}{3}}=a^{\frac{1}{3}}=A$,

$\frac{m}{n}AQ=\frac{1}{3}a^{\frac{1}{3}}\times\pm\frac{b}{a}=a^{\frac{1}{3}}\times\pm\frac{b}{3.a}=B,$

$\frac{m-n}{2n}BQ=\frac{-2}{6}\times a^{\frac{1}{3}}\times\pm\frac{b}{3.a}\times\pm\frac{b}{a}=a^{\frac{1}{3}}\times\frac{-2b^2}{3.6a^2}=C,$

$\frac{m-2n}{3n}CQ=\frac{-5}{9}\times a^{\frac{1}{3}}\times\frac{-2b^2}{3.6.a^2}\times\pm\frac{b}{a}=a^{\frac{1}{3}}\times\pm\frac{2.5.b^3}{3.6.9a^3}=D,$

$\frac{m-3n}{4n}DQ=\frac{-8}{12}\times a^{\frac{1}{3}}\times\frac{2.5.b^3}{3.6.9.a^3}\times\pm\frac{b}{a}=\frac{a^{\frac{1}{3}}\times-2.5.8b^4}{3.6.9.12a^4}=E$

$\therefore(a\pm b)^{\frac{1}{3}}=a^{\frac{1}{3}}\times(1\pm\frac{b}{3a}-\frac{2b^2}{3.6a^2}\pm\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4}\pm\&c.$

(6). Here $P=a$, $q=-\frac{b}{a}$, $m=1$, $n=4$. $P^{\frac{m}{n}}=a^{\frac{1}{4}}=A$,

$\frac{m}{n}AQ=\frac{1}{4}\times a^{\frac{1}{4}}\times-\frac{b}{a}=a^{\frac{1}{4}}\times-\frac{b}{4a}=B,$

$\frac{m-n}{2n}BQ=\frac{-3}{8}\times a^{\frac{1}{4}}\times-\frac{b}{4a}\times-\frac{b}{a}=a^{\frac{1}{4}}\times\frac{-3b^2}{4.8a^2}=C,$

$\frac{m-2n}{3n}CQ=\frac{-7}{12}\times a^{\frac{1}{4}}\times\frac{-3b^2}{4.8a^2}\times-\frac{b}{a}=a^{\frac{1}{4}}\times-\frac{3.7b^3}{4.8.12a^3}=D,$

$\frac{m-3n}{4n}DQ=\frac{-11}{16}\times a^{\frac{1}{4}}\times-\frac{3.7b^3}{4.8.12a^3}\times-\frac{b}{a}=\frac{a^{\frac{1}{4}}\times-3.7.11b^4}{4.8.12.16a^4}=E$

$\therefore(a-b)^{\frac{1}{4}}=a^{\frac{1}{4}}\times(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-\&c.$

(7). Here $P=a$, $Q=\frac{x}{a}$, $m=2$, $n=3$. $P^{\frac{m}{n}}=a^{\frac{2}{3}}=A$,

$\frac{m}{n}AQ=\frac{2}{3}a^{\frac{2}{3}}\times\frac{x}{a}=a^{\frac{2}{3}}\times\frac{2x}{3a}=B,$

$\frac{m-n}{2n}BQ=\frac{-1}{6}\times a^{\frac{2}{3}}\times\frac{2x}{3x}\times\frac{x}{a}=a^{\frac{2}{3}}\times-\frac{x^2}{9a^2}=C,$

$\frac{m-2n}{3n}CQ=-\frac{4}{9}\times a^{\frac{2}{3}}\times\frac{-x^2}{9a^2}\times\frac{x}{a}=a^{\frac{2}{3}}\times\frac{4x^3}{9^2a^3}=D,$

$\frac{m-3n}{4n}DQ=-\frac{7}{12}\times a^{\frac{2}{3}}\times\frac{4x^3}{9^2a^3}\times\frac{x}{a}=a^{\frac{2}{3}}\times\frac{-4.7x^4}{9^2.12a^4}=E,$

$\therefore(a+x)^{\frac{2}{3}}=a^{\frac{2}{3}}\left\{1+\frac{2x}{3a}-\frac{x^2}{9a^2}+\frac{4x^3}{9^2a^3}-\frac{4.7x^4}{9^2.12a^4}+\&c.\right.$

6′. Here P=1, Q=$-x$, $m=2$, $n=5$.

$P^{\frac{m}{n}}=1=A,\ \frac{m}{n}AQ=\frac{2}{5}\times-x=\frac{-2x}{5}=B,$

$\frac{m-n}{2n}BQ=\frac{-3}{10}\times-\frac{2x}{5}\times-x=\frac{-2.3x^2}{5.10}=C,$

$\frac{m-2n}{3n}CQ=\frac{-8}{15}\times-\frac{2.3x^2}{5.10}-x=\frac{-2.3.8x^3}{5.10.15}=D,$

$\frac{m-3n}{4n}DQ=\frac{-13}{20}\times-\frac{2.3.8x^3}{5.10.15}\times-x=\frac{-2.3.8.13x^4}{5.10.15.20}=E,$

$\therefore(1-a)^{\frac{2}{5}}=1-\frac{2x}{5}-\frac{2.3x^2}{5.10}-\frac{2.3.8x^3}{5.10.15}-\frac{2.3.8.13x^4}{5.10.15.20}-\&c.$

6″. Here P=a, Q=$\pm\frac{x}{a}$, $m=-1$, $n=2$. $P^{\frac{m}{n}}=a^{-\frac{1}{2}}=A$

$\frac{m}{n}AQ=\frac{-1}{2}a^{-\frac{1}{2}}\times\pm\frac{x}{a}=a^{-\frac{1}{2}}\times\mp\frac{x}{2.a}=B,$

$\frac{m-n}{2n}BQ=\frac{-3}{4}\times a^{-\frac{1}{2}}\times\mp\frac{x}{2a}\times\pm\frac{x^2}{a}=a^{-\frac{1}{2}}\times\frac{3x^2}{2.4.a^2}=C,$

$\frac{m-2n}{3n}CQ=\frac{-5}{6}\times a^{-\frac{1}{2}}\times\frac{3x^2}{2.4.a^2}\times\pm\frac{x}{a}=a^{-\frac{1}{2}}\times\mp\frac{3.5x^3}{2.4.6a^3}=$

$\frac{m-3n}{4n}DQ=\frac{-7}{8}\times a^{-\frac{1}{2}}\mp\frac{3.5x^3}{2.4.6a^3}\times\pm\frac{x}{a}=a^{-\frac{1}{2}}\times\frac{3.5.7x^4}{2.4.6.8a^4}=E;$

$\therefore(a\pm x)^{-\frac{1}{2}}=a^{-\frac{1}{2}}(1\mp\frac{x}{2.a}+\frac{3x^2}{2.4a^2}\mp\frac{3.5x^3}{2.4.6a^3}+\frac{3.5.7x^4}{2.4.6.8a^4}\mp\&c.$

6‴. Here P=a, Q=$\pm\frac{x}{a}$, $m=-1$, $n=3$. $P^{\frac{m}{n}}=a^{-\frac{1}{3}}=A,$

$\frac{m}{n}AQ=\frac{-1}{3}a^{-\frac{1}{3}}\times\pm\frac{x}{a}=a^{-\frac{1}{3}}\times\mp\frac{x}{3.a}=B,$

$\frac{m-n}{2n}BQ=\frac{-4}{6}\times a^{-\frac{1}{3}}\times\pm\frac{x}{3a}\times\pm\frac{x}{a}=a^{-\frac{1}{3}}\times\frac{4x^2}{3.6a^2}=C,$

$\frac{m-n}{3n}CQ=\frac{-7}{9}\times a^{-\frac{1}{3}}\times\frac{4x^2}{3.6a^2}\times\pm\frac{x}{a}=a^{-\frac{1}{3}}\times\mp\frac{4.7x^3}{3.6.9a^3}=D$

$\frac{m-3n}{4n}DQ=\frac{10}{12}\times a^{-\frac{1}{3}}\mp\frac{4.7x^3}{3.6.9a^3}\times\pm\frac{x}{a}=a^{-\frac{1}{3}}\times\frac{4.7.10x^4}{3.6.9.12a^4}=E;$

$\therefore(a\pm x)^{-\frac{1}{3}}=a^{-\frac{1}{3}}\times(1\mp\frac{x}{3.a}+\frac{4x^2}{3.6a^2}\mp\frac{4.7x^3}{3.6.9a^3}+\frac{4.7.10x^4}{3.6.9.12a^4}\pm\&c$

6⁗. Here P=1, Q=x, $m=-1$, $n=5$. See Index for last.

$P^{\frac{m}{n}}=1=A,\ \frac{m}{n}AQ=-\frac{1}{5}\times x=-\frac{x}{5}=B,$

$\frac{m-n}{2n}BQ=-\frac{6}{10}\times-\frac{x}{5}\times x=\frac{6x^2}{5.10}=C,$

$\frac{m-2n}{3n}CQ=-\frac{11}{15}\times\frac{6x^2}{5.10}\times x=\frac{-6.11x^3}{5.10.15}=D,$

$\frac{m-3n}{4n}DQ=-\frac{16}{20}\times\frac{-6.11x^3}{5.10.15}\times x=\frac{6.11.16x^4}{5.10.15.20}=E;$

$\therefore(1+x)^{-\frac{1}{5}}=1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11x^3}{5.10.15}+\frac{6.11.16x^4}{5.10.15.20}-\&c.$

We have seen that $(a\pm x)^2=a^2\pm2ax+x^2$; $(a\pm x)^3=a^3\pm3a^2x+3ax^2\pm x^3$; $(a\pm x)^4=a^4\pm4a^3x+6a^2x^2\pm4ax^3+x^4$, and in general, $(a\pm x)^m=a^m\pm ma^{m-1}x+\frac{m\cdot(m-1)}{1.2}a^{m-2}x^2\pm\frac{m\cdot(m-1)\cdot(m-2)}{1\cdot2\cdot3}a^{m-3}x^3+\&c.$, where m is any number whatever, whole or fractional, positive or negative.

This theorem will, therefore, not only enable us to find any given power of a binomial, but will furnish a ready way of extracting any proposed root of the same. Any binomial, as $(a+x)^m$ may be expanded into a series of the form $A+Bx+Cx^2+\&c.$ whatever be the value of m, where the first term is independent of x, and the following terms are whole positive increasing powers of x, with coefficients B, C, &c. independent of x.

Let m be any whole number, and let $(1+x)^{m-1}=1+Px+Qx^2+\&c.$; $\therefore\ (1+x)^m=(1+Px+Qx^2+\&c.)\cdot(1+x)$.

$$\left.\begin{array}{l}=1+Px+Qx^2+\&c.\\ \quad+x+Px^2+\&c.\end{array}\right\}=1+(P+1)\cdot x+(P+Q)\cdot x^2+\&c.$$

which shews that, by the introduction of another factor, the coefficient of the second term is increased by 1. Now $(1+x)^2=1+2x+\&c$; $(1+x)^3=1+3x+$ $\therefore$ generally $(1+x)^m=1+mx+\&c.$ & $(1+x)^{-m}$, or by actual division. $\frac{1}{(1+x)^m}=1-mx+\&c.$ Now since $a+x=a\cdot\left(1+\frac{x}{a}\right)$, $\therefore(a+x)^m=a^m\left(1+\frac{x}{a}\right)^m=a^m\left(1+m\frac{x}{a}+\&c\right)$

Hence, whatever be the value of m, the two first terms of the expansion $(a+x)^m$ are $a^m+ma^{m-1}x$. Assume $(a+x)^m=a^m+ma^{m-1}x+Ax^2+Bx^3+\&c.$, where A, B, &c. do not involve x, and remain to be determined. On this supposition

$$\{a+(x+z)\}=a^m+ma^{m-1}(x+z)+A(x+z)^2+B(x+z)^3+\&c.$$

$$\left.\begin{array}{l}=a^m+ma^{m-1}x+Ax^2+Bx^3+\&c.\\ \quad+ma^{m-1}z+2Axz+3Bx^2z+\&c.\\ \quad\quad+\&c.\end{array}\right\}\cdots\cdots\cdots p$$

Now $\{(a+x\}+z) = (a+x)^m + m(a+x)^{m-1}z + Az^2 + Bz^3 + \&c. =$

$$\left.\begin{array}{l} = a^m + ma^{m-1}x + Ax^2 + Bx^3 + \&c. \\ \quad + mz\{a^{m-1} + (m-1)\cdot a^{m-2}x + A^1x^2 + B^1x^3 + \&c.\} \\ \quad\quad + \&c. \end{array}\right\} \ldots\ldots q,$$

Where a^1, B^1, &c. are the values which A, B, &c. assume when $(m-1)$ is put for m. Now these two series p, q, expressing the values of $(a+x+z)^m$ must be identical; therefore, by equating coefficients $2A = m\cdot(m-1)^{m-2}$, $\therefore A = \frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}$; also $A^1 =$ $\frac{(m-1)\cdot(m-2)}{1\cdot 2}a^{m-3}$. Hence $2B = mA^1 = \frac{m\cdot(m-1)\cdot(m-2)}{1\cdot 2}a^{m-3}$. $\therefore B = \frac{m\cdot(m-1)\cdot(m-2)}{1\cdot 2\cdot 3}a^{m-3}$; &c. $\therefore (a+x)^m = a^m + ma^{m-1}x +$ $\frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}x^2 + \frac{m\cdot(m-1)(m-2)}{1.2.3}a^{m-3}x^3 + \&c.$

In the same manner it will appear that $(a-x)^m = a^m - ma^{m-1}x$ $+ \frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}x^2 - \frac{m\cdot(m-1)\cdot(m-2)}{1\cdot 2\cdot 3}a^{m-3}x^3 + \&c. \ldots \pm x^m$.

And this formula will hold good whatever be the value of m; whether it be integral or fractional, positive or negative.

Extract the cube root of 750, or of 9^3+21. Here $e=9$, $m=3$, and $x=21$ $\therefore \sqrt[3]{(750)} = 9\left\{1 + \frac{21}{2250} + \frac{4(21)^2}{18(750)^2} + \&c.\right\} =$ 9·085592, .

If e^m be large compared with x, these expressions converge with great rapidity; and in general it is not necessary to take more than four terms of the series.

Since $(a+x)^m = a^m + ma^{m-1}x + \frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}x^2$ &c., and $(a-x)^m = a^m - ma^{m-1}x + \frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}x^2 - \&c.$ $\therefore$ by addition, $\frac{(a+x)^m + (a-x)^m}{2} = a^m + \frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}x^2 + \&c. =$ sum of the odd terms of the expansion of $(a+x)^m$. By subtraction $\frac{(a+x)^m - (a-x)^m}{2} = ma^{m-1}x + \frac{m\cdot(m-1)\cdot(m-2)}{1\cdot 2\cdot 3}a^{m-3}x^3 + \&c. =$ sum of the even terms. If $a=x=1$, $(1+1)^m$ or $2^m = 1 + m + \frac{m\cdot(m-1)}{2}$ $+ \&c. =$ sum of the coefficients of the expansion $(a+x)^m$. The sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms: For if $a=x=1$,

$(a-x)^m = 0$; $\therefore \frac{(a+x)^m + (a-x)^m}{2} = \frac{(a+x)^m - (a-x)^m}{2} = 2^{m-1}$.

A trinomial $(a+b+x)$ may be raised to any power, thus:
$a+b+x=a+(b+x)$; $\therefore$ $(a+b+x)^m=(a+(b+x))^m=a^m+ma^{m-1}$
$(b+x)+\frac{m\cdot(m-1)}{1\cdot2}a^{m-2}(b+x)^2+\frac{m\cdot(m-1)\cdot(m-2)}{1\cdot2\cdot3}a^{m-3}(b+x)^3+\text{\&c.}$
by considering $(b+x)$ as one quantity. And in general

$$(a+b+c+d+\text{\&c.})^m=a^m+ma^{m-1}b+ma^{m-1}c+ma^{m-1}d+\text{\&c.}$$

$+\frac{m\cdot(m-1)}{1\cdot2}a^{m-2}b^2+\frac{m\cdot(m-1)}{1\cdot2}a^{m-2}bc+\text{\&c.}+m\cdot\frac{(m-1)\cdot(m-2)}{1\cdot2\cdot3}a^{m-3}b^3+$
&c., which is De Moivre's Theorem.

On the Multiplication and Division of Series.

117. Any two series may be multiplied or divided by each other, by substituting the numbers or letters of the given example, in the place of their corresponding terms, in the following general formulæ, and then finding the result accordingly.*

$$\begin{array}{l} A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\text{\&c.} \\ \underline{B_0+B_1x+B_2x^2+B_3x^3+B_4x^4+\text{\&c.}} \end{array}$$

$$\text{Product}=\left|\begin{array}{l|l|l|l|l} A_0B_0+A_1B_0 & x+A_2B_0 & x^2+A_3B_0 & x^3+A_4B_0 & x^4+\text{\&c.} \\ \quad\;\; +A_0B_1 & \;\; +A_1B_1 & \;\; +A_2B_1 & \;\; +A_3B_1 & \\ & \;\; +A_0B_2 & \;\; +A_1B_2 & \;\; +A_2B_2 & \\ & & \;\; +A_0B_3 & \;\; +A_1B_3 & \\ & & & \;\; +A_0B_4 & \end{array}\right.$$

And if $B_0=A_0$, $B_1=A_1$. $B_2=A_2$, &c. the formula for the square of the upper of the above series, or
$(A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\text{\&c.})^2$ will evidently be in that case as follows:

$$\text{Square}=\left|\begin{array}{l|l|l|l|l} A_0^2+A_1A_0 & x+A_2A_0 & x^2+A_3A_0 & x^3+A_4A_0 & x^4+\text{\&c.} \\ \quad\;\; +A_0A_1 & \;\; +A_1A_1 & \;\; +A_2A_1 & \;\; +A_3A_1 & \\ & \;\; +A_0A_2 & \;\; +A_1A_2 & \;\; +A_2A_2 & \\ & & \;\; +A_0A_3 & \;\; +A_1A_3 & \\ & & & \;\; +A_0A_4 & \end{array}\right.$$

$$\text{Also, }\frac{A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\text{\&c.}}{1+B_1x+B_2x^2+B_3x^3+B_4x^4+\text{\&c.}}$$

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* In the mode of notation here followed, it is to be observed, that A_0, A_1, A_2, &c. are the coefficients of the several powers of x, so taken, that the inferior figures 0, 1, 2, 3, &c. by which they are denoted shall always agree with the indices of the unknown quantities that are attached to them; by which means the law of the continuation of the terms is, in all cases, rendered obvious, and a conformity between the coefficients and powers of x constantly preserved.

$$\text{Quotient} = \left|\begin{array}{l|l|l|l|l} A_0 + A_1 & x + A_2 & x^2 + A_3 & x^3 + A_4 & x^4 + \&c. \\ \quad -B_1C_0 & -B_2C_0 & -B_3C_0 & -B_4C_0 & \\ & -B_1C_1 & -B_2C_1 & -B_3C_1 & \\ & & -B_1C_2 & -B_2C_2 & \\ & & & -B_1C_3 & \end{array}\right.$$

And, if A_0, in the upper series, be taken $= 1$, and the rest of the terms be put $= 0$, or made to vanish, we shall have for the reciprocal of the lower series, $\dfrac{1}{1+B_1x+B_2x^2+B_3x^3+B_4x^4+\&c.}$

$$\text{Reciprocal} = \left|\begin{array}{l|l|l|l|l} 1-B_1C_0 & x-B_2C_0 & x^2-B_3C_0 & x^3-B_4C_0 & x^4-\&c. \\ & -B_1C_1 & -B_2C_1 & -B_3C_1 & \\ & & -B_1C_2 & -B_2C_2 & \\ & & & -B_1C_3 & \end{array}\right.$$

where it is to be observed, that in each of the two latter series C_0 denotes the first term, C_1 the coefficient of the second term, C_2 that of the third, and so on, according to the order C_3, C_4, &c.*

1. Find the product of the two series $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+$ &c. and $1-\frac{1}{3}x+\frac{1}{5}x^2-\frac{1}{7}x^3+\frac{1}{9}x^4-\frac{1}{11}x^5+$&c.

Here $A_0=1$, $A_1=\frac{1}{2}$, $A_2=\frac{1}{3}$, &c. and $B_0=1$, $B_1=-\frac{1}{3}$, $B_2=\frac{1}{5}$, &c.

Whence $A_0B_0=1$; $A_1B_0+A_0B_1=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$;

$A_2B_0+A_1B_1+A_0B_2=\frac{1}{3}-\frac{1}{6}+\frac{1}{5}=\frac{11}{30}$

$A_3B_0+A_2B_1+A_1B_2+A_0B_3=\frac{1}{4}-\frac{1}{9}+\frac{1}{10}-\frac{1}{7}=\frac{121}{1260}$

$A_4B_0+A_3B_1+A_2B_2+A_1B_3+A_0B_4=\frac{1}{5}-\frac{1}{12}+\frac{1}{15}-\frac{1}{14}+\frac{1}{9}=\frac{281}{1260}$, &c.

$\therefore 1+\frac{1}{6}x+\frac{11}{30}x^2+\frac{121}{1260}x^3+\frac{281}{1260}x^4+$ &c., the product required.

2. Divide the series $1+\frac{1}{6}x+\frac{11}{30}x^2+\frac{121}{1260}x^3+\frac{281}{1260}x^4+$ &c. by the series $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+$ &c.

Here $A_0=1$, $A_1=\frac{1}{6}$, $A_2=\frac{11}{30}$, &c.; and $B_1=\frac{1}{2}$. $B_2=\frac{1}{3}$, $B_3=\frac{1}{4}$, &c.

Whence $A_0=1=C_0$; $A_1-B_1C_0=\frac{1}{6}-\frac{1}{2}=-\frac{1}{3}=C_1$;

$A_2-B_2C_0-B_1C_1=\frac{11}{30}-\frac{1}{3}+\frac{1}{6}=\frac{1}{5}=C_2$;

$A_3-B_3C_0-B_2C_1-B_1C_2=\frac{121}{1260}-\frac{1}{4}+\frac{1}{9}-\frac{1}{10}=-\frac{1}{7}=C_3$,

$A_4-B_4C_0-B_3C_1-B_2C_2-B_1C_3=\frac{281}{1260}-\frac{1}{5}+\frac{1}{12}-\frac{1}{15}+\frac{1}{14}=\frac{1}{9}=C_4$;

&c. Therefore $1-\frac{1}{3}x+\frac{1}{5}x^2-\frac{1}{7}x^3+\frac{1}{9}x^4-$ &c. $=$ quotient.

1. Multiply $1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\frac{1}{16}x^4+\frac{1}{32}x^5+$ &c. by $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+$ &c. Ans. $1+x+\frac{5}{6}x^2+\frac{2}{3}x^3+\frac{8}{15}x^4+\frac{13}{30}x^5+$ &c.

2. Find the square of the series $1-\frac{1}{2}x+\frac{2}{3}x^2-\frac{3}{4}x^3+\frac{4}{5}x^4-\frac{5}{6}x^5+$

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* The first term of the denominators of the two last examples given above are, for the sake of simplicity, taken, equal to unity; to which form any series can always be reduced; as is evident from the following instance,

$$a+bx+cx^2+dx^3+ex^4+\&c. = a\left(1+\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3+\frac{e}{a}x^4+\right)\&c.$$

which method may also be used in other cases, whenever it will render a formula more perspicuous, or any process of investigation less embarrassing.

$\frac{5}{6}x^5$—&c. Ans. $1-x+\frac{13}{12}x^2-\frac{13}{6}x^3+\frac{503}{180}x^4-\frac{53}{15}x^5+$&c.

3. Divide $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+$&c. by $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-$&c. Ans. $1+x+\frac{1}{2}x^2+\frac{5}{12}x^3+\frac{7}{24}x^4+\frac{191}{720}x^5+$&c.

4. Find the reciprocal of $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+\frac{1}{6}x^5+$&c.

Ans. $1-\frac{1}{2}x-\frac{1}{12}x^2-\frac{1}{24}x^3-\frac{19}{720}x^3-\frac{27}{1440}x^5-$&c.

On the Multinomial Theorem.

118. The multinomial theorem is a general expression, or formula, for determining any power or root of a given quantity, consisting of any number of terms, the form of which is as follows:

$$(A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\ldots\ldots+A_nx^n)^m=A_0{}^m+$$

$$\left.\begin{array}{l} mA_1B_0 \end{array}\right|\frac{x}{A_0}\ \left.\begin{array}{l} +2mA_2B_0 \\ +(m-1)A_1B_1 \end{array}\right|\frac{x^2}{2A_0}\ \left.\begin{array}{l} +3mA_3B_0 \\ +(2m-1)A_2B_1 \\ +(\ m-2)A_1B_2 \end{array}\right|\frac{x^3}{3A_0}\ \left.\begin{array}{l} +4mA_4B_0 \\ +(3m-1)A_3B_1 \\ +(2m-2)A_2B_2 \\ +(\ m-3)A_1B_3 \end{array}\right|\frac{x^4}{4A_0}\text{ \&c}$$

where $B_0=A_0{}^m$; also B_1, B_2, B_3, &c. are the coefficients of the terms immediately preceding those in which they first appear.

Or the same series may be exhibited in a more convenient and practical form, as follows;

$$(1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\text{\&c.})^m=1$$

$$+\frac{mA}{1}x+\frac{2mB+(m-1)Aa}{2}x^2+\frac{3mC+(2m-1)Ba+(m-2)Ab}{3}x^3$$

$$+\frac{4mD+(3m-1)Ca+(2m-2)Bb+(m-3)Ac}{4}x^4$$

$$\frac{5mE+(4m-1)Da+(3m-2)Cb+(2m-3)Bc+(m-4)Ad}{5}x^5$$

&c. where a, b, c, &c. are the coefficients of the terms immediately preceding those in which they are first used.

And the manner of applying the theorem to any particular case is by substituting the numbers, or letters, in the given examples, for A_0, A_1, A_2, &c. and m, as in the binomial theorem before given.

1. Find the cube of the series $1+x+x^2+x^3+x^4+x^5+$&c.

Here $A_0=1$, $A_1=1$, $A_2=1$, &c. and $m=3$;

whence $A_0{}^m=1^m=1=B_0$; $mA_1B_0=3\times1\times1=3=B_1$;

$$\frac{2mA_2B_0+(m-1)A_1B_1}{2A_0}=\frac{6+2\times3}{2}=6=B_2.$$

$$\frac{3mA_3B_0+(2m-1)A_2B_1+(m-2)A_1B_2}{3A_0}=\frac{9+15+6}{3}=10=B_3;$$

$$\frac{4mA_4B_0+(3m-1)A_3B_1+(2m-2)A_2B_2+(m-3)A_1B_3}{4A_0}=15=B_4\text{ \&c.}$$

Therefore $1+3x+6x^2+10x^3+15x^4+21x^5+$&c. = required cube.

2. Find the cube root of the series $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+\frac{1}{6}x^5+$ &c. Here $A_0=1, A_1=\frac{1}{2}, A_2=\frac{1}{3}, A_3=\frac{1}{4}$, &c.; and $m=\frac{1}{3}$, $\therefore\ A_0^m = 1^m = 1 = B_0$; $mA_1B_0 = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{6} = B_1$;

$$\frac{2mA_2B_0+(m-1)A_1B_1}{2A_0} = \frac{1}{2}\left(\frac{2}{9}-\frac{1}{18}\right) = \frac{1}{12} = B_2$$

$$\frac{3mA_3B_0+(2m-1)A_2B_1+(m-2)A_1B_2}{3A_0} = \frac{1}{3}\left(\frac{1}{4}-\frac{1}{54}-\frac{1}{72}\right) = \frac{35}{648} = B_3,$$

$$\frac{4mA_4B_0+(2m-1)A_2B_1+(2m-2)A_2B_2+(m-3)A_1B_3}{4A_0} = \frac{433}{1944} = B_4,$$

&c.; therefore $1+\frac{1}{6}x-\frac{1}{12}x^2+\frac{35}{648}y^3-\frac{433}{1944}x^4+$ &c. Ans.

3. What is the square root of the series $1+x+x^2+x^3+$ &c ?

Here a, b, c, &c. are each $=1$; also $Q=\frac{1}{2}$; therefore

$a^n=1=A$, $Qb=\frac{1}{2}=B$, $\frac{(n-1)Ab}{2a}+Qc=\frac{3}{8}=C$,

$\frac{(n-2)Bb+(2n-1)}{3a}+Qd=\frac{5}{16}=D$,

$\frac{(n-3)Cb+(2n-2)Bc+(3n-1)Ad}{4a}+Qe=\frac{35}{128}=E$; &c.

$\therefore (1+x+x^2+x^3+\text{\&c.})^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{124}x^4+$ &c.

4. Find the square of the series $1-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9-$ &c.

5. Find the square of the series $1+x+x^2+x^3+x^4+$ &c.

Ans. $1-\frac{2}{3}x^3+\frac{23}{45}x^5-\frac{44}{105}x^7+\frac{12}{315}x^9$ &c. Ans. $1+2x+3x^2+4x^3+5x^4$

6. Find the square root of the series $1+x+x^2+x^3+x^4+x^5+$ &c.

7. Find the square of the series $y-y^3+y^5-y^7+y^9-$ &c.

Ans. $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4$. Ans. $y^2-2y^4+3x^6-4y^8+5y^{10}$.

8. Find the cube of the series $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-$ &c.

9. Find the 5th power of the series $1+\frac{3}{x}+\frac{5}{x^2}+\frac{7}{x^3}+\frac{9}{x^4}+\frac{11}{x^5}$

10. Find the square root of the series $\frac{1}{1-\frac{1}{2}x^2+\frac{1}{4}x^4-\frac{1}{6}x^6+\frac{1}{8}x^8-\text{\&c.}}$

Of the Reversion of Series.

119. Reversion of Series is the finding the value of the root, or unknown quantity contained in an infinite series, by means of another infinite series, in which some other unknown quantity only is contained; the method of doing which is as follows:

Rule I. If the series consisis of all the powers of the unknown quantity, as $ax+bx^2+cx^3+dx^4+ex^5+\text{\&c}=y$, substitute the particular values of the coefficients in the given example, for a, b, c, d, &c. in the following formula, and the result will bethe value of x, as required. $x=\frac{y}{a}-b\frac{y^2}{a^3}+(2b^2-ac)\frac{y^3}{a^5}$ $-(5b^3-5abc+a^2d)\frac{y^4}{a^7}+(14b^4-21ab^2c+3a^2c^2+6a^2bd-a^3e)\frac{y^5}{a^9}$ $-(42b^5-84ab^3c+28a^2bc^2+28a^2b^2d-7a^3cd-7a^3be+a^4f)\frac{y^6}{a^{11}}+$ &c.

where it is to be observed, that if y be a large number, the series will often diverge, and $\therefore$ in that case will be of no practical use.

1. Given $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+$ &c. $=z$, to find the value of x in terms of z. Here $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{3}$, $d=\frac{1}{4}$, $e=\frac{1}{5}$, &c. and $y=z$, whence $\frac{1}{a}=1$, $-b=-\frac{1}{2}$, $+(2b^2-ac)=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$
$-(5b^3-5abc+a^2d)=-(\frac{5}{8}-\frac{5}{6}+\frac{1}{4})=-\frac{1}{24}$
$+(14b^4-21ab^2c+3a^2c^2+6a^2bd-a^3e)=\frac{1}{120}$
$-(42b^5-84ab^3c+28a^2bc^2+28a^2b^2d-7a^3cd-7a^3be+a^4f)$
$=-(\frac{21}{16}-\frac{7}{2}+\frac{14}{9}+\frac{7}{4}-\frac{7}{12}-\frac{7}{10}+\frac{1}{6})=-\frac{1}{720}$, &c. &c. Therefore

Ans. $x=z-\frac{1}{2}z^2+\frac{1}{6}z^3-\frac{1}{24}x^4+\frac{1}{120}z^5-\frac{1}{720}z^6+$ &c.

2. Given $x-x^2+x^3-x^4+x^5-x^6+x^7-$&c. $=z$ to find the value of x in terms of z. Ans. $x=z+z^2+z^3+z^4+z^5+z^6+$&c.

3. Given $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-\frac{1}{6}x^6+\frac{1}{7}x^7-$ &c. $=z$ to find x in terms of z. Ans. $x=z+\frac{1}{2}z^2+\frac{1}{2.3}z^3+\frac{1}{2.3.4}z^4+\frac{1}{2.3.4.5}z^5$

4. Given $x-\frac{1}{2}x^2+\frac{1}{4}x^3-\frac{1}{8}x^4+\frac{1}{16}x^5-$&c.$=y$, to find the value of x in terms of y. Ans. $x=y+\frac{1}{2}y^2+\frac{1}{4}y^3+\frac{1}{8}y^4+\frac{1}{16}y^5+\frac{1}{32}y^6+$&c.

5. Given $y+\frac{y^2}{r}+\frac{2y^3}{3r^2}+\frac{2y^4}{3r^3}+\frac{2y^5}{3r^4}+\frac{32y^6}{45r^5}+\frac{244y^7}{315r^6}$&c. $=z$, to find the value of y in terms of z.

$$\text{Ans. } y=z-\frac{z^2}{r}+\frac{4z^3}{3r^2}-\frac{7z^4}{3r^3}+\frac{14z^5}{3r^4}-\frac{452z^6}{45r^5}+\frac{7148z^7}{315r^6}-\text{\&c.}$$

RULE II. If the series consist of the odd powers of the unknown quantity only, as $ax+bx^3+cx^5+dx^7+ex^9+$&c.$=y$, substitute the particular values of the coefficients, in any given example, for a, b, c, d, &c. in the following formula, and the result will be the value of x, as required. $x=\frac{y}{a}-b\frac{y^3}{a^4}+(3b^2-ac)\frac{y^5}{a^7}-(12b^3+a^2d-8abc)\frac{y^7}{a^{10}}+(55b^4-55ab^2c+10a^2bd+5a^2c^2-a^3e)\frac{y^9}{a^{13}}-$&c.

Where it is to be observed, that in this case, as well as in the former, the given and reverted series must be both of the same form, or otherwise they are not convertible into each other.

Given $x-\frac{1}{2.3}x^3+\frac{1}{2.3.4.5}x^5-\frac{1}{2.3.4.5.6.7}x^7+\frac{1}{2.3.4.5.6.7.8.9}x^9$ $-$ &c. $=z$, to find the value of x in terms of z. Here $a=1$, $b=-\frac{1}{6}$, $c=\frac{1}{120}$, $d=-\frac{1}{5040}$, $e=\frac{1}{362880}$, $y=z$.
$\therefore \frac{1}{a}=1$, $-b=\frac{1}{6}=\frac{1}{2.3}$; $(3b^2-ac)=(\frac{1}{12}-\frac{1}{120})=\frac{3}{40}=\frac{3}{2.4.5}$;
$-(12b^3+a^2d-8abc)=-(-\frac{1}{18}-\frac{1}{5040}+\frac{1}{90})=\frac{3}{2.4.6.7}$;
$+(55b^4-55ab^2c+10a^2bd+5a^2c^2-a^3e=$
$\left(\frac{55}{1296}-\frac{11}{864}+\frac{1}{3024}+\frac{1}{2880}-\frac{1}{362880}\right)=\frac{3.5.7}{2.4.6.8.9}$; therefore

$$x = z + \frac{1}{2.3}z^3 + \frac{3}{2.4.5}z^5 + \frac{3.5}{2.4.6.7}z^7 + \frac{3.5.7}{2.4.6.8.9}z^9 + \&c.$$

2. Given $x + x^3 + x^5 + x^7 + x^9 + \&c. = y$, to find the value of x in terms of y. Ans. $x = y - y^3 + 2y^5 - 5y^7 + 14y^9 - \&c.$

3. Given $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \frac{1}{13}x^{13} - \&c.$ to find the value of x in terms of z. Ans $x = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{351}z^7 + \frac{62}{2835}z^9$

4. Given $x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{61}{5040}x^7 + \frac{277}{72576}x^9 + \&c. = z$ to find x in terms of z. Ans. $x = z - \frac{1}{6}z^3 + \frac{1}{24}z^5 - \frac{61}{5040}z^7 + \frac{277}{72576}z^9$.

Rule III. When two series are equal to each other, as

$$ax + bx^2 + cx^3 + dx^4 + \&c. = \alpha y + \beta y^2 + \gamma y^3 + \delta y^4 + \&c.$$

and it is required to find the unknown quantity in one of them, in terms of the unknown quantity in the other, substitute the particular values of the coefficients in the given example, in the place of the known letters, in the following general formula, and the result will be the root or value required.

$$x = \alpha\frac{y}{a} + (\beta - bA^2)\frac{y^2}{a} + (\gamma - 2bAB - cA^3)\frac{y^3}{a} + (\delta - bB^2 - 2bAC - 3cA^2B - dA^4)\frac{y^4}{a} + (\varepsilon - 2bBC - 2bAB - 3cAB^2 - 3cA^2C - 4dA^3B - eA^5)\frac{y^5}{a} + (\Sigma - 2bBD - bC^2 - 2bAE - CB^3 - 6cABC - 3cA^2D - 6dA^2B^2 - 4dA^3C - 5eA^4B - fA^6)\frac{y^6}{a} - \&c.,$$

where it is to be observed, that A, B, C, &c. are the coefficients of the first, second, third, *&c.* terms. The same form may also be rendered equally applicable to the case in which the odd powers of x and y only are concerned, by putting each of the coefficients of the even powers $= 0$.

1. Given $x - \frac{1}{2}x^2 + \frac{1}{2.3}x^3 + \frac{1}{2.3.4}x^4 + \frac{1}{2.3.4.5}x^5 - \&c. = \frac{1}{2}y + \frac{1}{3}y^2 + \frac{1}{4}y^3 + \frac{1}{5}y^4 + \frac{1}{6}y^5 + \&c.$ to find x in terms of y.

Here $a = 1$, $b = -\frac{1}{2}$, $c = \frac{1}{2.3}$, $d = -\frac{1}{2.3.4}$ &c.; and $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{4}$, $\delta = \frac{1}{5}$, &c.; whence $\alpha = \frac{1}{2} = A$, $\beta - bA^2 = \frac{1}{3} + \frac{1}{2} \times \frac{1}{4} = \frac{11}{24} = B$

$\gamma - 2bAB - cA^3 = \frac{1}{4} + \frac{11}{48} - \frac{1}{48} = \frac{11}{24} = C$,

$\delta - bB^2 - 2bAC - 3cA^2B - dA^4 = \frac{1381}{2880} = D$,

$\varepsilon - 2bBC - 2bAB - 3cAB^2 - 3cA^2C - 4dA^3B - eA^4 = \frac{1543}{3840}$, &c.

Therefore $\frac{1}{2}y + \frac{11}{24}y^2 + \frac{11}{24}y^3 + \frac{1381}{2880}y^4 + \frac{1543}{3840}y^5 + \&c.$

2. Given $x + x^2 + x^3 + x^4 + x^5 + \&c. = y + \frac{1}{2}y^2 + \frac{1}{4}y^3 + \frac{1}{8}y^4 + \frac{1}{16}y^5$, to find x in terms of y. Ans. $x = y - \frac{1}{2}y^2 + \frac{1}{4}y^3 - \frac{1}{8}y^4 + \frac{1}{16}y^5$.

3. Given $x + \frac{x^3}{6d^2} + \frac{3x^5}{40d^4} + \frac{5x^7}{112d^6} + \&c = ny + \frac{ny^3}{6d^2} + \frac{3ny^5}{40d^4} + \frac{5ny^7}{112d^6}$ to find x in terms of y. Or $z = ny + \frac{1-n^2}{2.3d^2}Ay^2 + \frac{9-n_2}{4.5d^2}By^3 \&c.$

Where A, B, &c. are the foregoing terms with their proper sign.

121. If the series given be $ax+bx^2+cx^3+dx^4+ex^5+\&c.=y$, for finding the value of x in terms of y, it may be demonstrated as follows; assume $x=Ay+By^2+Cy^3+Dy^4+Ey^5+\&c.$ &c.

Then, if this value be substituted for x and its powers, in the above equation, we shall have

$$\left|\begin{array}{l|l|l|l|l}
aAy+aB & y^2+aC & y^3\ +aD & y^4+aE & y^5+\&c. \\
\quad\ +bA^2 & \quad\ +2bAB & \quad\ +2bAC & \quad\ +2bAD & \\
 & \quad\ +cA^3 & \quad\ +bB^2 & \quad\ +2bBC & \\
 & & \quad\ +3cA^2B & \quad\ +3cA^2C & \\
 & & \quad\ +dA^4 & \quad\ +3cAB^2 & \\
 & & & \quad\ +4dA^3B & \\
 & & & \quad\ +eA^5 &
\end{array}\right.$$

And, consequently, if the coefficient of the first term be put $=1$, and each of the rest $=0$, there will arise the following results; $aA=1$; $aB+bA^2=0$; $aC+2bAB+cA^3=0$;
$aD+2bAC+bB^2+3cA^2B+dA^4=0$;
$aE+2bAD+2bBC+3cA^2C+3cAB^2+4dA^3B+eA^5=0$,

Where, finding the values of A, B, C, &c. in the first, second, third, &c. of these equations, and substituting them, as they arise in the succeeding equations, we shall have

$A=\frac{1}{a}$; $B=-\frac{b}{a^3}$; $C=\frac{2b^2-ac}{a^5}$; $D=-\frac{5b^3-5abc+a^2d}{a^7}$

$E=\frac{14b^4-21ab^2c+3a^2c^2+6a^2bd-a^3e}{a^9}$ &c. &c. which are the coefficients of y and its powers, in the reverted series.

123. When the series consists of the odd powers of x, or is of the form $ax+bx^3+cx^5+dx^7+ex^9+\&c.=y$, the value of x, in terms of y may be obtained in a similar manner, by assuming $x=Ay+By^3+Cy^5+Dy^7+Ey^9+\&c.$

For if this value be substituted for x and its powers, in the above equation, as in the former case, we shall have

$$y=\left|\begin{array}{l|l|l|l|l}
aAy+aB & y^3+aC & y^5+AD & y^7+aE & y^9+\&c. \\
\quad\ +bA^3 & \quad\ +2bA^2B & \quad\ +3bA^2C & \quad\ +3bA^2D & \\
 & \quad\ +cA^5 & \quad\ +3bAB^2 & \quad\ +6bABC & \\
 & & \quad\ +5cA^4B & \quad\ +bB^3 & \\
 & & \quad\ +dA^7 & \quad\ +5cA^4C & \\
 & & & \quad\ +10cA^3B^2 & \\
 & & & \quad\ +7dA^6B & \\
 & & & \quad\ +eA^9 &
\end{array}\right.$$

And consequently, by putting the coefficient of the first term $=1$, and the rest $=0$, as before, there will arise the following results: $aA=1$; $aB+bA^3=0$; $aC+2bA^2B+cA^5=0$
$aD+3bA^2C+3bAB^2+5cA^4B+dA^7=0$
$aE+3bA^2D+6bABC+bB^3+5cA^4C+10cA^3B^2+7dA^6B+eA^9=0$;
where, finding the values of A, B, C, &c. in the first, second,

third &c. of these equations, as in the former case, and then substituting them, as they arise, in the succeeding equations, we shall have $A=\frac{1}{a}$; $B=-\frac{b}{x^4}$; $C=\frac{3b^2-ac}{a^7}$; $D=-\frac{12b^3+a^2d-8abc}{a^{10}}$; $E=\frac{55b^4-55ab^2c+10a^2bd+5a^2c^2-a^3e}{a^{13}}$, &c. which are the coefficients of y and its powers, in the reverted series, for this case.

124. When there are two series, consisting of the successive powers of x and y, as $ax+bx^2+cx^3+dx^4+$&c.$=\alpha y-\beta y^2+\gamma y^3+\delta y^4$

Let there be assumed, as in the instances before given, $x=Ay+By^2+Cy^3+Dy^4+Ey^5+$&c. Then if this value be substituted for x and its powers, in the left hand member of the proposed equation, we shall have $\alpha y+\beta y^2+\gamma y^3+\delta y^4+\varepsilon y^5+$&c. $=aAy+(aB+bA^2)y^2+(aC+2bAB+cA^3)y^3+(aD+2bAC+bB^2+3cA^2B+dA^4)y^4+(aE+2bAD+2bBC+3cA^2C+3cAB^2+4dA^3B+eA^5)y^5+$ &c. and, consequently, by equating the coefficients of the homologous terms, there will arise the following results: $aA=\alpha$, $aB+bA^2=\beta$, $aC+2bAB+cA^3=\gamma$, $aD+2bAC+bB^2+3cA^2B+da^4=\delta$, $aE+2bAD+2bBC+3cA^2C+3cAB^2+4dA^3B+eA^5=\varepsilon$, where, finding the values of A, B, C, &c. as in the former cases, and substituting them, as they arise, in the succeeding equations, we shall have $A=\frac{\alpha}{a}$, $B=\frac{\beta-bA^2}{a}$; $C=\frac{\gamma-2bAB-cA^3}{a}$;

$D=\frac{\delta-bB^2-2bAC-3cA^2B-dA^4}{a}$;

$E=\frac{\varepsilon-2bBC-2bAD-3cAB^2-3cA^2C-4dA^3B-eA^5}{a}$, &c. &c.

which are the coefficients of the terms of the series that give the value of x in terms of y.

Of Permutations and Combinations.

Permutation is a change in the order or position of any number of things, so that no two parcels of them shall be situated alike.

Thus, the two letters *a*, *b*, admit of two permutations, viz. *ab* and *ba*.

The three letters, *a*, *b*, *c*, admit of six permutations, viz. *abc*, *bac*, *cab*, *acb*, *bca*, *cba*.

By combination of quantities is meant the different collections that can be formed of them, without regarding the order or position in which the quantities are placed.

Thus, the number of combination which can be made of three things, *a*, *b*, *c*, taken two and two together, is 3; viz. *ab*, *ac*, *bc*.

The number of combinations of four things, *a*, *b*, *c*, *d*, taken two and two together, is 6; viz. *ab*, *ac*, *ad*, *bc*, *bd*, *cd*. If taken

three at a time, the number of combinations is 4; viz. *abc*, *acd*, *bcd*, *abd*.

Theorem 1. The number of permutations that can be formed out of (n) quantities taken two and two together, is $n.(n-1)$; taken three and three together, the number is $n.(n-1).(n-2)$.

Let a, b, c, d, e, &c. be the (n) quantities, which are to be taken two at a time. Now, by placing a before each of the other ($n-1$) quantities, we form ($n-1$) permutations. Also, by placing b before each of the remaining ($n-1$) quantities, we form ($n-1$) permutation; and this process can be repeated (n) times, therefore in all, we obtain, $n(.n-1)$ permutations.

Again, of ($n-1$) quantities, when two are taken at a time, we have $(n-1).(n-2)$ permutations, as appears from what has been already said; and since a can be prefixed to each of these quantities, there may arise $(n-1).(n-2)$ permutations in which a stands first; and this process may be repeated (n) times, ∴ upon the whole we obtain $n(n-1).(n-2)$ permutations when three quantities are taken at a time.

By following the same method, it appears, that in (n) things, if r of them be taken together, there are $n.(n-1).(n-2).(n-3)\ldots.(n-r+1)$ permutations.

Theorem 2. In (n) quantities, containing (p) quantities of one sort, (q) quantities of another sort, &c. the permutations will be

$$\frac{n.(n-1).(n-2)\ \ldots.3.2.1}{1.2.3\ldots.p\times1.2.3\ldots.q\times1.2.3\ldots.r\&c.}$$

Now when the quantities are all different from one another, the number of permutations $=n.(n-1).(n-2)\ldots.3.2.1$, taking ($n$) at a time, by Theorem 1; but if any quantity recur twice, the number of permutations is diminished 2.1 times:* and if any quantity recur thrice, the number of permutations is diminished 3.2.1 times: and in general if there be (p) quantities alike, the number of permutations is diminished $1.2.3\ldots.p$ times. The same may be shown with respect to the (q) quantities. Hence the number of permutations which can be formed

$$=\frac{n(.(n-1).(n-2)\ldots\ldots..3.2.1}{1.2.3\ldots.p\times1.2.3\ldots.q\times1.2.3\ldots.r\&c.}$$

1. Required the number of variations or permutations that can be formed of the letters in the word "Bacchanalia."

Here the letter c recurs twice, a four times, and there are 11 letters in all; ∴ the number of variations required

* Since each combination of two quantities admits of 1.2 permutations, and each combination of three quantities admits of 1.2.3 permutations, ∴ the quantities 1.2, 1.2.3, &c. become divisors of that number which would represent the number of permutations if the quantities were all different.

$$=\frac{11.10.9.8.7.6.5.4.3.2.1}{1.2\times1.2.3.4}=831600.$$

2. How many different numbers can be made of the following digits; 1, 2, 2, 7, 7, 7, 5, 5, 5, 5? Ans. 12600.

3. How many permutations may be made of the letters in the word "examination." Ans. 4989600.

Theorem 3. The number of combinations that can be formed out of (n) things, taken two and two together, is $\frac{1}{2}\{n.(n-1)\}$; when taken three and three together $\frac{n(n-1)\cdot(n-2)}{1.2.3}$

Now since each combination ab admits of two permutations, $\therefore$ when the quantities are taken two at a time, there will be 1.2 times as many permutations as combinations. Now by Theorem 1, the number of permutations of (n) things, taken two at a time, $=n.(n-1)$; hence the number of permutations $=1\cdot2$ times the number of combinations; $\therefore$ the number of combinations $=\frac{n.(n-1)}{1.2}$.

Again, by Theorem 1, there are $n.(n-1).(n-2)$ permutations in (n) things, taken three at a time; but each combination of 3 things admits of 1.2.3 permutations, $\therefore$ $n.(n-1).(n-2)=1.2.3$ times the number of combinations, i. e. the number of combinations $=\frac{n.(n-1).(n-2)}{1.2.3}$. By following the same method, it appears that the number of permutations of (n) quantities, taken (r) at a time, $=n(n-1).(n-2)...(n-r+1)$. Also each combination of (r) things admits of $1.3.3.4....r$ permutations; $\therefore$ the number of permutations $=1.2.3....r$ times the number of combinations; hence the number of combinations in (n) quantities, taking r of them at a time, $=\frac{n.(n-1).(n-2)....(n-r+1)}{1.2.3....r}$.

Theorem 4. The total number of combinations which can be formed of (n) things taken, one at a time, two at a time, &c. $=2^n-1$.

First let the quantities be taken one at a time, then the number of combinations is ....n. If taken two at a time, the number is $n.\frac{n-1}{2}$. If taken 3 at a time, the number is $n.\frac{(n-1).(n-2)}{1\cdot2\cdot3}$.

If taken four at a time, the number is $n.\frac{(n-1).(n-2)\cdot(n-3)}{1.2.3.4}$,

$\therefore$ the total number $=n+\frac{n.(n-1)}{1.2}+\frac{n.(n-1).(n-2)}{1.2.3}+$&c.

$=(1+1)^n-1=2^n-1.$

On Logarithms.

123. Logarithms are the powers to which a given number must be raised, in order to become equal to other given numbers.

We know that $10^1=10$, $10^2=100$, $10^3=1000$, &c.; therefore, if 10 were the given number, 1, 2, 3, &c. are the powers to which 10 must be raised, in order to become equal to 10, 100, 1000, &c. Now, suppose it were required to find a number (x) such, that $10^x=5$; it is evident that x is not so great as 1, because $10^1=10$: suppose $x=\frac{2}{3}$, then $10^{\frac{2}{3}}=100^{\frac{1}{3}}$ = a number less than 5. $\therefore$ x is more than $\frac{2}{3}$ or .666666; but if x were = .69897, then $10^x=2$ very nearly; this number .69897 is the log. of 5.

It is likewise found that $10^{.301030}=2$

$$10^{.477121}=3$$
$$10^{.698970}=5$$
$$10^{1.041392}=11, \text{ \&c.}$$

i. e. .301030 is the log. of 2; .477121 is the log. of 3; .698970 is the log. of 5, and 1.041392 is the log. of 11, &c.

Since 10 is the base of our present system of arithmetic, the same number has been chosen for the base of the logarithmic system now in use. It is evident that any other base (a) might be taken, and then it would be required to find all the values of x, which would make a^x = each of the different numbers whose logarithms we wish to find.

Let N, n be any two numbers whatever; and suppose $a^x=\mathrm{N}$, then x is the log. of N. And $a^{x'}=n$, then x' is the log. of n. Also $\mathrm{N}\times n=a^x\times a^{x'}=a^{x+x'}$.

Now $(x+x')$ is the log. of $\mathrm{N}\times n$ by the definition;

$$\therefore \text{Log. N}+\text{log. } n=\text{log. }(\mathrm{N}\times n).$$

Hence the log. of the product of two numbers = the sum of the logarithms of the numbers. Thus, log. 3×5=log. 3+ log. 5;

$$\text{Log.}(9\times8\times7)=\text{log.}(9\times8)+\text{log.}7=\text{log.}8+\text{log.}9+\text{log.}7.$$
$$\text{Log.}pqr=\text{log.}pq+\text{log.}r=\text{log.}p+\text{log.}q+\text{log.}r$$

Also $\frac{\mathrm{N}}{n}=\frac{a^x}{a^{x'}}=a^{x-x'}$. $\therefore$ Log. $\frac{\mathrm{N}}{n}=x-x'$ by the definition of logarithms, =log.N—log.n.

124. Hence the logarithm of the quotient of any two numbers is equal to the difference of the logarithm of those numbers.

If N be less than n, x is less than x'; $\therefore$ therefore the logarithms of any proper fraction is negative.

Since $\mathrm{N}=a^x$; $\therefore$ $\mathrm{N}^m=a^{mx}$, and the log. of $\mathrm{N}^m=mx=m$log. N. Also $\mathrm{N}^{\frac{1}{n}}=a^{\frac{x}{n}}$; $\therefore$ Log. $\mathrm{N}^{\frac{1}{n}}=\frac{x}{n}=\frac{1}{n}\cdot x=\frac{1}{n}\cdot$log.N.

125. The properties of logarithms here given are true in every

system; but we have selected the system whose base is 10, because it is the most convenient for all purposes of calculation.

It is also evident, that if a series of numbers be taken in geometrical progression, their logarithms will be in arithmetic progression. Take the common scale

$$10^{-5},\ 10^{-4},\ 10^{-3},\ 10^{-2},\ 10^{-1},\ 1,\ 10^{1},\ 10^{2},\ 10^{3},\ 10^{4},\ \&c.$$

Or $\frac{1}{100000}$, $\frac{1}{10000}$, ...1, 10, 100, 1000, &c.

Or .00001, .0001, .001, .01, .1, 1, 10, 100, 1000, &c.; each of these series forms a geometric progression, and the logarithms of the terms of the series are —5, —4, —3, —2, —1, 0, 1, 2, 3, &c. which is an arithmetic progression. And in general, $a^x, a^{x+r}, a^{x+2r}, a^{x+3r}$, &c. form a geometric series of numbers whose logarithms are x, $x+r$, $x+2r+$&c. in arithmetic progression.

$$\text{Since } N=10^x;\ \therefore \frac{N}{10}=\frac{10^x}{10}=10^{x-1};\ \frac{N}{10^2}=\frac{10^x}{10^2}=10^{x-2},$$

&c. hence, by dividing any number by 10, its logarithm is diminished by 1; dividing by 100, diminishes the logarithms by 2, &c. therefore taking any number, 3854, whose logarithm is 3.5859117,

The log of 385.4 $=2.5859117$

38.54 $=1.5859117$

3.854 $=0.5859117$

.3854 $=\bar{1}.5859117$

.03854 $=\bar{2}.5859117$

.003854 $=\bar{3}.5859117$ &c.

Prob. 1. To find the logarithm of any given number.

Let $a^x=N$; then if x be found in terms of N and a, it will be the logarithm of N to the base a. Suppose $N=1+n$

$a=1+b$

$$\text{Then } (1+b)^x=1+n, \text{ and } (1+b)^{\frac{x}{y}}=(1+n)^{\frac{1}{y}}.$$

$$\text{Now } (1+b)^{\frac{x}{y}}=1+\frac{x}{y}\cdot b+\frac{x}{y}\cdot\left(\frac{x}{y}-1\right)\cdot\frac{b^2}{1.2}+\frac{x}{y}\cdot\left(\frac{x}{y}-1\right)\cdot\left(\frac{x}{y}-2\right)\cdot\frac{b^3}{1.2.3}+\&c.$$

$$\text{Also } (1+n)^{\frac{1}{y}}=1+\frac{1}{y}\cdot n+\frac{1}{y}\cdot\left\{\frac{1}{y}-1\right\}\cdot\frac{n^2}{1\cdot 2}+\frac{1}{y}\cdot\left\{\frac{1}{y}-1\right\}+\left\{\frac{1}{y}-2\right\}\cdot\frac{n^3}{1.2.3}$$

Now suppose y to become very great; then $\frac{x}{y}$ and $\frac{1}{y}$ become very small when compared with —1, —2, &c. If y be increased without limit we obtain

$$\left\{b-\tfrac{1}{2}b^2+\tfrac{1}{3}b^3-\tfrac{1}{4}b^4+\&c.\right\}\frac{x}{y}=\frac{1}{y}\left\{n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4\ \&c.\right\}$$

$$\therefore x=\frac{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.}=M\cdot\left\{n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4\right.$$

$+\&c.\}$. If $M = \frac{1}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.}$ = the modulus of the system.

Hence log. N; or log. $(1+n) = M\{n-\frac{1}{2}n^2-\frac{1}{3}n^3-\frac{1}{4}n^4+ \&c\}$.

If n be any whole number, the terms of this series increase, and ∴ the series itself does not converge. To obtain a converging series we have by a similar method, log. $(1-n) = M\cdot\{-n-\frac{1}{2}n^2-\frac{1}{3}n^3-\frac{1}{4}n^4- \quad \&c.\}$

Subtract this series from the other, and

Log. $(1+n)-\log.(1-n)=2M\cdot\{n+\frac{1}{3}n^3+\frac{1}{5}n^5+ \quad \&c.\}$

$$\text{Or log. } \frac{1+n}{1-n}=2M\cdot\{n+\frac{1}{3}n^3\times\frac{1}{5}n^5\times \quad \&c.$$

$$\text{Suppose } n=\frac{1}{N-1}\text{; then } \frac{1+n}{1-n}=\frac{N}{N-2}$$

$$\therefore \text{ Log. } \frac{N}{N-2}=2M\cdot\left\{\frac{1}{N-1}+\frac{1}{3.(N-1)^3}+\frac{1}{5(N-1)^5}+\&c.\right\}$$

$$\therefore \text{ Log.} N=2M\cdot\left\{\frac{1}{N-1}+\frac{1}{3\cdot(N-1)^3}+\frac{1}{5(N-1)^5}+\&c.\right\}+\log.$$

$(N-2)$; an expression which converges with sufficient rapidity.

It may be easily shewn that log. $(N+1)=2\log. N-\log.(N-1)-2M\left\{\frac{1}{2N^2-1}+\frac{1}{3}\cdot\frac{1}{(2N^2-1)^3}+\&c.\right\}$; two terms of which series will give the logarithms to 14 places of decimals, whenever N is $=100$ or upwards.

This series is found by supposing $n=\frac{1}{2N^2-1}$.*

Prob. 2. To construct a Table of Logarithms.

Since a may be arbitrarily assumed, let it be so taken that

$$(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\&c.=1$$

Then the first number whose logarithm we must find is 2.

Now log. $4=\log.2^2=2\log.2$.

$$\therefore \text{ Log. 4 or 2 log. } 2=2\left\{\frac{1}{3}+\frac{1}{3^3}+\frac{1}{5.3^5}+\&c.\right\}+\log. 2.$$

$$\therefore \text{ Log. } 2=2\left\{\frac{1}{3}+\frac{1}{3^3}+\frac{1}{5.3^5}+\&c.\right\} \ldots\ldots\ldots = .693147$$

$$\text{Log. } 3=2\left\{\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\&c.\right\} \ldots\ldots =1.098612$$

$$\text{Log. } 4=2\text{ log. } 2\ldots\ldots\ldots\ldots\ldots\ldots =1.386294$$

$$\text{Log. } 5=2\left\{\frac{1}{4}+\frac{1}{3.4^3}+\frac{1}{5.4^5}+\&c.\right\}+\log.3=1.609437$$

$$\text{Log. } 6=\log. 3+\log. 2\ldots\ldots\ldots\ldots =1.791759$$

* See Peacock's examples in the diff. and int. calculus, p. 61.

$$\text{Log. } 7 = 2\left\{\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5.6^5}+\&c.\right\}+\log.5=1.245892$$

$$\text{Log. } 8 = \log 4 + \log 2 \text{ or } = 3 \log 2 \cdots\cdots = 2.079441$$

$$\text{Log. } 9 = \log 4 \quad 2 \text{ lgo } 3 \cdots\cdots\cdots\cdots = 2.197224$$

$$\text{Log.}10 = \log 5 + \log 2 \cdots\cdots\cdots\cdots\cdots = 2.302585$$

$$\text{Log.}11 = 2\left\{\frac{1}{10}+\frac{1}{3.10^3}+\frac{1}{5.10^5}+\&c.\right\}+\log 9=\&c.$$

The logarithms here calculated are called Napier's logarithms, from the name of their noble and ingenious inventor; but are commonly known by the name of hyperbolic logarithms, because they can be represented by the areas of the equilateral hyperbola, contained between its assymptotes. To find the value of the base (a) of this system, whose modulus equals 1.

Since log N, or log $(1+n)=n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c$; therefore by reversion of series,

$$n=\text{Log N}+\tfrac{1}{2}(\log N)^2+\frac{1}{2.3}(\log N)^3+\frac{1}{2.3.4}(\log N)^4+\&c.*$$

$$\therefore 1+n \text{ or } N=1+\log N+\tfrac{1}{2}(\log N)^2+\frac{1}{2.3}(\log N)^3+\&c.$$

Let log N=1; then $N=1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{1.2.3.4}+\&c.$ $=2.7182818$ = the number whose hyperbolic log is 1, or = the base of the system.

According to this system $(2.7182818)^{.693147}=2$

$(2.7182818^{1.098612})=3$, &c.

Again, since Hyp log 2.7182818=1

therefore Hyp log $(2.7182818)^2=2$

Hyp log $(2.7182818)^3=3$, &c. Hence in this system the numbers whose logarithms are 1, 2, 3, &c. are decimal numbers, and therefore very inconvenient for the ordinary purposes of calculation. To change hyperbolic logarithms into common logarithms, or those generally used.

In this system the base $(a)=10$, and its log = 1. Now in the system whose modulus = M,

Log $N=M\{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\&c.\}$ and in the Naperian system, Hyp log $N=n=\frac{1}{2}n^2+\frac{1}{3}n^3-\&c.$ because M=1.

Therefore N=M×Hyp log N; and if N=10, we have

$$1=M\times\text{Hyp log }10;\ \therefore M=\frac{1}{\text{Hyp log }10}=\frac{1}{2.305585}=\cdot43429448$$

and common log N=·43429448×Hyp log N.

Now, since Hyperbolic log $2=2\left|\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5.3^5}+\&c.\right|$

* By reversion of series. See pp. 294—5.

$$\therefore \text{common log } 2 = {\cdot}86858896\left(\frac{1}{3}+\frac{1}{3^3}+\frac{1}{5.3^5}+\&c.\right) = 0.3010300$$

Log.

$$3 = {\cdot}86858896\left(\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\&c.\right) = 0{\cdot}4771213$$

$$4 = 2\log 2 \quad - \; - \; - \; - \; - \quad = 0{\cdot}6020600$$

$$5 = \log\frac{10}{2} = \log 10 - \log 2 = 1 - \log.2 \quad - \; - \quad = 0{\cdot}6979700$$

$$6 = \log 3 + \log 2 \quad - \; - \; - \; - \quad = 0{\cdot}7781513$$

$$7 = {\cdot}86858896\left(\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5.6^5}+\frac{1}{7.6^7}\right) + \log 5 = 0.8450980$$

$$8 = \log.\ 2^3 = 3\log.2 \quad - \; - \; - \; - \; - \; - \; - \; - \; - \quad = 0.9030900$$

$$9 = \log.\ 3^2 = 2\log.3 \quad - \; - \; - \; - \; - \; - \; - \; - \; - \quad = 0.9542425$$

$$10 = \log. \quad - \; - \; - \; - \; - \; - \; - \; - \; - \quad = 1.0000000$$

$$11 = {\cdot}86858896\left(\frac{1}{10}+\frac{1}{3.10^3}+\frac{5}{5.10^5}+\&c.\right) + \log 9 = 1{\cdot}0413927$$

$$12 = \log.\ 3 + \log.4 \quad - \; - \; - \; - \; - \; - \; - \; - \; - \quad = 1{\cdot}0791812$$

$$13 = {\cdot}86858896\left(\frac{1}{12}+\frac{1}{3{\cdot}12^3}+\frac{1}{5.12^5}+\right) + \log 11 = 1{\cdot}1139434$$

$$14 = \log.\ 7 + \log.2 \quad - \; - \; - \; - \; - \; - \; - \; - \quad = 1.1461280$$

$$15 = \log.\ 5 + \log.3 \quad - \; - \; - \; - \; - \; - \; - \; - \quad = 1.1760913$$

$$16 = \log.\ 4^2 \quad \log.4 \quad - \; - \; - \; - \; - \; - \; - \; - \quad = 1.2041200$$

$$17 = {\cdot}86858896\left(\frac{1}{16}+\frac{1}{3.16^3}+\frac{1}{5.16^5}+\&c.\right) + \log 15 = 1.2304489$$

$$18 = \log.\ 9 + \log.2 \quad - \; - \; - \; - \; - \; - \; - \; - \; - \quad = 1.2552725$$

$$19 = {\cdot}85858896\left(\frac{1}{18}+\frac{1}{3.18^3}+\frac{1}{5.18^5}+\&c.\right) + \log 17 = 1{\cdot}2787536$$

$$20 = \log.10 + \log.2 \quad - \; - \; - \; - \; - \; - \; - \; - \; - \quad = 1.3010300$$

$$21 = \log.\ 7 + \log.3 \quad - \; - \; - \; - \; - \; - \; - \; - \; - \quad = 1.3222193$$

$$22 = \log.11 + \log.2 \quad - \; - \; - \; - \; - \; - \; - \; - \; - \quad = 1.3424227$$

$$23 = {\cdot}86858890\left(\frac{1}{22}+\frac{1}{3.22^3}+\frac{1}{5.22^5}+\&c.\right) + \log 21 = 1{\cdot}3617278$$

The logarithms of 24, 25, 26, 27, 28, can also be found by addition; but the log of 29 must be calculated by the series, as must the log of every prime number. See Index.

The logarithm of 1=0 and log 10=1; hence the logarithm of any number between 1 and 10 is less than 1. Also the log of 100=2; therefore the log of any number between 10 and 100 will be 1, with some decimal part annexed. The log of any number between 100 and 1000 will be 2, with some decimal part. These whole numbers are called the indices of the numerals, and can be readily found from the rule just given; they are not given in logarithmic tables, but are left to be supplied by the student.

The great advantages attending the common system of logarithms arise from the facility with which we can find the index of the logarithm of any number; and that if any number be multiplied or divided by 10, 100, &c., its logarithm is increased or diminished by 1, 2, &c.; the decimal part remaining the same.

On the method of finding the Increase of Population in any country under given circumstances of Births and Mortality.

126. Let P represent the population of any country at a given period; $\frac{1}{m}$ the fractional part of the population which die in a year, (or ratio of mortality;) $\frac{1}{b}$ the proportion of births in a year. Then, if A represents the state of the population at the end of n years, $\log A = \log P + n \times \log\left(1 + \frac{m-b}{mb}\right)$.

The rate of increase of population in one year $= \frac{1}{b} - \frac{1}{m} = \frac{m-b}{mb}$; $\therefore 1 : 1 + \frac{m-b}{mb} :: P : P\left(1 + \frac{m-b}{mb}\right) =$ state of the population at the end of the first year.

But it is increased every year in the same proportion; therefore $1 : 1 + \frac{m-b}{mb} :: P\left(1 + \frac{m-b}{mb}\right) : P\left(1 \times \frac{m-b}{mb}\right)^2 =$ state of the population at the end of the second year. In the same manner we may prove that the state of the population at the end of n years will be $P\left(1 + \frac{m-b}{mb}\right)^n$. Hence $A = P\left(1 + \frac{m-b}{mb}\right)^n$; and $\log. A = \log. P + n \times \log\left(1 + \frac{m-b}{mb}\right)$. From which we deduce

$$\log. P = \log A - n \times \log\left(1 + \frac{m-b}{mb}\right).$$

$$n = \frac{\log A - \log P}{\log\left(1 + \frac{m-b}{mb}\right)};\ \log.\left(1 + \frac{m-b}{mb}\right) = \frac{\log A - \log P}{n}.$$

Of the quantities A, P, m, b, n, any four being given, the fifth may therefore be found.

1. Suppose the population of Great Britain in the year 1800 to have been ten millions; that $\frac{1}{40}$th part die annually; that the births are to the deaths as 40 to 30; and that no emigration takes place during the present century. What will be the state of its population in the year 1900?

Here $P = 10000000$, $n = 100$, $m = 40$, $b = 30$, and

$\therefore 1 + \frac{m-b}{mb} = \frac{121}{120}$. Now $\log A = \log P + n \times \log\left(1 \times \frac{m-b}{mb}\right)$

$=\log 10000000\times 100\times\log\dfrac{121}{120}=7\cdot 3604200=\log .22931000.$

Hence A=22931000. Ans.

2. Suppose the population of France in the year 1792 to have been 27000000; the ratio of mortality during the eighteenth century to have been $\frac{1}{30}$th, and the number of births $\frac{1}{26}$th. What was the state of its population in the year 1700?

Here A=27000000, $n=92$, $m=30$, $b=26$,

$$\therefore 1+\frac{m-b}{mb}=\frac{196}{195}.\ \text{Log P}=\log \text{A}-n\times\log\left(1+\frac{m-b}{mb}\right)$$

$=\log 27000000-92\times\log\frac{196}{195}=7.2269858=\log 16864980$, nearly; therefore P=16864980.

3. Suppose the population of North America to have been five millions in the year 1800; in how many years will it amount to sixteen millions, taking the ratio of mortality at $\frac{1}{45}$th, and the annual proportion of births at $\frac{1}{24}$th?

Here A=16000000, P=5000000, $m=45$, $b=24$;

$$\therefore 1+\frac{m-b}{mb}=\frac{367}{360};\ n=\frac{\log \text{A}-\log \text{P}}{\log\left(1+\dfrac{m-b}{mb}\right)}$$

$$=\frac{\log 16000000-\log 5000000}{\log\frac{367}{360}}=\frac{.5051500}{.0084636}=60.39.\text{years}.$$

4. A province in the year 1760 was estimated at 500000 persons. In the year 1800 it amounted to 720000. From the bills of mortality it appeared that, upon an average, $\frac{1}{50}$th part of the population had died annually; no register had been kept of the births. What was the annual proportion of them during this period?

Here A=720000, P=500000, $n=50$, $n=40$.

$$\text{Log.}\left(1+\frac{m-b}{mb}\right)=\frac{\log \text{A}-\log \text{P}}{n},\ \text{or}\ \log\left(1+\frac{50-b}{50b}\right)=$$

$$\frac{\log 720000-\log 500000}{40}=.0039590=\log 1.009.$$

$$\text{Hence } 1+\frac{50-b}{50b}=1.009=1+\frac{9}{1000}\ \text{and}\ \frac{50-b}{50b}=\frac{9}{1000}.$$

$$\therefore 50000-1000b=450b,\ \text{or}\ b=\tfrac{50000}{1450}=34.4.$$

The annual proportion of births therefore was about $\frac{1}{34}$th.

127. But in any country, under given circumstances of births and mortality, the fraction $\dfrac{m-b}{mb}$ is always a given quantity. Let it be represented by $\dfrac{1}{p}$; then the relation between the four quan-

tities A, P, p, n, is expressed by $A=P\left(1+\frac{1}{p}\right)^n$. If $A=mP$, we have $mP=P\left(1+\frac{1}{p}\right)^n$; or $m=\left(1+\frac{1}{p}\right)^n$; and taking the logarithm $\log m=n\times\log\left(1+\frac{1}{p}\right)$. Hence we deduce the six following formulæ; I. $\text{Log. } A=\log P+n\log\left(1+\frac{1}{p}\right)$.

II. $\text{Log } P=\log A-n\log\left(1+\frac{1}{p}\right)$. III. $n=\dfrac{\log A-\log P}{\log\left(1+\frac{1}{p}\right)}$.

IV. $\text{Log}\left(1+\frac{1}{p}\right)=\dfrac{\log A-\log P}{n}$. V. $n=\dfrac{\log m}{\log\left(1+\frac{1}{p}\right)}$,

for finding the period in which the population would be increase m times.

VI. $\text{Log}\left(1+\frac{1}{p}\right)=\dfrac{\log m}{n}$, for finding the rate, $\frac{1}{p}$, at which the population would be increased m times in n years.

The following questions are intended to illustrate the use of these formulæ, in the order in which they stand.

5. Suppose the population of a country to begin with six persons, and to increase annually by $\frac{1}{16}$th of the whole; what will be the state of its population at the end of 200 years?

Here $P=6$, $n=200$, $\frac{1}{p}=\frac{1}{16}$; $1+\frac{1}{p}=1+\frac{1}{16}=1\cdot0625$; $\therefore$ $A=\log P+n\log\left(1+\frac{1}{p}\right)=\log 6+200\times\log(1.0625)=\log 6.0439510=\log 1106400$. Hence $A=1106400$ persons.

6. If, as stated in the third Example, the population of North America was five millions in the year 1800, and the rate of increase has been $\frac{7}{360}$ths for fifty years previous. What was the state of its population in the year 1750? Ans. 1908930 persons.

Here $A=5000000$, $\frac{1}{p}=\frac{7}{360}$; $1+\frac{1}{p}=1+\frac{7}{360}=\frac{367}{360}$ and $n=50$; $\therefore$ $P=\log A-n\times\log\left(1+\frac{1}{p}\right)=\log 5000000-50\times\log\frac{367}{360}=\log 5000000-50\times(\log 367-\log 360)=6.2807900=\log 1908930$; hence $A=1908930$ persons.

7. An empire to be 40 millions, and the annual increase $\frac{1}{195}$th; how long will it be before it amounts to 50 millions?

Ans. 43.6 years.

Here A$=$50000000, P$=$40000000; $p=\frac{1}{195}$; $1+p=1+\frac{1}{195}=\frac{196}{195}$

$$\therefore n=n=\frac{\log.A-\log.P}{\log.\left(1+\frac{1}{p}\right)}=\frac{\log 50000000-\log 40000000}{(\log 196-\log 185)}=$$

$$\frac{.096910}{.002221}=43.6 \text{ years.}$$

4. What must be the rate of increase, that the population of a country may be changed from 1106400 persons to 5 millions in 100 years. Ans. About $\frac{1}{66}$th annually.

Here A—5000000, P.$=$ 1106400, $n=100$; $\therefore \left(1+\frac{1}{p}\right)=$

$$\frac{\log A-\log P}{n}=\frac{\log 50000000-1106400}{100}=\cdot 006550=\log 1.015;$$

hence $1+\frac{1}{p}=1.015$, and $\frac{1}{p}=\cdot 015=\frac{15}{\cdot 1000}=\frac{1}{66}$, nearly.

5. By means of the formula $n=\dfrac{\log m}{\log\left(1+\frac{1}{p}\right)}$, verify the following Table.

$\frac{1}{p}$	Period of doubling.	Period of trebling.	Period of being increased 10 times.
$\frac{1}{120}$	83.5 years.	132.3 years.	277.4 years.
$\frac{1}{52}$	36.3 years.	57.6 years.	120.8 years.

Here $m=2$; $\frac{1}{p}=\frac{1}{120}$; $1+\frac{1}{p}=1+\frac{1}{120}$; therefore

$$n=\frac{\log m}{\log\left(1+\frac{1}{p}\right)}=\frac{\log 2}{\log 121-\log 120}=\frac{\cdot 301030}{\cdot 003604}=83\tfrac{1}{2} \text{ years.}$$

Here $m=3$; then $\dfrac{\log 3}{\log 121-\log 120}=\dfrac{.477121}{.00364}=132\tfrac{1}{3}$ years, nearly. And here $m=10$; then $\dfrac{\log 10}{\log 121-\log 120}=\dfrac{1}{\cdot 003604}=$ 277.3 years. Again, here $m=2$, $\frac{1}{p}=\frac{1}{52}$; $1+\frac{1}{p}=1+\frac{1}{52}=\frac{53}{52}$; $\therefore$

$$n=\frac{\log m}{\log(1+\frac{1}{p})}=\frac{\log 2}{\log 53-\log 52}=\frac{\cdot 301030}{\cdot 008273}=36\tfrac{1}{3} \text{ years, nearly.}$$

Here $m=3$; then $\dfrac{\log 3}{\log 53-\log 52}=\dfrac{\cdot 477121}{\cdot 00827}=57\cdot 6$ years.

$n=10$; $\dfrac{\log 10}{\log 53-\log 52}=\dfrac{1}{\cdot 008273}=120\cdot 8$ years.

6. What must be the annual increase of population in any country, that it may double itself every century?

Here $\left.\begin{matrix} m=2 \\ n=100 \end{matrix}\right\} \therefore l=\frac{1}{p}=\frac{\log m}{n}=\frac{\log 2}{100}=\frac{\cdot 301030}{100}=\cdot 003010=$ $\log 1\cdot 00696$. Here $1+\frac{1}{p}=1\cdot 00696$, and $\frac{1}{p}=\cdot 00696$, which is the arithmetic means between $\frac{1}{143}$d and $\frac{1}{144}$th.

128. Supposing that a census of the whole population of a country is taken every n years, and that it is found to have increased π per cent. during that interval, then if P represents the amount of the population at the commencement of the n years, $P+\frac{\pi P}{100}$ will represent the amount of the population at the end of the n years. If the annual increase be $\frac{1}{p}$, then (by Art. 127) the amount of the population at the end of n years is

$$P\left(1+\frac{1}{p}\right)^n;\text{ hence } P\left(1+\frac{1}{p}\right)^n=P+\frac{\pi P}{100}=P\left(1+\frac{\pi}{100}\right)$$

$$\text{or } \left(1+\frac{1}{p}\right)^n=1+\frac{\pi}{100}=\frac{100+\pi}{100}$$

$$\therefore n\cdot\log\left(1+\frac{1}{p}\right)=\log\,(100+\pi)-\log\,100=\log\,(100+\pi)-2,$$

since $\log 100=2$, and $\log\left(1+\frac{1}{p}\right)=\frac{1}{n}\{\log\,(100+\pi)-2\}$.

Substitute this value of $\log\left(1+\frac{1}{p}\right)$ in the expression $\dfrac{\log m}{\log\left(1-\frac{1}{p}\right)}$ (Formula V. Art. 127,) and we have $\dfrac{\log m}{\frac{1}{n}\{\log\,(100+\pi)-2\}}$ for the number of years in which the population of a country will be increased m times, if it goes on increasing at the same rate as it has done for the last n years preceding the period at which the census is taken.

129. If the census be taken every ten years, and the period of doubling be required, then $n=10$, $m=2$, and the foregoing expression becomes $\dfrac{\log.\ 2}{\frac{1}{10}\{\log\,(100+\pi)-2\}}$. By substituting in it for π the particular value of the per centage, the following table exhibits the corresponding period of doubling. See index.

ON *Figurate and Polygonal Numbers.*

130. Figurate numbers are such as arise from taking the successive sums of the series of the natural numbers 1, 2, 3, 4, &c., and then the successive sums of these last, and so on; and Polygonal numbers are those which are formed of the successive sums of the terms of any arithmetical progression, beginning with unity; each of them being usually divided into orders, according to the scale of their generation, which, as far as regards those of the first class, may be shown as follows:

Order.	*Figurate Numbers.*	*General Terms.*
Nat. ser ies.	1, 2, 3, 4, 5, 6, &c.	n
1st order.	1, 3, 6, 10, 15, 21, &c.	$\frac{n(n+1)}{1.2}$
2d order.	1, 4, 10, 20, 35, 56, &c.	$\frac{n(n+1)(n+2)}{1.2.3}$
3d order.	1, 5, 15, 35, 70, 126, &c.	$\frac{n(n+1)(n+2)(n+3)}{1.2.3.4}$

where it is to be observed, that the general terms here given are so called, because if 1, 2, 3, &c. be respectively substituted, in each of them, for n, we shall obtain the several terms of the series.

And if, instead of the natural numbers 1, 2, 3, 4, &c. which give, by their addition, triangular numbers, an arithmetical series be taken, the common difference of which is 2, the sum of its successive terms will be the series of square numbers. If the common difference be 3, the series will be pentagonal numbers; if 4, hexagonal; and so on. Thus.

Arith. Series.	*Or.*	*Polygonal Numbers.*		*Gen. Terms.*
1, 2, 3, 4, &c.	1	Tri.	1, 3, 6, 10, 15, 21, &c.	$\frac{1}{2}\{n(n+1)\}$
1, 3, 5, 7, &c.	2	Sqrs.	1, 4, 9, 16, 25, 36, &c.	$\frac{1}{2}\{n(2n+1)\}$
1, 4, 7, 10, &c.	3	Pent.	1, 5, 12, 22, 35, 51, &c.	$\frac{1}{2}\{n(3n-1)\}$
1, 5, 9, 13, &c.	4	Hex.	1, 6, 15, 28, 45, 66, &c.	$\frac{1}{2}\{n(4n-2)\}$

where the number denoting any order is the common difference of the arithmetical series from which the polygonal numbers, belonging to that order, are generated.

In like manner, if we take the successive sums of the several polygonal numbers thus formed, and then the successive sums of these last, and so on, a great variety of other orders of this kind may be obtained.

Hence, also, in general, if n be made to denote the number of terms of the series, a figurate number of any order m, which will consist of $m+1$ factors, may be expressed by the following formula:

$$\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\cdots\cdots\cdots\frac{n+m}{m+1}$$

And supposing n to be the number of terms of the series, as before, a polygonal number of the order $m-2$, or one of which the number of sides of the polygon is denoted by m, may be expressed by $\frac{(m-2)n^2-(m-4)n}{2}$, so that figurate Nos. of any order, may be also determined without computing those of the preceding orders, by taking as many factors, in the first of these formulæ, as is denoted by the number of the given order plus 1, and making n equal to the term that is to be found.

And a polygonal number of any order, or number of sides, may be ascertained from the second of these formulæ by substituting the number denoting that order for $m-2$, or the number of sides of the polygon, for m, and taking n equal to the term required.

1. Required the 15th term of the 1st order of figurate numbers 1, 3, 6, 10, 15, &c. Here the number of factors being two, and $n=15$, we shall have, by the first formula,

$\frac{n}{1}\times\frac{n+1}{2}=\frac{15(15+1)}{2}=\frac{15\times16}{2}=15\times8=120$, the term required.

2. Find the 20th term of the 4th order of figurate numbers. Here, the number of factors being 5, and $n=20$, we shall have

$\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}\times\frac{n+4}{5}=\frac{20}{1}\times\frac{21}{2}\times\frac{22}{3}\times\frac{23}{4}\times\frac{24}{5}=$ 52504, the ans. term required.

3. It is required to find the 12th term of the fifth order of polygonal numbers, being those called heptagonal, or such as would be represented by a figure of seven sides.

Here $m-2$ being $=5$, or $m=7$, and $n=12$, we shall have, by the 2d formula $\frac{1}{2}\{(m-2)n^2-(m-2)n\}=\frac{1}{2}\{(7-2)\times144-(7-4)\}\times12=5\times72-3\times6=360-18=342$, the term required.

1. Find the 13th term of the 8th order of figurate numbers.

2. It is required to find the 36th term of that order of polygonal numbers which is denoted by a figure of twenty-five sides.

3. It is required to find the first seven terms of the 6th order of figurate numbers. Ans. 293930. Ans. 14526. Ans. 3011

4. It is required to find the first twelve terms of the order of polygonal numbers called nonagonal, or such as are denoted by a figure of nine sides. Ans.

On Interest and Annuities.

131. Interest is the consideration paid for the use or forbearance of the payment of money.

Rate of interest is the consideration paid for the use of a certain sum for a certain time. Thus, \$5 per cent. per annum means, that \$5 are to be paid by the borrower to the lender for the use of \$100 for a year.

When the interest of the principal, or sum lent, is taken, it is called simple interest; but if the interest, as soon as it becomes due, be considered as principal, and interest be charged upon the whole, it is called compound interest as before.

132. Amount is the whole sum due at the expiration of any time, principal and interest together.

Discount is the abatement made for the payment of money before it becomes due.

133. The present worth of any debt due some time hence is such a sum as, being put out to interest for that time, will amount to the debt.*

134. To find the amount of a given sum in any time, at simple interest.

Let p = principal or money lent, r = interest of \$1 for a year, n = time for which interest is required, m = amount.

Now it is evident that the interest of a given sum, at a given rate, must be proportional to the time. Hence $\overset{\text{Yr.}}{1} : \overset{\text{Yrs.}}{n} :: r : nr$ = interest of \$1 for n years, and the interest of \$P being P times as great, $\therefore$ Prn = the whole interest of \$P for (n) years at the proposed rate. Now amount = principal + interest;

therefore $\quad M = P + Prn = P\{1 + nr\}$.

In this simple equation, any three of the quantities P, n, r, M, being given, the fourth may be found.

1. Find the amount of \$280·5 for 3 years, 148 days, at 5 per ct. per annum. Here $P = 280·5$, $r = \frac{5}{100}$ — \$·05, n=3.40547 years.
$\therefore$ amount $= 280.5\{1 + .1702735\} = 328·26171675 = \$328$

2. Find the principal which, being put out to interest for 189 days at 4 per cent. will amount to \$200. Ans. \$196. nearly.

135. To find the amount of an anniity or pension, left unpaid any number of years, allowing simple interest upon each sum from the time it becomes due.

Let A be the annuity, then, at the expiration of one year, A becomes due, and at the end of the second year the interest of the first annuity is Ar; also at the end of this year the principal sum due is 2A, and its interest at the end of the third year is $2Ar$, &c. hence, at the end of n years, the sum due is

$nA + rA + 2rA + \&c. \ldots (n-1)rA$, $\therefore$ $M = nA + \frac{1}{2}\{n(m-1)rA.\}$

When the pensions are payable half yearly,

$M = nA + n(2n - 1)\cdot\frac{1}{4}Ar$, and if pensions payable quarterly

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Hence true discount is charged on any sum when the difference between the sum and its present worth is taken.

$$M = nA + 2n\cdot(4n - 1)\cdot\tfrac{1}{16}Ar.$$

1. If an annuity of \$70 be forborne five years, what will it amount to at 5 per cent.? [\$385.

$$M = nA + \frac{n.(n-1)}{2}rA = 5\times70 + \frac{5\times4}{2}\times\cdot05\times70 = 350 + 35 =$$

2. Required the same as in the last question, supposing half-yearly payments to be made. $M = nA + n\cdot(2n-1)\frac{1}{4}Ar = 350 + 5\times9\times70\times.0125 = \$389.375$. If the payments are made quarterly, the amount will be \$391.56$\frac{1}{4}$

136. To find the present value of an annuity to continue a certain number of years, allowing simple interest.

Let $x =$ the present value, then if x and the annuity at the same rate of interest amount to the same sum, they are of equal value. Now, $x + nrx =$ amount of x in n years at rate r; also $nA + \frac{n\cdot(n-1)}{2}rA =$ amount of the annuity. $\therefore x + nrx = nA + \frac{n\cdot(n-1)}{2}rA$, and $x = \frac{2nA + n\cdot(n-1)\cdot rA}{2}$ $=$ the present value.

1. What is a pension of \$30 per annum, for 5 years, worth in ready money, at $4\frac{1}{2}$ per cent.?

$$x = \frac{300 + 30\times30\times\cdot045}{2 + \cdot45} = \$133\cdot469 = \$133.47$$

2. What is the present worth of \$50 per annum, payable quarterly for 6 years, at 5 per cent? Ans. \$263.94

In the equation $x = \frac{2nA + n\cdot(n-1)rA}{2(1+nr)}$, if n be infinite, x is infinite, which leads us to suppose that for a limited annuity to continue for ever, an infinitely great sum ought to be paid; a conclusion which shews the necessity of estimating the value of an annuity upon different principles.

On Compound Interest.

137. Interest is a certain sum paid for the use of money for any stated period, and when the interest of this money is regularly received, the money, or principal, is said to be at simple interest. But when, instead of being regularly received, it is allowed to go to the increase of the principal, then the interest of the whole is called compound interest.

138. An annuity is a yearly income, or pension.

139. The present value of an annuity is that sum which, if put out at compound interest, shall amount to sufficient to pay the annuity at the time it becomes due.

140. Let P be the principal, or sum put out to compound interest, r the fraction which expresses the rate of interest per cent.*

*ct. A the amount at the end of n years, the interest being paid yearly; then the following Theorems may be established by means of logarithms.

141. Log A = log. P + $n \times$ log (1 + r). Theorem 1.

For since $1, at the end of the first year, because 1+r, and that the amount is increased each year in the same ratio, we have, by the rule of proportion,

$1 : 1+r :: P : P(1+r)$ = amount of P at end of first year.
$1 : 1+r :: P(1+r) : P(1+r)^2$ = ...second year.
$1 : 1+r :: P(1+r)^2 : P(1+r)^3$ = ...third year.

So that, at the end of n years, the amount is $P(1+r)^n$.

Hence $A = P(1+r)^n$; and, taking the logarithm,

$\text{Log } A = \log P + n \times \log (1+r)$. From which we deduce

$\text{Log } P = \log A - n \times \log (1+r)$.

$$\text{Log } (1+r) = \frac{\log A - \log P}{n}; \text{ and } n = \frac{\log A - \log P}{\log (1+r)}$$

Any three of the quantities A, P, r, n, being found, the fourth may therefore be found.

142. Let $A = mP$, then $n = \dfrac{\log m}{\log (1+r)}$. Theorem 2.

For, in this case, $mP = P(1+r)^n$.

Divide by P, then $m = (1+r)^n$.

Take the logarithm, $\log m = n \times \log (1+r)$; $\therefore n = \dfrac{\log m}{\log (1+r)}$.

By means of this Theorem we ascertain the period or number of years in which a sum of money would double, treble, &c. or amount to m times itself, when put out at compound interest, at r rate per cent.

143. Suppose the interest to be paid half yearly, and at the same time converted into principal, then will $\log A = \log P + 2n \times \log (1 + \frac{1}{2}r)$. For in this case, $2n$ must be substituted for n, and $\frac{1}{2}r$ for r. Hence, at the end of n years, $A = P(1+\frac{1}{2}r)^{2n}$; and, taking the logarithm, $\log A = \log P + 2n \times \log (1+\frac{1}{2}r)$.

144. Suppose now, that besides the interest being converted into principal at the end of every year, the sum P is at the same time invested in capital; then the amount A, at the end of n years, will be $\dfrac{PR(R^n-1)}{R-1}$, (if $R = 1+r$.) In this case the principal P is put out for n, $n-1$, $n-2$, &c. years, in succession; the amount

* That is, the fraction which expresses the ratio of the interest to the principal. Let the interest, for example, be 5 or 6 per cent. Then this fraction r will be $(\frac{5}{100} = \frac{1}{20})$, or $(\frac{6}{100} = \frac{3}{50})$.

therefore is the sum of the several amounts of P put out for n, $n-1$, $n-2$, &c. years;

$\therefore A = P(1+r)^n + P(1+r)^{n-1} + P(1+r)^{n-2} + \&c. \ldots + P(1+r)$

$= (\text{if } 1+r=R) \quad PR^n + PR^{n-1} + PR^{n-2} + \&c. \ldots + PR$

$= P(R^n + R^{n-1} + R^{n-2} + \&c. \ldots + R) = P \times$ (geom. prog. first term R common ratio R) $= \dfrac{P(R^{n+1}-R)}{R-1} = \dfrac{PR(R^n-1}{R-1}$.

Otherwise, to find the amount of a given sum in any number of years at compound interest. Art. 142.

Let r represent the interest of \$1 for one year, and put \$1+r $=R=$ the amount in one year.

Then \$1 : R :: R : $R^2 =$ the amount in 2 years,

\$1 : R :: R^2 : $R^3 =$ the amount in 3 years.

Therefore R^n is the amount of \$1 in n years, and consequently the amount of \$$p$ is pR^n, $\therefore$ calling the amount a, we have $\log a = \log p + n \log R$, and $\log p = \log a - n \log R$.

Cor. 1. $\text{Log } R = \dfrac{\log a - \log p}{n}$, and $n = \dfrac{\log a - \log p}{\log R}$.

Therefore any one of the quantities a, p, R, n, may be found from having the others given; and $\therefore$ Cor. 2, if $a = mp$, then

$$n = \frac{\log mp - \log p}{\log R} = \frac{\log m + \log p - \log p}{\log R} = \frac{\log m}{\log R}.$$

145. If the interest, instead of being due yearly, is supposed to become due half-yearly, quarterly, or after any other given period, then n, of course, instead of representing years, represents some number of those periods, r being the interest for one period.

1. How much would \$300 amount to in 4 years at 4 per cent. per annum compound interest?

Here $p = 300$, $R = 1 + \frac{4}{100} = 1.04$ and $n = 4$;

$\therefore \log a = \log p + n \log R = \log 300 + 4 \log 1\cdot04 = 2\cdot5452545$; the number answering to which in the tables is $350\cdot95\frac{3}{4}$, Ans.

2. How much money must be placed out at compound interest to amount to \$1000 in 20 years, the interest being 5 per cent.?

Here $a = 1000$, $R = 1 + \frac{5}{100} = 1\cdot05$, and $n = 20$.

$\therefore \log p = \log a - n \log R = 1000 - 20 \log 1\cdot05 = 2\cdot576214$, the number answering to which is $376\cdot89$, Ans.

3. At what interest must \$300 be placed out, to amount to \$350, 95·7 in 4 years? Here $p = 300$, $a = 350\cdot957$, and $n = 4$;

$\therefore \log R = \dfrac{\log a - \log p}{n} = \dfrac{\log 350\cdot957 - \log 300}{n} = \cdot0170333$, the number answering to which is $1\cdot04$, and 4.

$\therefore r = \cdot04$, and $\cdot04 \times 100 = 4$, the rate per cent. = Ans.

4. In how many years will \$400 amount to \$540 at 4 per cent.

compound interest? Here $p=400$, $a=540$, & $R=1+\frac{4}{100}=1\cdot04$.

$$\therefore n=\frac{\log a-\log p}{\log R}=\frac{\log 540-\log 400}{\log 1\cdot04}=\frac{.1303338}{.0170333}=7\cdot65 \text{ years.}$$

5. What will \$600 amount to in 6 years at $4\frac{1}{2}$ per cent. compound interest, supposing the interest to be receivable half yearly?

Here $p=600$, $n=12$, and $R=1+\frac{2\cdot25}{100}=1\cdot0225$;

$\therefore \log a=\log p+n \log R=\log 600+12 \log 1\cdot0225=2\cdot8941109$ the number answering to which is \$783·63, the amount Ans.

6. In what time will a sum of money double itself at 5 per cent. compound interest? Here $m=2$, and $R=1\cdot05$;

$$\therefore n=\frac{\log m}{\log R}=\frac{\log 2}{\log 1\cdot05}=\frac{\cdot3010300}{\cdot0211893}=14\cdot206=14\tfrac{1}{5} \text{ years.}$$

146 Again find the amount of a given sum at compound interest. Let $r=$ the interest of 1 \$ for a year,

$P=$ the sum lent, $n=$ the number of years.

Then at the end of one year, $P(1+r)$ is due, and this forms the principal for the second year; therefore, at the end of the second year $P(1+r)\times(1+r)$ or $P\cdot(1+r)^2$ is due, by Prob. I. in simple interest. This forms the principal for the third year, therefore at the end of the third year $P\cdot(1+r)^2\times(1+r)$ or $P\cdot(1+r)^3$ becomes due, and so on for any number of years; hence, at the expiration of n years the amount (M) will be $P\cdot(1+r)^n$.

If P, r, M, begiven to find n, we have $\log M=\log P\cdot(1+r)^n$ $=\log P+n\log(1+r)$. $\therefore n=\dfrac{\log M-\log P}{\log(1+r)}$.

1. Find the amount of \$450 at 5 per cent. per annum for 3 years. $M=P\cdot(1+r)^n=450(1+\cdot05)^3=450(1\cdot05)^3$
$=450\times1\cdot157625=\$520\cdot93+$ Ans.

2. What is the amount of \$500 at $4\frac{1}{2}$ per cent. for 4 years?
Ans. \$590·57 cts. 3m.

If interest be payable half yearly, $M=P(1+\frac{1}{2}r)^{2n}$
quarterly, $M=P(1+\frac{1}{4}r)^{4n}$.

3. Required the same as in Ex. 1, supposing the interest to be paid half-yearly. Ans. \$521·86 +.

147. *Prob.* To find the amount of an annuity forborne any number of years, at compound interest. Let A be the annuity or sum due at the end of the first year.

Then \$ 1 : $(1+r)$\$: : A : $A\cdot(1+r)=$ interest due at end of second year $\therefore$ $A+A(1+r)=$ sum due at the end of 2d year.

In the same way, $1:1+r::A+A(1+r):A(1+r)+A(1+r)^2=$ the interest due at the end of the 3d year.

$\therefore A+A(1+r)+A(1+r)^2=$ sum due at the end of the 3d year.

And at the expiration of (n) years the amount $=$ $A\{1+(1+r)+(1+r)^2+\ldots(1+r)^{n-1}\}$. Hence $M=A\frac{(1-r)^n+1}{r}$

1. What will be the amount of an annuity of \$30 per annum, payable yearly, for 4 years, at 5 per cent.?

$$M=30.\frac{(1.05)^4-1}{.05}=\frac{6.465187}{.05}=\$129.30+$$

If half yearly payments are made, If quarterly,

$$M=\frac{(1+\frac{1}{2}r)^{2n}-1}{r.}A. \qquad M=\frac{(1+\frac{1}{4}r)^{4n}-1}{r}\cdot A.$$

148. *Prob.* To find the present value of an annuity to be paid (n) years, allowing compound interest.

Let $x=$ present value of the annuity A; then $x(1+r)^n=$ amount of x in n years, and $\frac{(1+r)^n-1}{r}\cdot A=$ amount of A in the same time by Art. 146.

$\therefore x\cdot(1+r)^n=\frac{(1+r)^n-1}{r}\cdot A$, and $x=\frac{(1+r)^n-1}{r(1+r)^n}\cdot A=$ present value required.

If the number of years be infinite, $(1+r)^n$ is infinite, and $x=\frac{A}{r}=$ value of the perpetuity \$A per annum. If half-yearly payments are made, $x=\frac{(1+\frac{1}{2}r)^{2n}-1}{r(1+\frac{1}{2}r)^{2n}}\cdot A$, and $x=\frac{(1+\frac{1}{4}r)^{4n}-1}{r(1+\frac{1}{4}r)^{4n}}.A$ when the annuity is payable quarterly.

1. What is the present worth of an annuity or pension of \$30 per annum for 5 years at 4 per cent.?

$$\text{Present worth}=\frac{(1\cdot04)^5-1}{\cdot04\times(1\cdot04)^5}\times30=\frac{1-\frac{1}{(1\cdot04)^5}}{\cdot04}\cdot30=133\cdot554.$$

2. If an estate in fee be sold for \$7250, in ready money, what is the yearly rent, allowing 4 per cent. to the buyer?

Here n is infinite; therefore present worth $=\frac{A}{\cdot04}$, or $7250=\frac{A}{\cdot04}$

Therefore A$=$\$290$=$ yearly value of the estate.

3. A freehold estate be purchased for \$9159, what is the yearly rent, allowing the purchaser $3\frac{1}{2}$ per ct for his money? Ans 320.56

149. To find the present value of an annuity to commence at the expiration of (p) years, and to continue (q) years.

Let A be the annuity; then the present worth of A to begin immediately, and to continue ($p+q$) years $=\frac{(1+r)^{p+q}-1}{r\cdot(1+r)^{p+q}}\cdot A$

Also the present worth of the same annuity to continue (p) years from the present time $=\frac{(1+r)^p-1}{r\cdot(1+r)^p}\cdot A$; $\therefore$ the present value required $=\left\{\frac{(1+r)^{p+q}-1}{r(1+r)^{p+q}}-\frac{(1+r)^p-1}{r\cdot(1+r)^p}\right\}\cdot A$, $=$ present worth.

1. In what time will a sum of money double and treble itself, at five per cent. compound interest? By 196, (since $r=\frac{1}{20}$,)

If $m=2$, then time of doubling $=\frac{\log 2}{\log 1.05}=\frac{.3010300}{\cdot 0211893}=14.2$ years.

If $m=3$, then time of trebling $\frac{\log 3}{\log 1\cdot 05}=\frac{\cdot 3771213}{\cdot 0211893}=22\cdot 5$ years.

2, Supposing the interest to be paid half-yearly, what will be the amount of \$500 in 8 years, at 5 per cent. compound interest?

Here $P=500$, $r=\frac{1}{20}$, $1+\frac{1}{2}r=1\cdot 025$, $n=8$. By 137, $\log A=\log P+2n=\log(1+\frac{1}{2}r)=\log 500+16\times\log(1\cdot 025)=2\cdot 8705525=\log 742\cdot 25$, Ans.

3. Suppose a person to place out annually \$100 for 10 successive years, and suffer the whole to accumulate at the rate of five per cent. compound interest. What sum would he have at the end of the tenth year? Here $P=100$, $R=1\cdot 05$, $n=10$.

By 144, $A=\frac{PR(R^n-1)}{R-1}=\frac{105\{(1\cdot 05)\}^{10}-1}{\cdot 05}=2100\{(1\cdot 05)\}^{10}-1$

Now $\log(1\cdot 05)^{10}=10\times\log 1\cdot 05=\cdot 2118930=\log 1\cdot 6289$; $\therefore(1\cdot 05)-1=\cdot 6289$. Hence $A=2100\times\cdot 6289=\$1320$.

ON Vanishing Fractions.

150. Vanishing fractions, and other similar expressions, are such as in certain cases become equal to $\frac{0}{0}$; which symbol, though apparently nugatory or of no value, must not be rejected as useless, being of frequent occurrence in several algebraical and fluxional investigations, where it will often be found to denote some fixed quantity, or to be 0, or infinite, according to the nature of the question.*

Thus, if a be made to represent the first term of any regular geometrical series, r the ratio, or common multiplier, and n the number of terms, we shall have, by the rule in the Art. Geom.

$$\frac{a-ar^n}{1-r}=a+ar+ar^2+ar^3+ar^4+\ldots ar^{n-2}+ar^{n-1},$$

* The idea of fractions of this kind first originated about the year 1702, in a contest between *Varignon* and *Rolle*, two members of the Academy of Sciences at Paris, concerning the principles of the Differential Calculus, of which *Rolle* was a strenuous

where the left hand member of the equation is an universal expression for the sum (s) of the series on the right hand, whatever may be the values of a, r, and n; as will appear by dividing the numerator by the denominator.

Let therefore the ratio, or multiplier r be taken $=1$, in which case its power, r^n, or 1^n, will also $=1$, and the expressions for the sum will be $S=\frac{a-a}{1-1}=\frac{0}{0}$. But when $r=1$, the original series becomes of the form $S=a+a+a+a+$&c.....to n terms, of which the sum is, evidently, $=na$; and therefore in this case it follows that $\frac{0}{0}=na$.

And in the same way it might be shown that this symbol is the representative of various other quantities, according to the nature of the expression from which it is derived. But it will be here sufficient to observe, that the true value of any fractional expression of this kind may be obtained, as follows.

Rule I. If both the terms of the given fraction be rational, divide each of them by their greatest common measure; then, if the hypothesis which is found to reduce the original expression to the form $\frac{0}{0}$, be applied to the result, it will give the true value of the fraction in the state under consideration.

2. When any part of the fraction is irrational, observe what the unknown quantity is equal to when the numerator and denominator both vanish, and put it $=$ that quantity $+$ or $-$ i; then, if this be substituted for the unknown quantity, and the roots of the surds be extracted, to a sufficient number of places, the result, when i is put $=0$, will give the true value of the fraction.

1. Find the value of the fraction $\frac{x^2-a^2}{x-a}$, when x is equal to a.

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opposer. Among other arguments against the truth of this doctrine, which had then been recently introduced, he proposed an example of drawing a tangent to a certain curve, at the point where the two branches intersect each other; and as the fractional expression for the subtangent, according to that method, had both its numerator and denominator equal to 0, he regarded such a result as absurd, and adduced it as a proof of the fallacy of this mode of solution. But the mystery was soon afterwards explained by *John Bernoulli*; and, upon the renewal of the dispute, still further by *Saurin*, who showed that $\frac{0}{0}$, in the case here mentioned, had a real value. *See Montucla,* Histoire des Mathematiques, Vol. III, p. 114.

Here, if we put $x=a$, there will arise $\frac{a^2-a^2}{a-a}=\frac{0}{0}$.
But, by division, $\frac{x^2-a^2}{x-a}=x+a$; and if x be now put $=a$, we shall have $\frac{a^2-a^2}{a-a}=2a$; whence $\frac{0}{0}$, or the given fraction, in its vanishing state, is $=2a$.

2. Find the value of the fraction $\frac{x-x^5}{1-x}$, when x is equal to 1.

Here, as before, if we put $x=1$, there will arise $\frac{1-1}{1-1}=\frac{0}{0}$; but, by division, $\frac{x-x^5}{1-x}=x+x^2+x^3+x^4$; where, if x be now put $=1$, we shall have the result $=4$, which is therefore in this case, the true value of the symbol $\frac{0}{0}$

3. Find the value of the expression $y=\frac{b(x-\sqrt{ax})}{x-a}$, when x is equal to a. Here, if x be taken $=a+e$, according to the rule, we shall have $y=\frac{b\{a+e-\sqrt{(a^2+ae)}\}}{e}$. And by extracting the square root of a^2+ae, and then dividing by e,
$y=b\{\frac{1}{2}+\frac{1}{2\cdot4}\left(\frac{e}{a}\right)-\frac{3}{2\cdot4\cdot6}\left(\frac{e}{a}\right)^2$ &c.$\}$, whence, putting the indeterminate quantity $e=0$, there will arise $y=\frac{1}{2}b$, which is the true value of the expression in the case proposed.

4. Find the value of the fraction $\frac{x^m-a^m}{x-a}$, when $x=a$.

Here $\frac{x^m-a^m}{x-a}=(x-a)^{m-1}+ma(x-a)^{m-2}+\frac{m(m-1)}{1\cdot2}a^2(x-a)^{m-3}$
$+\frac{m(m-1)(m-2)}{1\cdot2\cdot3}a^3(x-a)^{m-4}+\ldots\ldots+ma^{m-1}$.

And if we now take $x=a$, both in the original expression and in the quotient, there will arise $\frac{0}{0}=ma^{m-1}$; where it is to be observed, that by this hypothesis, all the terms of the quotient will vanish, except the last; whence the undeveloped fraction, which, in the supposed case, takes the form $\frac{0}{0}$, has for its true value the definite quantity ma^{m-1}.

5. Let there be taken, as another example of this kind, the following equation $y=\frac{P(x-a)^m}{Q(x-a)^n}$; where P and Q are supposed to be certain functions or combinations of x, which do not become 0 for the same value of x.

Then taking $x=a$, the expression, according to this hypothesis,

will become of the form $\frac{P\times 0}{Q\times 0}=\frac{0}{0}$. But by considering the indices m, n, of the proposed fraction, under each of the relations $m>n$, $m=n$, $m<n$, we shall have by division the three following results; $y=\frac{P(x-a)^{m-n}}{Q}$, $y=\frac{P}{Q}$, $y=\frac{P}{Q(x-a)^{n-m}}$

And consequently, by now taking $x=a$, there will arise

$$y=\frac{P\times 0}{Q},\ y=\frac{P}{Q},\ y=\frac{P}{Q\times 0}$$

Whence, agreeably to the subjoined note, the value of the symbol $\frac{0}{0}$, in this case, will be nothing, finite or infinite, according to the conditions above mentioned.

1. What is the value of $\frac{a\sqrt{ax-x^2}}{a-\sqrt{ax}}$ whe $x=a$.

Put $x=a+e$, then expunging x, $\frac{a\sqrt{(ax)}-x^2}{a-\sqrt{ax}}=$

$\frac{a\times(a^2+ae)^{\frac{1}{2}}-(a+e)^2}{a-(a^2+ae)^{\frac{1}{2}}}=\frac{a\times a+\frac{1}{2}e\ \&c.-a^2-2ae\ \&c.}{a-a-\frac{1}{2}e\ \&c.}=$

$\frac{a^2+\frac{1}{2}ae\&c.-a^2-2ae\&c.}{-\frac{1}{2}e}=\frac{-\frac{3}{2}ae}{-\frac{1}{2}e}=\frac{3ae}{e}=3a$, the value of the (fraction.

2. What is the value of $\frac{\sqrt{[2a^3x-x^4]}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$, when $x=a$.

Let the fraction $=y$, and put $x=a-e$, then

$y=\frac{\sqrt{\{2a^3\times(a-e)-(a-e)^4\}}-a\sqrt[3]{(a^3-a^2e)}}{a-\sqrt[4]{\{a\times(a-e)^3\}}}$. But

$\sqrt{\{2a^3\times(a-e)-(a-e)^4\}}=\sqrt{(2a^4-2a^3e-a^4+4a^3e}=(a^4+2a^3e)^{\frac{1}{2}}$ $=a^2+ae$ &c. Also $a\sqrt[3]{(a^3-a^2e)}=a\times(a-\frac{1}{3}e)=a^2-\frac{1}{3}ae$ &c.

And $\sqrt[4]{\{a\times(a-e)^3\}}=\sqrt[4]{(a^4-3a^3e)}=a-\frac{3}{4}e$ &c.

Whence $y=\frac{a^2+ae\&c.-a^2+\frac{1}{3}ae\&c.}{a-a+\frac{3}{4}e\&c.}=\frac{\frac{4}{3}ae}{\frac{3}{4}e}=\frac{16a}{9}$.

3. Let $\frac{a\sqrt[3]{(4a^3+4x^3)}-ax-a^2}{\sqrt{(2a^2+2ax)}-x-a}=y$, what is its value when $x=a$. Let $a-e=x$. And expunging x,

$\frac{a\sqrt[3]{\{4a^3+4\times(a-e)^3\}}-a^2+ae-a^2}{\sqrt{\{2a^2+2\times(a-e)^2\}}-x-a}=y$. But $a\sqrt[3]{\{4a^3+4\times(a-e)^3\}}$ $=a\times(8a^3-12a^2e+12ae^2)^{\frac{1}{3}}=2a^2-ae+\frac{1}{2}e^2$&c. And $\sqrt{(2a^2+2x^2)}$ $=(4a^2-4ae+2e^2)^{\frac{1}{2}}=2a-e+\frac{ee}{4a}$&c.

$$y = \frac{2a^2 - ae + \frac{1}{4}e^2 \&c. - 2a^2 + ae}{2a - e + \frac{e^2}{4a} \&c. - 2 + e} = \frac{+\frac{1}{4}e^2}{+\frac{e^2}{4a}} = \frac{2ae^2}{e^2} = 2a.$$

Here, if I had gone no farther than the first power of e, it is evident by inspection that all the terms would have vanished, by which nothing could have been concluded.

If e remains at last in the numerator, the value of the fraction is 0; and if e remains in the denominator, the fraction is infinite. But if all the terms vanish out of both numerator and denominator, the series must then be carried to more places, to have a solution.

Note. In addition to the above article, it will here be proper to show the signification of some other operations and symbols, which often occur in the solution of problems; as, for instance,

1. If 0 be multiplied or divided by any finite quantity, the product or quotient will be 0. Thus $0 \times a = 0$, or $\frac{0}{a} = 0$.

2. If any finite quantity be divided by 0, the quotient will be infinite; and if it be divided by an infinite quantity, the quotient will be 0. Thus $\frac{a}{0} = \infty$; and $\frac{a}{\infty} = 0$.

3. Adding or subtracting any finite quantity to or from an infinite quantity, makes no alteration in it. Thus $\infty \pm x = \infty$.

4. It is also to be observed, that the greater any number or quantity, a is, when taken positively, the less is $-a$;

Thus, -1 is > -2; $-3 < -4$; and so on.

5. That 0 added to or subtracted from any quantity makes it neither great nor less; that is, $a + 0 = a$, and $a - 0 = a$.

Also, if nought be multiplied or divided by any quantity, both the product and quotient will be nought; because any number of times 0, or any part of 0, is 0; that is, $0 \times a$, or $a \times 0 = 0$, and $\frac{0}{a} = 0$.

6. From the last property it likewise follows, that nought divided by nought is a finite quantity of some kind or other. For since $0 \times a = 0$, or $0 = 0 \times a$, it is evident from the common rule of

1. Find $\frac{x^3 - a^3}{x - a}$; $\frac{a\sqrt{ax - x^2}}{a - \sqrt{ax}}$; and $\frac{\sqrt{x} - \sqrt{a} + \sqrt{(x-a)}}{\sqrt{(x^2 - a^2)}}$, the value of each fraction when x is equal to a. Ans. $3a^2$, $3a$, and $\frac{1}{\sqrt{2a}}$.

2. Find the value of $\frac{1 - x^n}{1 - 1}$, or $\frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2}$, when x is equal to 1. Ans. n, or $\frac{1}{2}\{n(n+1)\}$.

The Decomposition of Rational Fractions into their equivalent Simple Fractions.

151. This problem being of considerable utility in the Integral Calculus, and other branches of analysis, where it is of frequent occurrence, it will be here proper to lay before the reader such of the more simple methods of effecting the decomposition required, as can be employed without the assistance of fluxions; which, as far as regards most of the useful cases of this kind, are the following:

RULE I. Let the fraction $\frac{N}{D}$, that is to be decomposed, have at least one dimension less in its numerator than in the denominator; to which state it can always be reduced, when necessary, by division.

2. Find all the simple factors of which the denominator (D) is composed, either by trial, or by putting the sums of its terms $=0$, and then determining the roots r, r', r'', &c. of the equation so formed, in the usual way.

3. Put A, A′, A″, &c. for the numerators of the several partial fractions into which the given fraction is resolvable; and let $x-r$, $x-r'$, $x-r''$, &c. be their denominators; then, in the case where the factors, thus found, are all unequal, we shall have $\frac{N}{D}=\frac{A}{x-r}+\frac{A'}{x-r'}+\frac{A''}{x-r''}+\frac{A'''}{x-r'''}+$ &c. taken to as many terms as there are factors in the denominator of the proposed fraction.

4. Reduce these partial fractions to a common denominator, (which will be the same as D), and add their numerators together; then comparing the coefficients of the several powers of a, in the numerator of this new fraction, with those of the like powers of x in the numerator of the proposed fraction, we shall obtain the necessary equations for determining the values of A, A′, A″, &c. as required.*

1. Convert the rational fraction $\frac{1+x}{x-x^2}$ into its equivalent simple fractions.

* The process here followed may be employed for all rational fractions, whose denominators are resolvable into unequal simple factors; but as this cannot be generally effected when the highest power of the unknown quantity exceeds the fourth degree, we are necessary limited in the application of the rule, in consequence of the imperfertion which still attends the general solution of equations.

Here the factors of the denominators $x-x^2$ being x and $1-x$, we shall have $\frac{1+x}{x-x^2}=\frac{A}{x}+\frac{A'}{1-x}$; or $\frac{1+x}{x-x^2}=\frac{A+(A'-A)x}{x-x^2}$

And consequently, by equating the coefficients of the like terms of the two numerators,

$$A=1 \text{ and } A'-A=1, \text{ or } A'=1+A=1+1=2;$$

whence $\frac{1+x}{1-x^2}=\frac{1}{x}+\frac{2}{1-x}=$ the fractions required.

2. Convert the rational fraction $\frac{1+x^2}{x-x^3}$ into its equivalent simple fractions.

Here the factors of the denominator $x-x^3$ being x, $1-x$, and $1+x$, we shall have $\frac{1+x^2}{x-x^3}=\frac{A}{x}+\frac{A'}{1-x}+\frac{A''}{1+x}$; or, by reducing the fractions to the same denominator,

$\frac{1+x^2}{x-x^3}=\frac{A(1-x^2)+A'(x+x^2)+A''(x-x^2)}{x-x^3}$; whence $1+x^2=A+(A'+A'')x+(-A+A'-A'')x^2$. And, by equating the coefficients of the like terms, $A=1$, $A'+A''=0$, and $-A+A'-A''=1$.

From which equations we have $A=1$, $A'=1$, and $A''=-1$.

$\therefore\ \frac{1+x^2}{x-x^3}=\frac{1}{x}+\frac{1}{1-x}\ \frac{1}{1+x}$ the simple fractions.

3. It is required to convert the rational fraction

$\frac{1-x}{1-4x+2x^2}$, or its equal $\frac{\frac{1}{2}-\frac{1}{2}x}{x^2-2x+\frac{1}{2}}$, into its partial fractions.

Here, by putting the denominator $x^2-2x+\frac{1}{2}=0$, the two roots of the equation will be found to be $1+\frac{1}{2}\sqrt{2}$, and $1-\frac{1}{2}\sqrt{2}$, and consequently the factors are $x-(1+\frac{1}{2}\sqrt{2})$, and $x-(1-\frac{1}{2}\sqrt{2})$; whence $\frac{\frac{1}{2}-\frac{1}{2}x}{x^2-2x+\frac{1}{2}}=\frac{A}{x-(1+\frac{1}{2}\sqrt{2})}+\frac{A'}{x-(1-\frac{1}{2}\sqrt{2})}$; or, by reduction, $\frac{\frac{1}{2}-\frac{1}{2}x}{x^2-2x+\frac{1}{2}}=\frac{Ax-(1+\frac{1}{2}\sqrt{2})A+A'x-(1-\frac{1}{2}\sqrt{2})A'}{x^2-2x+\frac{1}{2}}=$

$\frac{-(1+\frac{1}{2}\sqrt{2})A-(1-\frac{1}{2}\sqrt{2})A'+(A+A')x}{x^2-2x+\frac{1}{2}}$, where, putting

$-(1+\frac{1}{2}\sqrt{2})A-(1-\frac{1}{2}\sqrt{2})A'=\frac{1}{2}$, and $A+A'=-\frac{1}{2}$, we shall have, from these equations, $A=-\frac{1}{4}$, and $A'=-\frac{1}{4}$; therefore

$\frac{\frac{1}{2}-\frac{1}{2}x}{x^2-2x+\frac{1}{2}}=-\frac{\frac{1}{2}}{x-(1+\frac{1}{2}\sqrt{2})}\ \frac{\frac{1}{4}}{x-(1-\frac{1}{2}\sqrt{2})}$, as required.*

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*In cases of this kind, where the highest term of the equation, representing the denominator of the given fraction, has any coefficient prefixed to it, the numerator and denominator must be each

RULE II. When the factors of the denominator of the given fraction are all equal, or some of them equal and others unequal, the decomposition of it into simple factors cannot be effected in the way before pointed out, but must now be be taken of the form

$$\frac{N}{D}=\frac{B}{(x-p)^n}+\frac{B'}{(x-p)^{n-1}}+\ldots+\frac{B^{[n-1]}}{x-p}+\frac{A}{x-r}+\frac{A'}{x-r'},\ \&c.$$

where p represents the root of one of the equal factors, contained in D, and r, r', &c. those of the unequal factors.*

In which case, as in the former, if all the partial fractions, so formed, be brought to a common denominator, and the resulting fraction be reduced, when necessary, so as to have the same denominator D as the proposed fraction, the comparison of the coefficients of the like terms, as before, will give the values of the numerators sought.

1. Required to convert the rational fraction $\frac{1+x}{(1-x)^2}$ into its equivalent simple fractions. Here, according to the rule, we have $\frac{1+x}{(1-x)^2}=\frac{B}{(1-x)^2}+\frac{B'}{1-x}=\frac{B+B'(1-x)}{(1-x)^2}=\frac{B+B'-B'x}{(1-x)^2}$; and by equating the coefficients of the homologous terms, $B'=-1$, and $B+B'=1$, or $B=2$; whence

$$\frac{1+x}{(1-x)^2}=\frac{2}{(1-x)^2}-\frac{1}{1-x},$$ the simple fractions required.

2. It is required to convert the rational fraction $\frac{1-5x}{(1-x)(1+x)^3}$ into its equivalent simple fractions. Here, according to the above rule,

$$\frac{1-5x}{(1-x)(1+x)^3}=\frac{B}{(1+x)^2}+\frac{B'}{1+x}+\frac{A}{1-x}=$$
$$\frac{B\{(1+x)(1-x)\}+B'\{(1+x)^2(1-x)\}+A(1+x)^3}{(1-x)(1+x)^3}=$$

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divided by this coefficient, as above, in order to obtain the true factors; which equal division, it is evident, will not alter the value of the fraction.

* That the former rule cannot hold, in the case where the denominator of the given fraction has equal factors, is obvious; for if we take, for instance, $\frac{a+bx}{[x-p]^2}=\frac{B}{x-p}+\frac{B'}{x-p}$, the two partial fractions, on the right hand side of the equation, would form in fact but one equation, $\frac{B+B'}{x-p}$; from which no useful conclusion can be derived.

$$\frac{B(1-x)+B'(1-x^2)+A(1+2x+x^2)}{(1-x)(1+x)^2}=$$

$\frac{B+B'+A+(2A-B)x+(A-B')x^2}{(1-x)(1+x)^2}$; whence, by putting $B+B'+A=1$, $2A-B=-5$, and $A-B'=0$, we shall have, from these equations, $B=3$, $B'=-1$, and $A=-1$; therefore

$$\frac{1-5x}{(1-x)(1+x)^2}=\frac{3}{(1+x)^2}-\frac{1}{1+x}-\frac{1}{1-x},$$ are the simple fractions.

Note. When the factors of the denominator of the given fractions are few in number, we can always find the partial fractions into which it is convertible by the preceding methods; but as the calculation, in other cases, becomes more laborious, it will be here proper to show, that any one of the numerators of these fractions may be deduced immediately from N and D, independently of the rest, as follows:

RULE III. 1. When the factors of the denominator of the given fraction are all unequal, or of the form

$\frac{N}{D}=\frac{A}{x-r}+\frac{A'}{x-r'}+\frac{A''}{x-r''}+\frac{A'''}{x-r'''}$, &c. take that which constitutes the denominator of the simple fraction that is to be determined, and let S denote the product of all the rest of the factors; then if the root or value of x (found by putting the factor thus taken $=0$) be substited for x in the formula $\frac{N}{S}$, it will give the numerator of that fraction; and the same rule will hold for all the rest.

2. If the factors of the denominator are equal, or some equal and others unequal, as in the form

$$\frac{N}{D}=\frac{A}{x-r}+\frac{A'}{x-r}+\text{\&c.}\ldots\ldots+\frac{B}{(x-p)^n}+\frac{B'}{(x-p)^{n-1}}+\text{\&c.}$$

let S denote the product of all the factors in the denominator of the given fraction, except one, as before; then find the simple fractions, due to the unequal factors, by the first part of the rule, and for the unequal factors proceed as follows:

1. $B=\frac{N}{S}$, taking $x-p=0$, or $x=p$, put $P=\frac{N-BS}{x-p}$; then
2. $B'=\frac{P}{S}$, taking $x-p=0$, or $x=p$, put $Q=\frac{P-B'S}{x-p}$; then
3. $B''=\frac{Q}{S}$, taking $x-p=0$, or $x=p$, put $R=\frac{Q-B''S}{x-p}$; then
4. $B'''=\frac{R}{S}$, taking $x-p=0$, or $x=p$, &c. &c., observing when

the factor $x-p$, in any case of this kind, becomes $x+p$, to take x. in these formulæ, $=-p$, instead of $+p$; and if it be simply x, or $x\pm 0$, take $x=0$; which operations being performed, the sum of the fractions thus obtained, together with the former, will give all the simple fractions into which the given fraction is resolvable.

1. It is required to convert the rational fraction $\frac{1+x^2}{x+x^3}$ into its equivalent simple fractions. Here, the factors of the denominators being x, $1-x$ and $1+x$, we shall have, as in Example 2, Rule 1, $\frac{1+x^2}{x(1-x)(1+x)}=\frac{A}{x}+\frac{A'}{1-x}+\frac{A''}{1+x}$; whence, for x, or $x-0$, the first factor, $A=\frac{N}{S}=\frac{1+x^2}{(1-x)(1+x)}=\frac{1+x^2}{1-x^2}=1$, x being $=0$; and for $1-x$, the second factor, $A'=\frac{N}{S}=\frac{1+x^2}{x(1+x)}=\frac{1+x^2}{x+x^2}=1$, x being $=1$; also, for $1+x$, the third factor, $A''=\frac{N}{S}=\frac{1+x^2}{x(1-x)}=\frac{1+x^2}{x-x^2}=-1$, x being $=-1$. $\therefore \frac{1+x^2}{x-x^3}=\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}$, the simple factors, as before.

2. Convert the rational fraction $\frac{1}{x^3(1-x)^2(1+x)}$ into its equivalent simple fractions. Here, according to the rule, we have $\frac{1}{x^3(1-x)^2(1+x)}=\frac{A}{1+x}+\frac{B}{x^3}+\frac{B'}{x^2}+\frac{B''}{x}+\frac{C}{(1-x)^2}+\frac{C'}{1-x}$, where B, B', B'', are the numerators of the first set of factors, and C, C', those of the latter. Hence, for the unequal factor $1+x$, we have $A=\frac{N}{S}=\frac{1}{x^3(1-x)^2}=\frac{1}{x^3-2x^4+x^5}=-\frac{1}{4}$, x being $=-1$. And for x^3, x^2, and x, or $(x-0)^3$, $(x-0)^2$, and $x-0$, in the first set of equal factors. $B=\frac{N}{S}=\frac{1}{(1+x)(1-x)^2}=\frac{1}{1-x-x^2+x^3}=1$, x being $=0$.

Put $P=\frac{N-BS}{x-0}=\frac{x+x^2-x^3}{x}=1+x-x^2$; then $B'=\frac{P}{S}=\frac{1+x-x^2}{1-x-x^2+x^3}=1$, x, as before, being $=0$. Put $Q=\frac{P-B'S}{x-0}=\frac{2x-x^2}{x}=2-x^2$; then $B''=\frac{Q}{S}=\frac{2-x^2}{1-x-x^2+x^3}=2$, x, as before, being $=0$. Again, for $(1-x)^2$, and $1-x$, in the second set of equal

factors, $C=\frac{N}{S}=\frac{1}{x^3(1+x)}=\frac{1}{x^3+x^4}=\frac{1}{2}$, x being $=1$.

Put $R=\frac{N-CS}{1-x}=\frac{1-\frac{1}{2}x^3-\frac{1}{2}x^4}{1-x}=1+x+x^2+\frac{1}{2}x^3$; then

$C'=\frac{R}{S}=\frac{1+x+x^2+\frac{1}{2}x^3}{x^3+x^4}=\frac{7}{4}$, x, as above, being $=1$. Therefore

$$\frac{1}{x^3(1-x)^2(1+x)}=\frac{1}{x^3}+\frac{1}{x^2}+\frac{2}{x}+\frac{1}{2(1-x)^2}+\frac{7}{4(1-x)}\ \frac{1}{4(1+x)}.$$

1. Let $\frac{5-x}{1-x^2}$, $\frac{1-x-4x^2}{x-4x^2+3x^3}$, and $\frac{13+21x+2x^2}{1-5x^2+4x^4}$, each be reduced to its equivalent simple fractions.

$$\frac{2}{1-x}+\frac{3}{1+x};\ \frac{1}{x}+\frac{2}{1-x}+\frac{1}{1-3x};\ \frac{1}{1+x}\ \frac{6}{1-x}+\frac{2}{1+2x}+\frac{16}{1-2x}.$$

2. Let $\frac{1+5x+3x^2}{(1+x)^2(1+2x)^2}$ and $\frac{2+28x-x^2}{x(1+x)(2-x)(1-2x)^2}$, each be reduced to its equivalent simple fractions.

Ans. $-\frac{1}{(1+x)^2}\ \frac{5}{1+x}\ \frac{3}{(1+2x)^2}+\frac{10}{1+2x}$.

Ans. $\frac{1}{x}+\frac{1}{1+x}+\frac{1}{2-x}+\frac{2}{1-2x}+\frac{14}{(1-2x)^2}$.

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Of Recurring Series.

152 A Recurring Series is a rank, or progression of quantities, so constituted that each succeeding term is formed from the sums, or differences of some multiples of a certain number of the preceding terms, taken continually in the same way. Thus, if $1+6x+12x^2+48x^3+120x^4+$&c. be the given series, we shall have the third term $12x^2=x\times 2d$ term $+6x^2\times$ 1st term; the 4th term $48x^3=x\times$ 3d term $+6x^2\times$ 2d term; and so on. In which case the compound expression x, $6x^2$, or simply 1, 6, is called the scale of relation of the several terms.

And if $1+4x+6x^2+11x^3+28x^4+63x^5+$&c. be the given series, we shall have the 4th term $11x^3=2x\times 3d$ term $-x^2\times$2d term $+3x^3\times$ 1st term; the 5th term $28x^4=2x\times$ 4th term $-x^2\times$3d term $+3x^3\times$2d term; and so on.

Where $2x$, $-x^2$, $+3x^3$, or 2, -1, $+3$, is the scale of relation, the numbers composing it being the multipliers by which the several terms of the series, or their coefficients are produced.* These

* This branch of the science appears to have been first treated on by *Demoivre*, in his *Miscellanea Analytica*, and his *Doctrine of Chances;* and has since been considerably improved by *Euler*,

series, which are called recurrent, from its being necessary, in their formation, to have recourse to the prdceding terms, arise from the expansion of certain fractional expressions of the form $\frac{a}{1+a'x}$; $\frac{a+bx}{1+a'x+b'x^2}$; $\frac{a+bx+cx^2}{1+a'x+b'x^2+c'x^3}$ &c. in which the terms of each denominator, (abating unity,) with their signs changed, constitute the scale of relation of the term of the series, produced from the development of the fraction in question.

Thus, if A, B, C, &c. be made to denote the first, second, third &c. terms of the series, arising from the expansion of the first of these fractions, we shall have $\frac{a}{1+a'x}=a-a'Ax-a'Bx^2-a'Cx^3-a'Dx^4-\ldots a'Px^{n-1}$; where it is evident that each term, beginning with the second, is formed by multiplying that immediately preceding it by $-a'x$; in which case $-a'x$, or simply a', is the scale of relation of the several terms of the series. Also, if the second of these fractions be converted in like manner into a series, we shall have $\frac{a+bx}{1+a'x+b'x}=a-(a'A-b)x-(b'A+a'B)x^2-(b'B+a'C)x^3-$ &c. where each term, beginning with the third, is determined by means of the two that preeede it, multiplied respectively by $-b'x$, $-a'x^2$; which quantities, therefore, or simply $-b'$, $-a'$, constitute in this case the scale of relation of the terms of the series. And in the same manner it will be found, by expanding the third of the above fractions, that the coefficient of any power of x, in this case, depends upon the three that precede it, multiplied respectively by $-c'$, $-b'$, $-a'$; and therefore that these form the scale of relation of the series; and so on, for other fractions of this kind.

In all of which cases, it may be observed, that the coefficient of any term in the series, depends upon as many of the preceding coefficients as there are units in the highest power of x in the denominator; and that the law above mentioned only takes place after as many terms as are contained in the numerator.

Problem I. To find whether a given infinite series, of the form $a+bx+cx^2+dx^3+ex^4+$&c. be a recurring series; and if so, what is the scale of relation of its several terms.

Rule I. Suppose the series to be a recurring one, and that the scale of relation consists of two terms, which let be denoted by

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and other later writers, who have more fully developed its principles, as well as extended its application.

a', b'. Then, by beginning at the third term, we shall have, from what has been before observed, $ba'+ab'=c$, $ca'+bb'=d$, $da'+cb'=e$, &c. where the coefficients a, b, c, d, e, &c. being known, the numeral values of a', b', may be determined from the first two of these equations. And if the coefficient of each term, after the second, be equal to the sum of the two that precede it, multiplied respectively by the numbers thus found, the series is a recurring one, having a', b', for its scale of relation; but otherwise not.

2. If the scale of relation be supposed to consist of three terms, let them be denoted by a', b', c'. Then, by beginning at the 4th term, and employing the same principles as in the former instance we shall have $ca'+bb'+ac'=d$, $da'+cb'+bc'=e$, $ea'+db'+cc'=f$, &c. where, as before, the coefficients a, b, c, d, e, f, &c. being known, the numeral values of a', b', c', may be determined from the three first of these equations. And if the coefficient of each term, after the third, be equal to the sum of the three that precede it, multiplied respectively by these numbers, the series is a recurring one, having a', b', c', for its scale of relation; but otherwise not.

3. In like manner we may also proceed, to find whether a series, which is supposed to have a scale of relation of four terms, be a recurring one or not; and so on.*

1. It is required to find whether $1+x+5x^2+13x^3+41x^4+121x^5$

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* 1. Divide unity, or 1, by the sum (S) of the proposed series as far as to two terms in the quotient, which let be $p+qx$, and call the remainder $S''x^2$.

2. Divide in like manner the given series S by the latter S', as far as to two terms in the quotient, which let be $p'+q'x$, and call the remainder $S'x^2$.

3. Proceed in the same manner with S' and S'', calling the first two terms of the quotients $p''+q''x$, and the remainder $S'''x^2$, and so on; then, if the series be recurrent, the operation will at length terminate, otherwise not. And in the former of these cases, the scale of relation will consist of as many terms as there are divisions; the values of which may then be found, as in the rule given in the text. And when these values are known, we shall have for the sum of the series,

$$S=\cfrac{1}{p+qx+\cfrac{x^2}{p'+q'x+\cfrac{x^2}{p''+q''x+\cfrac{x^2}{p'''+q''x}}}}\ \text{\&c., where the}$$

rational fraction, answering to this continued fraction, may be determined in the usual way. See Continued Fractions.

$+365x^8+$&c. be a recurring series, and if so, what is its scale of relation.

Here $a=1$, $b=1$, $c=5$, $d=13$, &c.; whence, supposing the scale of relation to consist of two terms a', b', we shall have, by the first formula, $1\times a'+1\times b'=5$, and $5a'+1\times b'=13$, which equations, when resolved in the usual manner, give $a'=2$, and $b'=3$. And as the coefficient of each term after the second will be found upon trial to be equal to the sum of the two that precede it, multiplied respectively by 2 and 3, the series is a recurring one, having 2, 3, for its scale of relation.

2. It is required to find whether $1+2x+3x^2+3x^3+7x^4+5x^5+15x^6+9x^7+31x^8+17x^9+63x^{10}+$ &c. be a recurring series, and if so, what is its scale of relation.

Here $a=1$, $b=2$, $c=3$, $d=3$, $e=7$, $f=5$, &c.

And if two terms, a', b' only, be taken for the scale of relation, their values, as determined from the first formula, will not be found to succeed. But if the scale of relation be supposed to consist of the three terms a', b', c', we shall have, by the second formula, $3a'+2b'+c'=3$, $3a'+3b'+2c'=7$, and $7a'+3b'+3c'=5$, which equations, being resolved in the usual manner, give $a'=-1$, $b'=2$, and $c'=2$. And, as the coefficient of each term, after the third, will be found to be equal to the sum of the three that precede it, multiplied respectively by -1, 2 and 2, the series is a recurring one, having -1, 2, 2 for its scale of relation.

3. Required the scale of relation of the infinite recurring series $a+(a+d(x+(a+2d)x^2+(a+3d)x^3+(a+4d)x^4+$&c. Ans. 2, -1.

4. Required the scale of relation of the infinite recurring series $1+4x+10x^2+12x^3+10x^4+24x^5+66x^6+100x^7+$&c.

Ans. 2, -3, 4.

Problem II. Having given an infinite recurring series, of the form $a+bx+cx^2+dx^3+ex^4+$&c. to find its sum.

Rule I. 1. Find the scale of relation, when it is not obvious, by the former rule. Then, if the scale consists of only two terms, as $+a'$, $+b'$, the radix or sum (S) of the series, infinitely continued, will be $S=\dfrac{a+(b-aa')x}{1-a'x-b'x^2}$.

2. When the scale of relation consists of three terms, $+a'$, $+b'$ $+c'$, the sum of the series, infinitely continued, will be

$$S=\frac{a+(b-aa')x+(c-ba'-ab')x^2}{1-a'x-b'x^2-c'x^3}$$

3. In like manner, when the scale of relation consists of four terms, $+a'$, $+b'$, $+c'$, $+d'$, the sum of the series, infinitely continued, will be

$$S=\frac{a+(b-aa')x+(c-ba'-ab')x^2+(d-ca'-bb'-ac')x^3}{1-a'x-b'x^2-c'x^3-d'x^4}$$, and so

on, for any greater number of terms, observing to change the signs of the quantities a', b', c', &c. which are here supposed positive, from — to +, when any of them are negative.

1. Required the sum of the infinite recurring series $1+6x+12x^2+48x^3+120x^4+$ &c. the scale of relation of its coefficients being 1, 6. Here $a=1$, $b=6$, $a'=1$, and $b'=6$; by the rule,

$$S=\frac{a+(b-aa')x}{1-a'x-b'x^2}=\frac{1+(6-1\times1)x}{1-x-6x^2}=\frac{1+5x}{1-x-6x^2}$$ = sum of the series.

2. Required the sum of the infinite recurring series $1+4x+6x^2+11x^3+28x^4+63x^5+$ &c. the scale of relation of the coefficients being 2, —1, 3. Here $a=1$, $b=4$, $c=6$, $a'=2$, $b'=-1$, and $c'=3$; whence by the 2d case of the rule,

$$S=\frac{a+(b-aa')x+(c-ba'-ab')x^2}{1-a'x-b'x^2-c'x^3}=\frac{1+(4-2)x+(6-8+1)x^2}{1-2x+x^2-3x^3}=$$

$$\frac{1+2x-x^2}{1-2x+x^2-3x^3}=\frac{(1+x)^2-2x^2}{(1-x)^2-3x^3}$$ = sum of the series. And in the same way we may obtain the sum of any series of this kind, when the scale of relation consists of four, five, or more terms.

1. Required the sum of the infinite recurring series $2-4x+8x^2-16x^3+32x^4-$&c. the scale of relation being —2.

2. Required the sum of the infinite recurring series $1+2x+3x^2+4x^3+5x^4+6x^5$, &c. the scale of relation of the coefficients being 2, and — 1.

3. Required the sum of the infinite recurring series $1-\frac{3}{x}+\frac{5}{x^2}-\frac{7}{x^3}+\frac{9}{x^4}$, &c. the scale of relation of the coefficients being —2, —1. Ans. $\frac{2}{1+2x}$, $\frac{1}{(1-x)^2}$. and $\frac{x(x-1)}{(x+1)^2}$ Ans.

4. Required the sum of the infinite recurring series $1+3x+7x^2+13x^3+25x^4+51x^5+103x^6+$&c. the scale of relation of the coefficients being +2, —1, +2.

5. Required the sum of the infinite recurring series $1+8x+27x^2+64x^3+125x^4+216x^5+343x^6+$&c. the scale of relation being 4, —6, 4, —1 $\frac{1+x+2x^2}{(1-x)^2-2x^3}$. Ans. $\frac{1+4x+x^2}{(1-x)^4}$.

Rule III. When the sum of a limited number of terms only, of a recurring series, is required, it may be determined by first finding the sum of all the terms of the series, following the term proposed, and then subtracting the result from the whole sum, found by the former rule.

1. Let it be required to find the sum of n terms of the recurring series $1+2x+3x^2+4x^3+5x^4\ldots\ldots nx^{n-1}$.

Here, the general term being nx^{n-1}., the next terms will be $[n+1]x^n+[n+2]x^{n+1}+[n+3]x^{n+2}+$&c. where the scale of relation of the coefficients is 2, —1, as before.

Whence, by substituting $(n+1)x^n$, and $(n+2)x^{n+1}$, for a and b in the first case of the rule, and making $a'=2$, and $b'=-1$, we shall have

$$\frac{(n+1)x^n+(n+2)x^{n+1}-2(n+1)x^{n+1}}{1-2x+x^2}=\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}=\text{sum of}$$

the series $(n+1)x^n+(n+2)x^{n+1}+(n+3)x^{n+2}+(n+4)x^{n+3}+$ &c. continued *ad infinitum.* But from the answer to example 2, in the former rule, it appears that the sum of the whole infinite series $1+2x+3x^2+4x^3+5x^4+\&c.=\frac{1}{(1-x)^2}$.

Whence $\frac{1}{(1-x)^2}-\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}=\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}=$An.

1. Sum n terms of the infinite recurring series $1+4x+14x^2+46x^3+146x^4+454x^5\ldots\ldots(2.3^n-2^n)x^n$, the scale of relation of the coefficients being 5, -6.

Ans. $\frac{1-x-(2\cdot3^n-2^n)x^n+(4\cdot3^n-3\cdot2^n)x^{n+1}}{1-5x+6x^2}$.

2. Required the sum of n terms of the infinite recurring series $1+0x+2x^2+2x^3+6x^4+10x^5+22x^6+42x^7+86x^8\ldots\ldots\frac{2^n\pm2}{3}x^n$, using the upper sign when n is an even number, and the under sign when it is odd; and observing that the scale of relation is 1, 2.

Ans. $\frac{1-x-\frac{1}{3}(2^n\pm2)x^n-\frac{1}{3}(2^n\mp4)x^{n+1}}{1-x-2x^2}$.

3. Required the sum of n terms of the series $1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7\ldots\ldots\frac{2n+3\pm1}{4}x^n$, using the upper sign when n is an even number and the under sign when it is odd; and observing that the scale of relation is 1, 1, -1.

Ans. ——————————

Indeterminate Analysis.

153. Indeterminate Analysis is the resolution of a species of equations where the number of the unknown quantities is greater than the number of equations, and where the answers are restricted to whole and positive numbers.

1. If a whole number be subtracted from a whole number, the remainder is a whole number.

2. Any multiple of a whole number is a whole number.

Problem. Given the indeterminate equation $ax\pm by=c$, to find all the possible values of x and y in integer numbers, suppose the numbers a, b, c, prime to each other.

Find the value of one of the unknown quantities in terms of the other. Thus, if the equation be $ax-by=c$, then

$$x=\frac{by+c}{a};\text{ or, } ax+by=c,\text{ then } x=\frac{by-c}{a}.$$

Increase the values of y successively by the coefficient of x, and diminish the values of x successively by the coefficient of y, and all the values of x and y will be obtained.

Rule I. Of the two quantities (viz. the divisor and dividend,) divide that which has the greatest coefficient by the least, the divisor by the remainder, and thus continue the process until a remainder appear in which the coefficient of y is unity, observing in each step to take the quotient figure such, that the coefficient of y in the multiple of the divisor may be the nearest possible to the coefficient of y in the dividend, whether greater or less than the dividend, and to subtract the less from the greater, whether the least be the multiple or the divisor.

2. Divide the absolute number in the remainder by the coefficient of x, and if the sign of the dividend be negative, the remainder is the value of y; but if it be affirmative, subtract the remainder from the diviser, and the last remainder will be the value of y.

154. For since $x=\frac{by+c}{a}$, $\frac{by+c}{a}$ must be a whole number; and since $\frac{ay}{a}$ is a whole number, let $\frac{ky}{a}$ be the nearest multiple of $\frac{ay}{a}$ to $\frac{by}{a}$; then $\frac{ky}{a}$ is a whole number, consequently $\frac{by+c}{a}-\frac{ky}{a}=\frac{(b-k)y+c}{a}$ is a wh. No. let $b-k=l$, then $\frac{ly+c}{a}$ is a whole number, and the coefficient l is less than either a or b.

Again, let $\frac{my+d}{a}$ be such a multiple of $\frac{ly+c}{a}$ as will make the coefficient m of x the nearest possible to the coefficient a of $\frac{ay}{a}$; then since $\frac{my+d}{a}$ is a whole number, the difference $\frac{ay}{a}-\frac{my+d}{a}=\frac{(a-m)y-d}{a}$ is a wh. Let $a-m=n$, then $\frac{ny-d}{a}$ must also be a whole number, and n will be less than either l or a; but l is less than either a or b; therefore n is much less than either a or b. It is evident by continuing this process, the operation will be similar to that of finding a common measure, and will be performed without the use of the denominator a, and that as the coefficient of y is continually diminishing it may be reduced to unity.

1. Given $19x-14y=11$, to find the least affirmative values of x and y. Here $x=\frac{14y+11}{19}$

$$\begin{array}{l}
14x+11)19y\quad(1 \\
\quad\quad\quad\;\; \underline{14y+11} \\
\quad\quad\quad\;\; 5y-11)14y+11(3 \\
\quad\quad\quad\quad\quad\quad\quad\; \underline{15y-33} \\
\quad\quad\quad\quad\quad\quad\quad\quad y-44
\end{array}
\qquad\qquad
\begin{array}{l}
19)44(2 \\
\quad\;\underline{38} \\
\text{Rem } 6
\end{array}$$

And because the sign of the dividend 44 is negative, the remainder 6 is the least value of y

$$\text{Therefore } x=\frac{14y+11}{19}=\frac{84+11}{19}=\frac{95}{19}=5;$$

Whence the least value of y is 6, and the least value of x is 5.

155. If all the values of x and y were required in this example, it would be impossible to give them, as the number of answers would be infinite. For by increasing the value of x by the coefficient of y, and increasing the value of y by the coefficient of x, We have $\left\{\begin{array}{l} x=5,\ 19,\ 33,\ 47,\ 61,\ 75,\ 89,\ 103, \&c. \\ y=6,\ 25,\ 44,\ 63,\ 82,\ 101,\ 120,\ 139, \&c. \end{array}\right\}$

3. Given $19x=14y-11$, to find x and y in whole numbers. Here $y=\frac{19x+11}{14}=x+\frac{5x+11}{14}$, and $a=14$.

$$\begin{array}{l}
14)33(2 \\
\quad\;\;28 \\
\text{Rem. } 5
\end{array}
\qquad
\begin{array}{r}
5x+11 \\
\underline{3} \\
15x+33 \\
\underline{14x\quad\;\;} \\
x+33
\end{array}$$

Now $\frac{33}{14}$ gives a remainder $=5$; $\therefore 14-5=9=$ the least value of x, and since in this example the less x is, the less will y be, we have, by substitution,

$\frac{271+11}{14}=13=$ the least value of y, the number of solutions being indefinite.

4. Given $17x+29y=573$, to find all the positive values of x and y in whole numbers. Here $x=\frac{573-29y}{17}=-\frac{29y-573}{17}$.

$$\begin{array}{l}
17y)29y-573(2 \\
\quad\;\;\underline{34y\quad\quad\quad\quad\quad} \\
\quad\;\; 5y+573(17y\quad\quad(3 \\
\quad\quad\quad\quad\quad\;\underline{15y+1719} \\
\quad\quad\quad\quad\quad\; 2y-1719)5y+573(2 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\underline{4y-3438} \\
\quad\quad\quad\quad\quad\;\text{Rem. Divi. } y+4011
\end{array}
\qquad
\begin{array}{l}
17)4011(235 \\
\quad\;\;\underline{34} \\
\quad\;\;\;61 \\
\quad\;\;\;\underline{51} \\
\quad\;\;\;101 \\
\quad\;\;\;\;\underline{85} \\
\quad\;\;\;16 \text{ Rem.}
\end{array}$$

Then because the sign of 4011 in the dividend is affirmative, subtract the remainder 16 from 17, and the remainder is 1, which

is the least value of y. Whence $x = \frac{29y-573}{17} = 32$. Then diminishing x by the coefficient of y, in the original equation, and increasing y by the coefficient of x, We have $x = 32, 3,$
and $y = 1, 18.$

5. Given $21x+17y = 2000$, to find all the positive values of x and y in whole numbers. Here $x = -\frac{17y-2000}{21}$

$17y-2000)21y\ (1$	$21)10000(476$
$\quad 17y-2000$	84
$\quad 4y+2000)17y-2000(4$	160
$\quad\quad 17y+8000$	147
And because the dividend $\quad y-10000$	130
-10000 is negative, the remainder 4	126
is the least value of y.	4 Rem.

Whence $x = \frac{17y-2000}{21} = \frac{1932}{21} = 92$, the greatest value of x.

Then diminishing x by the coefficient of y, and increasing y by the coefficient of x, we have $x = 92, 75, 58, 41, 24, 7,$
and $y = 4, 25, 46, 67, 88, 109.$

6. Given $21x+17y = 2000$, or $4x+17p = 11$; $p = x+y-117$,
$4q + p = 3$; $q = x+4p-2$. Hence $p = 3-4q$:
$\therefore x = \frac{11-17p}{4} = 17q - 10$; $y = \frac{2000-21x}{17} = 130 - 21q$.

By taking q, 6, 5, 4, 3, 2, 1, we shall obtain the values of x and y, given above. The manner of proceeding, by considering the values of p and q on the right, will be easily understood; the method consists in continually dividing by the coefficients in each equation, (rejecting the whole numbers in the absolute terms,) until the coefficients of one of the quantities is 1.

7. Given $19x-14y = -11$, to find x and y in whole numbers.

$19x-14y = -11$, or	Hence $q = 4r-1$
$5x-14p = -11$; $p = -x+y$,	$p = 5r-1$
$5q-4p = -1$; $q = x-2p+2$,	$x = 14r-5$
$q-4r = -1$; $r = -q+p$;	$y = 19r-6$

where r may be taken at pleasure, if $r = 1$, $x = 9$, $y = 13$, &c.

As the work on the right hand is merely explanatory, it may be omitted, as in the 7th example.

8. Given $21x + 5y = 800$, to find x and y in whole numbers..

$21x+5y = 800$, $x+5p = 0$.	This method of solving Inde-
Hence $x = -5p$; $y = 160+21p$	terminate Equations is nearly
if $p = -1$, $x = 5$, $y = 139$,	the same as that given by Euler
if $p = -2$, $x = 10$, $n = 118$.	in his Algebra, vol. ii.;

9. A company of men and women club together for the payment of a reckoning. Each man pays C25, and each woman C16,

and it is found that all the women together have paid C1 more than the men. How many men and women were there? Let $x =$ the number of women, and $y =$ the number of men. Then the women will have expended $16x$, and the men $25y$, so that

$16x = 25y + 1$, whence $x = \frac{25y+1}{16}$.

$x = \frac{25y+1}{16} = \frac{175+1}{16} =$

Then $\begin{cases} x = 11, 36, 61, 86, 111, \&c. \\ y = 7, 23, 39, 55, 71, \&c. \end{cases}$

$16y)25+1)2$
$32y$
$7y-1)16y \quad (2$
$14y-2$
$2y+2)7y-0(3$
$6y+6$
$y-7$
or $y=7$

According to the first answer, the women expended, $16 \times 11 = 176$; and the men $25 \times 7 = 175$, according to the second answer, the women spent $16 \times 36 = 576$, and the men $25 \times 23 = 576$.

Obs. When the remainder is negative and not divisible by the coefficient of x, then it is the least value of y; but should the remainder be affirmative, subtract it from the coefficient of x, and it will give the least value of y.

156. Determine *a priori* the number of solutions that the equation $ax + by = c$, will admit of, &c. Let such integral values of x' and y' be found, that we may have $ax' - by' = 1$, which is always possible; then $acx' - bcy' = c$; $\therefore ax + by = acx' - bcy'$, and consequently, we must have $x = cx' - mb$, and $x = ma - cy'$, where m may be any number taken at pleasure, that will make these values of x and y positive integers; but if no such value of m can be found, it will be a proof that the proposed equation is impossible in positive integers, and, on the contrary, as many suitable values of m as can be found, so many solutions will the equation admit of and no more. Hence, because we must have $cx' > mb$, and $cy' < ma$, the whole number of solutions will be expressed by the differences between the integral parts of $\frac{cx'}{b}$, and $\frac{cy'}{a}$; because as m must be less than the first of these fractions, and greater than the second, the difference of their integral parts will evidently express the number of different values of m, except when $\frac{cx'}{b}$ is a complete integer; in which case, since $m < \frac{cx'}{b}$, the difference of the integral parts would be one more than the number of different values of m; therefore, when the expression $\frac{cx'}{b}$ is an integer, we must consider $\frac{b}{b}$ as a fraction, and reject it therefrom; but this must not be done with the other quantity $\frac{cy'}{a}$ because $m > \frac{cy'}{a}$.

157. The values of x and y, in the equation $ax-by=1$, can be easily found by dividing b by a, (b being supposed greater than a) a by the remainder of this division, and so on, as in the method of finding the greatest common measure of numbers. Then put down the last quotient but one, multiply it by the next quotient and add 1, multiply this by the next quotient and add the preceding sum, and so on, until all the quotients are taken; the last sum will be the value of x, and the sum preceding that the value of y.

158. We can find the number of solutions that the equation $ax+by=c$ or, $9x+13=2000$, admits of, when the remainder of $\frac{cx'}{b}$ is not less than that of $\frac{cy'}{a}$. For $\frac{cx'}{b}-\frac{cy'}{a}=\frac{c(ax'-by')}{ab}$, and $ax'-by'$ being 1, it is evident that $\frac{c}{ab}$ will in this case represent the number of solutions; this can be exemplified in this example, which falls under this case; for 2000 (c) being divided by 9 and 13, gives 17 for the number of solutions which the equation admits of. But as we cannot tell when the remainder of $\frac{cx'}{b}$ is greater or less than that of $\frac{cy'}{a}$, without first finding the values of x' and y' in the equation $ax'-by'=1$, we do not know, by this method, whether the true number of solutions is found, or whether it be one less than the number of solutions; thus, in the equation $7x+9y=100$, since $100\div(7\times9)=1$, we know that the number of solutions is either 1 or 2; by proceeding according to the usual method we shall find 2 to be the true number. This method, however, may be modified so that we can always ascertain the number of solutions which the equation $ax+by=c$ admits of. Let us suppose that $\frac{cx'}{b}=d+\frac{h}{b}$ and $\frac{cy'}{a}=f+\frac{k}{a}$; hence, by taking the difference, we have $\frac{c(ax'-by')}{ab}=d-f+\frac{ah-bk}{ab}$. Hence $\frac{c}{ab}=d-f+\frac{ah-bk}{ab}$

or $c=(d-f)ab+ah-bk$, or $ah-bk=c-(d-f)ab$.

Consequenly, the right hand of the equation consists of known quantities, and by determining the least values of h and k, from this equation, we thence know the value of the fractions $\frac{h}{b}$ and $\frac{k}{a}$, and if $\frac{h}{b}$ is an improper fraction; that is, h greater than b, then the number of solutions is *one more* than the number found by dividing c by ab.

1. Let the equation $7x+9y=100$, be proposed to find whether it admits of more than one solution. Here $d=f$ being 1, we have $ah-bk=c-ab$, or $7h-9k=37$; from which the least value of h

is found $=13$; and since $\frac{12}{8}$ is more than 1, the equation admits of two solutions.

2. Given $9x+13y=2000$, to find whether there are more than 17; $\left(\frac{2000}{9\times 13}\right)$ solutions which the equation admits of. Here $d-f=17$, and we have the equation $9h-13k(=c-17ab)=11$: from this we find the least value of h to be 7, and since $\frac{7}{13}$ does not amount to 1, the number of solutions is but 17.

159. Let us suppose $\frac{cx'}{b}=d+\frac{b-1}{b}$; and $\frac{cy'}{a}=d+\frac{1}{a}$; showing the greatest degree of possibility without there being an integral solution, since $b-1$ is the greatest remainder, and 1 the least. Then we have, by subtraction, $\frac{c}{ab}=\frac{ab-a-b}{ab}$, or $c=ab-a-b$; which shows that if c is greater than $ab-a-b$, the equation $ax+by=c$ is possible in integers.

So also if $\frac{cx'}{b}=d+\frac{b-1}{b}$, and $\frac{cy'}{a}=f+\frac{1}{b}$, we shall have $c=ab(d-f)+ab-a-b$; or putting $d-f=d$, $c=ab(d'+1)-(a+b)$, which shows the greatest value of c for any number of solutions; and by putting $\frac{cx'}{b}=d+\frac{1}{b}$, and $\frac{cy'}{a}=f+\frac{a-1}{a}$, we have $c=ab(d'-1)+(a+b)$, for the least value of c for any number of solutions $\{ab(d'+1)-(a+b)\}-\{ab(d'-1)+(a+b)\}=2\{ab-(a+b)\}$, expresses the difference between the greatest and least value of c. Thus, in the equation $8x+9y=559$, 559 is the greatest number for 7 solutions; and in the equation $8x+9y=449$, 449 is the least for 7 solutions.

160. If there be three unknown quantities, and only two equations for determining them, as $ax+by+cz,=d$, and $ex+fy+gz=h$; exterminate one of these quantities in the usual way, and find the values of the other two from the resulting equation, as before. Then if the values thus found be separately substituted in either of the given equations, the corresponding values of the remaining quantities will likewise be determined; thus,

1. Let there be given $x-2y+z=5$, and $2x+y-z=7$, to find the values of x, y, and z.

Here, by multiplying the first of these equations by 2, and subtracting the second from the product, we shall have

$3z-5y=3$, or $z=\frac{3+5y}{3}=1+y+\frac{2y}{3}=wh.$; and consequently, $\frac{2y}{3}$, or $\frac{3y}{3}-\frac{2y}{3}=\frac{y}{3}=wh.=p$, whence $y=3p$. And, by taking $p=1, 2, 3, 4,$ &c. we shall have $y=3, 6, 9, 12, 15,$ &c. and

$z = 6, 11, 16, 21, 26$, &c. But from the first of the two given equations, $x=5+2y-z$; whence, by substituting the above values for y and z, the results will give $x=5, 6, 7, 8, 9$, &c. There fore the first six values of x, y, and z are as below;

$$\left\{\begin{array}{l|l|l|l|l|l} x=5 & 6 & 7 & 8 & 9 & 10 \\ y=3 & 6 & 9 & 12 & 15 & 18 \\ z=6 & 11 & 16 & 21 & 26 & 31 \end{array}\right\}$$

where the law by which they may be continued is sufficiently obvious.

1. Given $3x+5y+7z=100$, to exhibit all the differen- valuesof x, y, and z in integers. Here z cannot be greater than $\frac{100-3-5}{7}=13\frac{1}{7}$; and by proceeding as in Prob. 1,

$$x=\frac{100-5y-7z}{3}=33-y-2z-\frac{2y+z-1}{3};$$

$$\begin{array}{l} 3y \\ 2y+z-1 \\ \hline y-z+1 \end{array}$$

Now, by taking $z=1$, y becomes $=0$, and $x=31$; but this answer is inadmissible, because $y=0$ is not an integer, but by adding 3, the coefficient of x, to this value of y, and subtracting 5, the coefficient of y, from the value of x, we shall obtain another answer, and by repeating this process continually, we shall obtain all the possible values of x and y, for this value of z; and in a similar manner are the values of x and y to be found when $z=2$, &c. when all the possible solutions will be found to be 41 in number, and to be as follow;

$z=1$	$y=$	3	6	9	12	15	18
	$x=$	26	21	16	11	6	1
$z=2$	$y=$	1	4	7	10	13	16
	$x=$	27	22	17	12	7	2
$z=3$	$y=$	3	5	8	11	14	
	$x=$	23	18	13	8	3	
$z=4$	$y=$	3	6	9	12		
	$x=$	19	14	9	4		
$z=5$	$y=$	1	4	7	10		
	$x=$	20	15	10	5		
$z=6$	$y=$	2	5	8	11		
	$x=$	16	11	6	1		
$z=7$	$y=$	3	6	9			
	$x=$	12	7	2			
$z=8$	$y=$	1	4	7			
	$x=$	13	8	3			
$z=9$	$y=$	2	5				
	$x=$	9	4				
$z=10$	$y=$	3					
	$x=$	5					
$z=11$	$y=$	1	4				
	$x=$	6	1				
$z=12$	$y=$	2					
	$x=$	2					

161. To determine the number of solutions that the equation $ax+by+cz=d$ will admit of, two at least of the coefficients a, b, c, being prime to each other.*

* When this is not the case, the proposed equation must be transformed to another that shall have two, at least, of its coefficients prime to each other. Thus, if the equation be $12x+15y+20z=100001$, by transposing $20z$, and dividing by 3, we have $4x+5y=33384-7z+\frac{z-1}{3}$; $\therefore \frac{z-1}{3}$ is an integer, which call u; then $z=3u+1$; whence, by substitution, the proposed equation becomes $12x+15u+20(3u+1)=100001$, which, by transposing

But the number of solutions that the equation $ax + by = c$ will admit of is expressed by the integral parts of $\frac{cx'}{b} - \frac{cy'}{a}$, x' and y' being determined from the equation $ax' - by' = 1$. Therefore, in the equation $ax + by = d - cz$, if we make $z = 1.\ 2.\ 3,\ 4,$ &c. successively, then the number of solutions in the equations

$$\left\{\begin{array}{ll} ax+by=d-c \text{ will be the integral parts of} & \frac{(d-c)x'}{b} - \frac{(d-c)y'}{a} \\ ax+by=d-2c \dots\dots\dots\dots\dots\dots & \frac{(d-2c)x'}{b} - \frac{(d-2c)y'}{a} \\ ax+by=d-3c \dots\dots\dots\dots\dots\dots & \frac{(d-3c)x'}{b} - \frac{(d-3c)y'}{a} \end{array}\right.$$

the sum of which will be the whole number of solutions that the equation admits of, that is, if we take the sum of the integral parts of the arithmetical series

$$\frac{(d-c)x'}{b} + \frac{(d-2c)x'}{b} + \frac{(d-3c)x'}{b} + \frac{(d-4c)x'}{b} + \frac{(d-5c)x'}{b} + \&c.$$

as also of the arithmetical series

$$\frac{(d-c)y'}{a} + \frac{(d-2c)y'}{a} + \frac{(d-3c)y'}{a} + \frac{(d-4c)y'}{a} + \frac{(d-5c)y}{a} + \&c.$$

the difference of the two will be the whole number of integral solutions. Now in each of these series the first and last terms, as also the number of terms, are known; for the general terms being $\frac{(d-cz)x'}{b}$, and $\frac{(d-cz)y'}{a}$, we shall have the extremes by taking the extreme limits of z, that is, $z = 1$, and $z < \frac{d-a-b}{a}$, which last value of z also expresses the number of terms in the series.

If, therefore, we find the sum of the two whole series, and then the sums of the fractional parts in each, by deducting these last sums, each from the corresponding whole sum, the sum of the integral parts of each series will be obtained. In summing the fractional parts, there will be no necessity to go through the whole series, for, as the denominator in each is constant, these fractions will necessarily recur in periods, and the number in each period can never exceed the denominator;* it will therefore only be ne-

the 20, becomes divisible by 3, and we then have $4x+5y+20u=33327$; in which equation x and y have, of course, the same values as in the one proposed, and therefore the number of solutions must be the same; but in this last one value of u may be 0, because $x=3u+1$. See Simpson's Alg. p. 199, Prob. 15.

* This will appear from considering the above series; for, if in the first series d and c be prime to each other, and neither of them

cessary to find the sum of the fractions in one period, and to multiply this sum by the number of periods, in order to get the sum of all the fractions, observing, however, that when there are not an exact number of periods, the overplus fractions must be summed by themselves, which may be readily done, since they will be the same as the leading terms of the first period; it must also be remembered that $\frac{b}{b}$ is to be considered as a fraction in the first series, as in Prob. 2.

1. It is required to determine the number of integral solutions that the equation $3x+5y+7z=100$ will admit of. Ans. 41.

$3x+5y=100-7z$ if we make $z=1, 2, 3, 4 \ldots 13$ in succession, then the number of solutions in the equation $3x+5y=a$ admits of is expressed by the integral parts of $\frac{ap}{3}-\frac{aq}{5}$, p and q being determined from the equation $3p-5q=1$, (where it is evident $p=2$, $q=1$,) $\therefore$ in the equation $3x+5y=100-7z$, if we take $z=1, 2, 3,$ to 13, which is the limit to the value of z, then the number of solutions in the equation $3x+5y=9$, and $3x+5y=16$, &c. will be contained in the integral parts of $\frac{9p}{5}-\frac{9q}{3}$; $\frac{16p}{5}-\frac{16q}{3}$ and the whole number of solutions will be expressed by the difference of the sums of the arithmetical series, viz.

$$\frac{9p}{5}+\frac{16p}{5}+\frac{23p}{5}+\frac{30p}{5}+\frac{37p}{5}\text{\&c.},\ \frac{9q}{3}+\frac{16q}{3}+\frac{23q}{3}+\frac{30q}{3}+\frac{37q}{9}+$$

$$\text{or, } \frac{2.9}{5}+\frac{2.16}{5}+\frac{2.23}{5}+\frac{2.30}{5}+\frac{2.37}{5}+\ \cdots\cdots\cdots\ \frac{2.93}{5}$$

$$\text{and } \frac{1.9}{3}+\frac{1.16}{3}+\frac{1.23}{3}+\frac{1.30}{3}+\frac{1.37}{3}+\cdots\ \cdots \quad \frac{1.93}{3}$$

and consequently the sum of these series are $265\frac{1}{5}$ and 221, respectively. Now as the fractions which remain amount to something, these must be deducted from the above sums. Now

$$\frac{3}{5}+\frac{2}{5}+\frac{1}{5}+\frac{5}{5}+\frac{4}{5}=\frac{15}{5}=3, \text{ and } \frac{0}{3}+\frac{1}{3}+\frac{2}{3}=\frac{3}{3}=1$$

and as the number of terms is 13, the sum of the first period of fractions is $(2\times3+1\frac{1}{5})=7\frac{1}{5}$, and of the second $1\times4+0=4$.

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prime to b, each term will be wholly integral, that is, the fractions will all be 0. If b be prime to d, and not to c, the fractions will be all equal. If b be prime to c, but not to d, then the fractions will recur after the first integral term, which can never lie beyond the bth term; and finally, if a, b, c, be all prime to each other, the series of fractions will always recur after the bth term. Similar observations evidently apply to the second series.

Hence we have $\left.\begin{array}{r}265\frac{1}{3}-7\frac{1}{3}=258\\ 221\ -4\ =217\end{array}\right\}$ $=41$, Ans. the whole number of solutions, as required.

$\overline{41}$Ans.

2. Given the equation $5x+7y+11z=224$, to find the number of solutions which it admits of in integers.

Here the greatest limit of $z<\frac{224-5-7}{11}$ is 19; also in the equation $5x'-7y'=1$, we have $x'=3$, and $y'=2$, also $a=5$, and $b=7$; therefore the two series, of which the sums are required, beginning with the least terms $\frac{(d-19c)x'}{b}$, and $\frac{(d-19c)y'}{a}$, will be $\frac{3.15}{7}+\frac{2.26}{7}+\frac{3.37}{7}+\frac{3.48}{7}+\frac{3.59}{7}+\ldots\ldots\ldots\ \frac{3.213}{7}$,

and $\frac{2.15}{5}+\frac{2.36}{5}+\frac{2.37}{5}+\frac{2.48}{5}+\frac{2.59}{5}+\ldots\ldots\ldots\ \frac{2.213}{5}$;

the common diff. in in the 1st being $\frac{3.11}{7}$, and in the 2d $\frac{2.11}{5}$, and the number of terms in each 19.

Now the sum of the first series is $928\frac{4}{7}$,

and the sum of the second ...... $866\frac{3}{5}$; also the first period of fractions, in the first series, is $\frac{3}{7}+\frac{1}{7}+\frac{6}{7}+\frac{4}{7}+\frac{2}{7}+\frac{7}{7}+\frac{5}{7}=4$, and the first period in the second series, is $0+\frac{2}{5}+\frac{4}{5}+\frac{1}{5}+\frac{3}{5}=2$, $\frac{7}{7}$ being considered as a fraction in the first period, but not $\frac{5}{5}$ in the second.

Hence the number of terms in each series being 19, we have two periods and five terms of the first series $=2\times4+$ the first five fractions $=10\frac{4}{7}$, for the sum of all the fractions, and therefore $928\frac{4}{7}-10\frac{4}{7}=918=$ sum of the integral terms of the first series; also in the second we have three periods and four terms $=3\times2+1\frac{3}{5}=7\frac{3}{5}$, and therefore $866\frac{3}{5}-7\frac{3}{5}-859=$ sum of the integral terms of the second series; whence $918-859=59$ is the whole number of integral solutions. In a similar manner may the number of solutions be obtained, when there are four or more unknown quantities.

3. Required to determine the number of integral solutions that the equation $7x+9y+23z=9999$ will admit of. Ans. 34365.

4. Required to determine the number of integral solutions that the equation $5x+7y+9z=93256$ will admit of. Ans. 13801148.

5. Required to determine the num. of integral solutions that the equation $5x+7y+9z=6665$ will admit of. Ans. 70

6. Required to determine the num. of integral solution that the equation $5x+21y+27z=20000$ will admit of. Ans. 70734

162. To find such a whole number as, being divided by other given numbers a, b, c, shall leave given remainders f, g, h,

Rule I. Call the number that is to be determined, x, the num-

bers by which it is to be divided a, b, c, &c. and the given remainders f, g, h, &c.

2. Subtract each of the remainders from x, and divide the differences by a, b, c, &c. and there will arise

$$\frac{x-f}{a},\ \frac{x-g}{b},\ \frac{x-h}{c},\ \text{\&c.} = \text{whole numbers.}$$

3. Put the first of these fractions $\frac{x-f}{a}=p$, and substitute the value of x as found in terms of p, from this equation, in the place of x in the second fraction.

4. Find the least value of p in this second fraction, by the last problem, which put $= r$, and substitute the value of x, as found in terms of r, in the place of x in the third fraction. 4

5. Find in like manner the least value of r, in this third fraction, which put $= s$, and substitute the value of x, as found in terms of s in the fourth fraction, as before; and so on to the last; when the value of x thus determined will give the whole number

1. To find a number which, divided by 3, 5, 7, and 2, will leave the remainders 2, 4, 6, 0, respectively.

Let the number be x, then $\frac{x-2}{3}$, $\frac{x-4}{5}$, $\frac{x-6}{7}$, and $\frac{x-0}{2}$ are whole numbers. Let $\frac{x-2}{3}=\text{P}$, and $x = 3\text{P}+2$; then $\frac{x-4}{5} = \frac{3\text{P}+2-4}{5} = \frac{3\text{P}-2}{5} = wh.$ subtract it from $\frac{5\text{P}}{5}$, and $\frac{2\text{P}+2}{5} = wh.$ Subtract this from $\frac{3\text{P}-2}{5}$; then $\frac{\text{P}-4}{5} = wh = \text{Q}$, and $\text{P}=5\text{Q}+4$ and $x=15\text{Q}+14$. Again $\frac{x-6}{7} = \frac{15\text{Q}+8}{7} = wh.$, and $\frac{\text{Q}+1}{7} = wh. = \text{R}$, and $\text{Q}=7\text{R}-1$, and $x=105\text{R}-1$. Lastly, $\frac{x-0}{2} = \frac{105\text{R}-1}{2} = wh.$, and $\frac{\text{R}-1}{2} = wh. = \text{S}$, and $\text{R}=2\text{S}+1$. Whence $x = 210\text{S}+104$, the number sought; and putting $\text{S}=0$, the least value of x is 104.

2. Find a whole number which, being divided by 16, 17, 18, 19, 20, will leave 6, 7, 8, 9, 10, remainders.

Let x = number. Then $\frac{x-6}{16}$, $\frac{x-7}{17}$, $\frac{x-8}{18}$, $\frac{x-9}{19}$, $\frac{x-10}{20}$ are whole numbers. Put $\frac{x-6}{16}=\text{P}$, then $x=16\text{P}+6$. Then $\frac{x-7}{17} = \frac{16\text{P}-1}{17} = wh.$ And thence $\frac{\text{P}+1}{17} = wh. = \text{Q}$, and $\text{P}=17\text{Q}-1$, and $x=272\text{Q}-10$. Also $\frac{x-8}{18} = \frac{272\text{Q}-18}{18} = wh.$ and $\frac{2\text{Q}}{18} =$

$wh. = R$, and $Q = 9R$, whence $x = 2448R - 10$.

Again $\frac{x-9}{19} = \frac{2448R-19}{19} = wh.$, and $\frac{-3R}{19} = wh.$, or $\frac{3R}{19} = wh.$ and $\frac{18R}{19} = wh.$ whence $\frac{R}{19} = wh. = S$, and $R = 19S$. Then $x = 46512S - 10$.

Lastly $\frac{x-10}{20} = \frac{46512S-20}{20} = wh.$ and $\frac{12S}{20} = wh. = T$, and $S = 5T$. Whence $x = 232550T - 10$. And if $T = 1$, then the least value of $x = 232550$.

3. To find a number (x) which, being divided by 3, 7, 14, 20, there shall remain 1, 3, 7, 14.

Here $\frac{x-1}{3}, \frac{x-3}{7}, \frac{x-7}{14}, \frac{x-14}{20}$ are whole numbers. Let $\frac{x-1}{3} = P$, and $x = 3P+1$. Then $\frac{x-3}{7} = \frac{3P-2}{7} = wh.$ and $\frac{6P-4}{7} = wh.$ Whence $\frac{P+4}{7} = wh. = Q$, and $P = 7Q - 4$, and $x = 21Q - 11$.

Also $\frac{x-7}{14} = \frac{21Q-18}{14} = wh.$ and $\frac{7Q-4}{14} = wh.$ and $\frac{14Q-8}{14} = wh.$ Whence $\frac{8}{14} = wh.$ which is absurd.

Hence the question is impossible for the three first suppositions; but will hold good for two of them. In which case $x = 21Q - 11$, where the least value of x is 10.

Rule II. When two divisors and their remainders are given, then find two fixed multipliers, M, N, such, that dividing them, $\frac{M}{a}$ leaves 0, and $\frac{M}{b}$ leaves 1 remaining, and $\frac{N}{a}$ leaves 1, and $\frac{N}{b}$ leaves 0 remaining. Then divide $\frac{Mg+Nf}{ab}$, and the remainder is x, the number sought.

Likewise, for three divisors and remainders, find three fixed multipliers, M, N, P, such, that, by dividing them, $\frac{M}{a}$ leaves 1, and $\frac{M}{bc}$ leaves 0 remaining. $\frac{N}{b}$ leaves 1, $\frac{N}{ac}$ leaves 0 remaining. $\frac{P}{c}$ leaves 1, $\frac{P}{ab}$ leaves 0 remaining. Then dividing $\frac{Mf+Ng+Ph}{abc}$, the remainder is x, required; and the like for more quantities.

To prove the truth of this. Since (Case 1,) $\frac{M}{a}$ as also $\frac{N}{b}$ leave 0 by division; therefore $\frac{Mg}{a}$, and $\frac{Nf}{b}$ leave 0. And since $\frac{M}{b}$, as

also $\frac{N}{a}$ leave 1. Therefore $\frac{M-1}{b}$ and $\frac{N-1}{a}$ leave 0. Therefore $\frac{Mg-g}{b}$ and $\frac{Nf-f}{a}$ leave 0; that is $\frac{Mg}{b}$ leaves g, and $\frac{Nf}{a}$ leaves f. Therefore $\frac{Mg+Nf}{b}$ leaves $0+f$, and $\frac{Mg+Nf}{b}$ leaves $g+0$. But since $Ng+Nf$ may exceed ab, and therefore is not the least number; therefore divide by ab, and the remainder is the least number required. And the same way, Case 2, or any other, is proved.

4. Having the cycle of the dominical letter f, and cycle of the moon g, to find the year of the Dionysian period.

Let x be the year sought. Then $\frac{x-f}{28}$ and $\frac{x-g}{19}$ are whole numbers. Here $a=28$, and $\frac{M}{28}=wh.=P$, and $M=28P$. Also $\frac{M-1}{16}=wh.=\frac{28P-1}{19}$; and multiplying by 2, $\frac{56P-2}{19}=wh.$ Also, $\frac{57P}{19}=wh.$ Therefore, $\frac{P+2}{19}=wh.=Q$, and $P=19Q-2$ Whence $M=28\times19(Q-2)=532Q-56$, and if $Q=1$, then $M=476$. Then $\frac{N}{b}=\frac{N}{19}=wh.=P$, and $N=19P$. Also $\frac{N-1}{28}=\frac{19P-1}{28}=wh.$ multiply by 3; then $\frac{57P-3}{28}=wh.$ and $\frac{56P}{28}=wh.$ therefore $\frac{P-3}{28}=wh.=Q$, and $P=28Q+3$. Whence $N=28\times19Q+57$, and if $Q=0$, $N=57$. Therefore $x=$ remainder of $\frac{476g+57f}{532}$, which serves in general for any numbers, f, g. Let $f=10$, $g=12$; then $x=430$.

5. Having the cycle of the Sunday letter f, the golden number g, and indiction h, to find the year of the Julian period.

Here $a=28$, $b=19$, $c=15$, $ab=532$, $ac=420$, $bc=285$, and $abc=7980$. Then $\frac{M}{285}=wh.=P$, and $M=285P$. Also $\frac{M-1}{28}=\frac{285P-1}{28}=wh.$ This at last gives $P=17$ the least; and then $M=4845$. Again $\frac{N}{420}=wh.=P$, and $N=420P$. Also $\frac{N-1}{19}=\frac{420P-1}{19}=wh.$ and $\frac{2P-1}{19}=wh.$ which will give $P=10$, and

N=4200. Lastly, $\frac{P}{532}$=wh.=Q. and P=532Q. Also $\frac{P-1}{15}$ $=\frac{532Q-1}{15}$=wh. and $\frac{7Q-1}{15}$ = wh. at last Q = 13, and P = 6916. Whence the remainder of $\frac{4845f+4200g+6916h}{7980}$ is $=x$. Let f=0, or 28, g=1, h=2; then x=2072.

Cor. When the operation brings out a lesser number divided by a greater, instead of a whole number, the problem is impossible.

6. Given the cycle of the sun 18 the golden number 8 and the Roman indiction 10, to find the year. Ans.1717.

The solar cycle is a period of 28 years, the lunar of 19, and the indiction a period of 15. The year before the christian era was the 9th of the solar cycle, the 1st of the lunar, and the third of the indiction cycle. Wherefore 9, 1, and 3 being severally added to any year x of Christ, and the sums divided respectively by 28, 19, and 15, the remainders will shew the several years of the cycles for that year. But, in the present case proposed, the remainders are 18, 8, and 10; hence then $\frac{x+9-18}{28}$; $\frac{x+1-8}{19}$; and $\frac{x+3-10}{15}$ must be integers; or $\frac{x-9}{28}$; $\frac{x-7}{19}$; and $\frac{x-7}{15}$ integers. Put the first of these equal to the integer m, that is $\frac{x-9}{28}=m$; then $x=28m+9$, a value of x answering the first condition. Write this in the 2d, then $\frac{28m+9-7}{19}$ or $\frac{28m+2}{19}=m+\frac{9m+2}{19}$ = an integer; $\therefore \frac{9m+2}{19}$ or $\frac{18m+4}{19}=m-\frac{m-4}{19}$ = an integer; $\therefore \frac{m-4}{19}=n$ an integer; hence $m=19n+4$; which substituted in the value of x, we have $x=28\times(19n+4)+9=532n+121$ a value of x answering the two first conditions. This value of x being written in the 3d original integer, we have $\frac{532n+121-7}{15}$ or $\frac{532n+114}{15}=35n+7+\frac{7n+9}{15}$ = an integer; hence $\frac{7n+9}{15}$ or $\frac{14n+18}{15}=n+1-\frac{n-3}{15}=$ an integer; $\therefore \frac{n-3}{15}=p$ an integer; $\therefore n=15p+3$; this written in the last value of x, it becomes $x=532\times(15p+3)+121=7980p+1717$, a general expression for the year answering all the three contions, in which p may be either 0, or any whole number. In the

present case, it is evident that the value of p must be nothing, and then $=$ 1717, the year. It is evident that such a combination cannot happen again till the year $9697=7980+1717$, when the value of p is 1; and that the successive years of its happening are found by the continual addition of 7980.

7. A person receiving a box of oranges, consisting of between 100 and 200, observed that when he told them out by 2, 3, 4, 5, and 6 at a time, he had none remaining, but when he told them out by 7 at a time, there remained 6. How many oranges were there in the box? Ans. 180.

If the number is divisible by 4, it is evidently divisible by 2; therefore we need only make $\frac{x}{4}$ $\frac{x}{5}$ $\frac{x}{6}$ and $\frac{x-5}{7}=$ whole numbers. Put $\frac{x}{4}=p$, then $x=4p$, and $\frac{x}{5}=\frac{4p}{5}$. But $\frac{5p}{5}-\frac{4p}{5}=\frac{p}{5}=$ wh.$=r$, or $p=5r$ and $x=20r$; therefore $\frac{x}{6}=\frac{20r}{6}=3+\frac{2r}{6}=$ wh. or $\frac{2r}{6}=\frac{r}{3}=$ wh.$=s$, whence $r=3s$, and $x=60s$; $\therefore$ $\frac{x-5}{7}=\frac{60s-5}{7}$ $=8+\frac{4s-5}{7}=wh$, or $\frac{4s-5}{5}=wh$; whence $\frac{4s-5}{7}\times 2=\frac{8s-10}{7}=$ $s-1+\frac{s-3}{7}=wh$, or $\frac{s-3}{7}=wh.=t$; therefore $s=7t+3$, and $x=$ $420t+180$; whence if $t=0$, $x=180$, the answer.

8. Find the least whole number, which being divided by 11, 19 and 29, shall leave the remainders 3, 5, and 10, respectively.

Another method of solving questions of this nature is to suppose x the given number; then if r, r', r'', &c. be the remainders, and d, d', d'', &c. the divisors, it is evident that $\frac{x-r}{d}$; $\frac{x-r'}{d'}$; $\frac{x-r''}{d''}$, must be whole numbers. To apply this to the above example, we must have $\frac{x-3}{11}$; $\frac{x-5}{19}$; $\frac{x-10}{29}$; whole numbers. Put $\frac{x-3}{11}=p$, then $x=11p+3$, and $\frac{11p-2}{19}$; $\frac{11p-7}{29}$ whole numbers, or $\frac{77p-14}{19}$; $\frac{88p-56}{29}$, or $\frac{p-14}{19}$; $\frac{p-27}{29}$ whole numbers. Put $p=19q+14$, then $\frac{1}{29}\{19q-13\}$ must be a whole number, or $\frac{57q-39}{29}=2q-1-\frac{q+10}{29}=$ whole number. Put $\frac{q+10}{29}=r$; $\therefore$ $q=29r-10$; if $r=1$, $q=19$, $p=19q+14=375$, $x=11p+3=4128$.

9. Find the least whole number, which being divided by 28, 19 and 15, shall leave for remainders, 19, 15, and 11, respectively.

If r, r', r'', denote generally the remainders of a number divided by 28, 19, and 15 respectively, we shall find by proceeding as above, $x=7980c-3135r'+4200r''-1064r'''$, where c may be any number taken at pleasure; supposing r, r', r'', to be 19, 15, 11, as in the ex. we have, $x=7980c-59565+63000-11704=7980c-8269$, if $c=2$, then $x=7691$. The formula exhibiting the value of x is useful for finding the year, when the cycle of the sun, the golden number, and the Roman indiction are given.

Hence $\frac{x+9-18}{28}$ $\frac{x+1-8}{19}$ $\frac{x+3-10}{15}$ or $\frac{x-9}{28}$ $\frac{x-7}{19}$ $\frac{x-7}{15}$ must be whole numbers, we have $x=7980c-28215+29400-7448=7980c-6263$; if $c=1$, $x=1717$ year. As in 6th Ex.

10. It is required to determine in what year of Christ the cycle of the sun was 8, the golden number or cycle of the moon 10, and the cycle of indiction 10. Ans. In the year 1567.*

11. Here $5x+7y+9z=93256$, or $x=\frac{93256-7y-9z}{5}=18651$ $-y-2z+\frac{1-2y+z}{5}=wh.$ or $\frac{1-2y+z}{5}=wh.$ $\therefore$ $\frac{1-2y+z}{5}\times 2=\frac{2-4y+2z}{5}=wh.$ and $\frac{5y}{5}+\frac{2-4y+2z}{5}=\frac{y+2+2z}{5}=wh=r$, and $y=5r-2z-2$; whence $x=18654-7r+z$.

Now it is evident that y will be a positive integer as long as $5r$ is greater than $2z+2$. But the greatest value of z is

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* The entire cycles here mentioned are 28, 19 and 15 years respectively; from which, agreeably to the times of their commencement, the cycle for any particular year may be found as follows:

1. For the sun. Add 9 to any given year of Christ, and divide the sum by 28; then the quotient is the number of years elapsed since his birth, and the remainder is the solar cycle for the given year; and if nothing remains, the cycle is 28.

2. For the moon. Add 1 to the given year of Christ, and divide the sum by 19, then the quotient is the number of lunar cycles elapsed in the interval, and the remainder is the cycle for the given year; and if nothing remains, the cycle is 19.

3. For the indiction. Divide the date of the year by 15, and add 3 to the remainder, and the result will be the indiction.

Whence, by reversing these observations, the year of Christ answering to any particular cycle, may be found.

$=\frac{93256-5-7}{9}=\frac{93244}{9}=10360\frac{4}{9}$, and the greatest value of $2z+2=20720\frac{8}{9}$; hence the greatest value of r, so that y may be positive, is $20720\frac{8}{9}\div5=4144\frac{8}{45}$; therefore all the positive values of y are included between the limits $r=1$, and $r=4144$.

If $r=1$, then $y=5-2z-2=3-2z$, where z can only be $=1$, so that in this case there is but one value of y.

If $r=2$, then $y=10-2z-2=8-2z$, where z may be either 1, 2 or 3, and in this case there will be three Answers.

If $r=3$, then $y=15-2z-2=13-2z$, where z may be either 1, 2, 3, 4, 5, or 6, and in this case there will be six Ans.

If $r=4$, then $z=18-2z$, and here there will be eight Ans.

If $r=5$, then $y=23-2z$, and there will be eleven Ans.

Whence it is evident that the number of answers or values of y, from $r=1$ to $r=4144$, is equal to the sum of the series $1+3+6+8+11\ldots\ldots$ to 4144 terms, which terms being united in pairs, form the arithmetical progression $4+14+24+34+$ &c. to 2072 terms, the sum of which is $\frac{4+10\times2071+4}{2}\times2072=21463848$, the number of the positive values of y. But as some of the corresponding values of x will either be 0, or negative, we must now find the number of those values, which must be deducted from the number found above. It is obvious that x will be a positive integer as long as $18654+z$ is greater than $7r$; that is, x will be positive, for all values of z, as long as $7r$ is not greater than 18654, or r not greater than $\frac{18654}{7}=2664\frac{6}{7}$.

But, if r be $=2665$, then $7r=18655$, and $x=z-1$; where, if $z=1$, then x is $=0$, and there is one value to be deducted.

If $r=2666$, then $x=z-8$; and in this case x is $=0$ when $z=8$, and negative when $z=1$, 2, 3, 4, 5, 6, or 7; consequently there are 8 values to be deducted.

If $r=2667$, then $x=z-15$; and in this case x is $=0$ when $z=15$, and negative when $z=1$, 2, 3, &c. to 14; hence there are 15 values to be deducted.

Whence it is evident that the number of answers to be deducted is expressed by the arithmetical progression $1+8+15+22+$ &c. to $4144-2664=1480$ terms, the sum of which is equal to $\frac{1+7\times1479+1}{2}\times1480=7662700$; hence $21463848-7662700=13801148$, the number of answers required.

7 How many different ways is it possible to pay £100 in crowns, guineas and moidors only. Ans. 70234 different ways.

Let x denote the number of crowns, y the number of guineas, and z the moidores; then $5x+21x+27z=20000$ by the ques-

tion. In this equation, if y and z be supposed $=0$, we shall have $x=\frac{20000}{5}=4000$; and if x and y be $=0$, then $z=\frac{20000}{27}=740$; consequently the greater value of x is 4000, and of z, 740.

But $x=\frac{20000-21y-27z}{5}=4000-4y-5z-\frac{y+2z}{5}=wh.$, or $\frac{y+2z}{5}=wh.=p$; whence $y=5p-2z$, and $y=\frac{20000-21y-27z}{5}=4000+3z-21p$.

By assigning different values to p, in these equations, for the values of x and y, we may find all the possible answers that the question will admit of; but as these answers are too many to be put down here, we shall only show how the number of them may be determined.

Now it is evident that y will be a positive integer as long as $5p$ is greater than $2z$; that is, as long as $5p$ is greater than $\frac{2z}{5}$. But since the greatest value of z is 740, p cannot exceed $\frac{740\times 2}{5}=296$; therefore all the positive values of y are included between the limits $p=0$, and $p=296$.

If $p=0$, then y and z are each $=0$, and $x=4000$; therefore, in this case, the question admits of one answer only.

If $p=1$, then $y=5-2z$; where z may be either 0, 1, or 2; and in this case the question admits of three answers.

If $p=2$, then $y=10-2z$, where z may be either 0, 1, 2, 3, 4, or 5. In this case the question admits of six answers.

By proceeding in this manner for each particular value of p, the number of answers will stand as in the following table;

$p=$	$y=$	Values of z that will give positive values of y.	Number of answers.
0	$0-2z$	0	1
1	$5-2z$	0,1,2	3
2	$10-2z$	0,1,2,3,4,5	6
3	$15-2z$	0,1,2,3,4,5,6,7	8
4	$20-2z$	0,1,2,3,4,5,6,7,8,9,10,	11
5	$25-2z$	0,1,2,3,4,5,6,7,8,9,10,11,12,	13
&c.	&c.	&c.	&c.

Whence it is evident that the number of answers from $p=0$ to $p=296$ will be equal to the series $1+3+6+8+11+13+$&c. continued to 297 terms, which terms (setting aside the first) being united in pairs, we shall have the arithmetical progression $9+19+29+39+$&c. to 148 terms, the sum of which is $=\frac{9+1470+9}{2}$

$\times$ 148 = 110112; and adding (1), the term omitted, we have 110113 for the number of answers, or positive values of y. But in these answers some negative values of x are included, the number of which must be deducted from the number of answers already found, in order to obtain the true number.

Now it is evident that x will be a positive integer as long as $4000+3z$ is greater than $21p$; that is, as long as p is less than $\frac{4000}{21}$ or 190; but when p is = to 190, x will be = 0; and when p is greater than 190, x will be negative in certain values of z.

If $p=191$, then $x=4000+3z-21p=3z-11$, where it is obvious that x will be negative when z is either 0, 1, 2 or 3; hence there will be four negative values of x.

If $p=192$, then $x=3z-32$, where x will be negative when z is either 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10; and in this case there will be 11 negative values of x.

If $p=193$, then $x=3z-53$, and there will be 18 negative values of x. And when $p=194$, there will be 25 negative values of x; and so on for other values of p; so that the number of all the negative values of x will be expressed by the arithmetical progression $4+11+18+25+$&c. continued to $296-190(=106)$ terms; whereof the sum is $\frac{4+7\times105+4}{2}\times 106=39379$, which, subtracted from 110113, found above, leaves 70734 for the number of answers required.

13. Find what year of Christ the cycle of the sun was 8, golden number or cycle of the moon 10, and the cycle of indiction 10.

Let x be the year required; then if $x+9$, $x+1$, and $x+3$ be divided respectively by 28, 19, and 15, the remainders will be the cycles of the sun, moon, and indiction.

Hence $\frac{x+9-8}{28}$, $\frac{x+1-10}{19}$; and $\frac{x+3-10}{15}$; or $\frac{x+1}{28}$; $\frac{x-9}{19}$ and $\frac{x-7}{15}$ must be the whole numbers. Put $\frac{x+1}{28}=p$, then $x=28p-1$; and this value of x, substited in the fraction, gives $\frac{28p-10}{19}=1+\frac{9p-10}{19}=wh.$; hence $\frac{9p-10}{19}=wh.$, and $\frac{9p-10}{19}\times2=\frac{18p-20}{19}=wh.$ But $\frac{19p}{19}=wh.$; therefore $\frac{19p}{19}-\frac{18p-20}{19}=\frac{p+20}{19}=\frac{p+1}{19}+1=wh.$; or $\frac{p+1}{19}=wh.$, which suppose $=r$; then $p=19r-1$, and $x=532r-29$. This value of x substituted in the 3d fraction, gives $\frac{532r-36}{15}=35r-2+\frac{7r-6}{15}=wh.$; hence

$\frac{7r-6}{15}=wh.$, and $\frac{7r-6}{15}\times 2=\frac{14r-12}{15}=wh.$ But $\frac{15r}{15}=wh.$ therefore $\frac{15r}{15}-\frac{14r-12}{15}=\frac{r+12}{15}=wh.=s$; hence $r=15s-12$ and $x=7980s-6413$, where, if s be taken $=1$, then $x=1567$, the year required.

14. Given $3x=8y-16$, to find the least values of x and y in whole numbers. Ans. $x=8$, $y=5$.

15. Given $14x=5y+7$, to find the least value of x and y in whole numbers. Ans. $x=3$, $y=7$.

16. Given $27x=1600-16y$, to find the least values of x and y in whole numbers. Ans. $x=48$, $y=19$.

17. Divide 100 into two such parts, that one of them may be divisible by 7, and the other by 11. Ans. 56 and 44.

18. Given $9x+13y=2000$, to find the greatest value of x, and the least value of y in whole numbers. Ans. $x=215$, $y=5$.

19. Given $11x+5y=254$, to find all the possible values of x and y in whole numbers. Ans. $x=19, 14, 9, 4$; $y=9, 20, 31, 42$.

20. Given $17x+19y+21z=400$, to find all the answers in whole numbers which the question admits of. Ans. 10 different ans.

21. Given $5x+7y+11z=224$, to find all the possible values of x, y, and z, in whole positive numbers. The number is 59.

22. Required, to find in how many different ways it is possible to pay £20 in half-guineas and half-crowns, without using any other sort of coin. Ans. 7 different ways.

23. I owe my friend a shilling, and have nothing about me but guineas, and he has nothing but louis d'ors; how must I contrive to acquit myself of the debt, the louis being valued at 17*s*. apiece, and the guineas at 21*s*.? Ans. I must give him 13 guineas, and he must give me 16 louis.

24. How many gallons of British spirits, at 11*s*, 15*s*, and 18*s*, a gallon, must a rectifier of compounds take to make a mixture of 1000 gallons that shall be worth 17*s* a gallon?

Ans. $111\frac{1}{9}$ at 12*s*; $111\frac{1}{9}$ at 15*s*; and $777\frac{7}{9}$ at 18*s*.

25. Given $7x-12y=19$, to find the least values of x and y in the whole numbers. Ans. $x=13$, $y=6$.

26. A person bought as many ducks and geese, together, as cost him 28*s*; for the geese he paid 4*s* 4*d* apiece, and for the ducks 2*s* 6*d* apiece. What number had he of each?

Ans. 3 geese and 6 ducks.

27. A person in exchange for a number of pieces of foreign gold, valued at 17*s*4*d* each, received a certain number of guineas under 50, and 1*s* over. What was the sum exchanged? Ans. $49\frac{1}{4}$£

28. Forty-one persons, consisting of men, women and children, spent, among them, 40*s*, of which each man paid 4*s*, each woman 3*s*, and each child 4*d*. How many were there of each?

Ans. 5 men, 3 women, and 33 children.

29. With guineas and moidores, the fewest, which way,
Three huudred and fifty-one pounds can I pay?
If paid every way 't will admit of, what sum
Do the pieces amount to?—my fortune's to come.

Ans. 9 guineas and 233 moidores; and 37 ways, is = £12987.

30. How many ways can a refiner mix three kinds of silver, of $11\frac{1}{4}$, $13\frac{3}{4}$, and $17\frac{1}{2}$ pennyweights fine per *oz.* so as to form a mass of 75 *oz.* of 15 pennyweights fine per ounce? Ans. 8 ways.

31. A person bought 100 head of cattle for \$200.; oxen at \$20 apiece, cows at \$10 apiece, calves at \$4 apiece, and sheep at \$1 apiece. How many of each sort did he purchase?

Ans. The question admits of 10 different answers, one of which is 4 oxen, 1 cow, 5 calves, and 90 sheep.

32. A company of men and women spent 1000 cents at a tavern; the men paid each 19 cents, and each woman 13 cents. How many men and women were there?

Let $x =$ the number of men, and $y =$ the number of women. Then we shall have $19x+13y=1000$. $x = +\dfrac{13y-1000}{19}$.

$$\begin{array}{l} 13y-1000)\,19x\quad(1 \\ \underline{13y-1000} \\ 6y+1000)\,13y-1000\,(2 \\ \underline{12y+2000} \\ y-3000 \end{array} \qquad \begin{array}{rr} & 19)3000(157 \\ & \underline{19} \\ *150 & 110 \\ \underline{133} & \underline{95} \\ 17 & *150 \end{array}$$

$x=\dfrac{1000-13y}{19}=\dfrac{1000-221}{19}=41$, the greatest value of x.

And 17 is the least value of y; then diminishing x by the coefficient of y, and increasing y by the coefficient of x, we have $\left\{\begin{matrix} x=41, & 28, & 15, & 2 \\ y=17, & 36, & 55, & 74 \end{matrix}\right\}$ Now, when $x=41$, $y=17$, so that the men spent 779 cents, and the women 221. When $x=28$, $y=36$, then the men spent 532 cents, and the women 468. When $x=15$, $y=55$, the men spent 285 cents, women 715. And when $x=2$, $y=74$, the men spent 38 cents, women 962.

33. To find a number which, being divided by 6, shall leave the remainder 2, and when divided by 13, shall leave the remainder 3.

Ans. 68.

34. Find the least whole number which, being divided by 39, shall leave the remainder 16, and when divided by 56, the remainder shall be 27. Ans. 1147.

35. Find the least whole number which, being divided by 7, 8, and 9, respectively, shall leave the remainders 5, 7 and 8.

Ans. 1727.

36. Find three numbers in the proportion of 5, 7 and 9, which, being divided by 11, 13 and 15, shall leave the remainders 1, 2, 3, respectively. Ans. 2685, 3759, 4833.

Compound Indeterminate Equations.

163. Equations of this kind, not higher than the second degree which admit of answers in whole numbers, are chiefly such as consist of the products or squares of two unknown quantities, together with the quantities themselves; being usually, as far as regards the plan of this part of the present article, of one of the four general forms given below.

RULE. If the proposed equation be of the form $xy=ay+by+c$, we shall have, for its solutions in whole numbers

$y=a+\frac{ab+c}{x-b}$, where $x-b$ must be a divisor of $ab+c$.

2. If the proposed equation be of the form $x^2+xy=ax+by+c$, we shall have for its solution, in this case, $y=a-b-x+\frac{c+b(a-b)}{x-b}$, where $x-b$ must be a divisor of $c+b(a-b)$.

3. If the proposed equation be of the form $x^2=y^2+ay+b$, we shall have, for its solution, in that case, $y=\frac{a^2-4b}{8n}+\frac{n-a}{2}$, and $x=\frac{a}{2}+y-n$, where $n-a$ must be an even number, and n be so taken that $8n$ may be a divisor of a^2-4b.

4. And if the proposed equation be of the form $x^2=ay^2+by+c^2$ we shall have for its solution, in this case, $y=\frac{b-2cn}{n^2-a}$, and $x=c+ny$, where n must be some whole number between $\sqrt{a}$ and $\frac{q}{2c}$ *....

1. Given $xy=42-2x-3y$, to find the several values of x and y in whole numbers. Here, by the first form, $a=-2$, $b=-3$, and $c=42$, whence $y=-2+\frac{6+42}{x+3}=-2+\frac{48}{x+3}$, where it is plain

* Every indeterminate equation of the second degree, containing two unknown quantities, can always be reduced to the form $x^2-ay^2=b$, consequently, on the solution of this depends that of every possible case that can arise in indeterminate equations of this kind; for some examples of which, see the latter part of this present article.

that x must be such a number, that, when added to 3, the sum shall be a divisor of 48. But the divisors of 48, that will give quotients greater than 2, are 16, 12, 8, 6, 4, and 2. Therefore the integral values of the two unknown quantities, are $x=16-3$, or $13 \mid =12-3$, or $9 \mid =8-3$, or $5 \mid =6-3$, or $3 \mid =4-3$, or 1 $y=\frac{48}{16}-2$, or $1 \mid =\frac{48}{12}-2$, or $2 \mid =\frac{48}{8}-2$, or $4 \mid =\frac{48}{6}-2$, or $6 \mid =\frac{48}{4}-2$, or 10, which are all the answers in whole positive numbers that the question admits of.

2. Given $x^2+xy=2x+3y+29$, to find the values of x and y in whole positive numbers. Here, by the second form, $a=2$, $b=3$, and $c=29$. Whence $y=-x-1+\frac{29+3(2-3)}{x-b}=-x-1+\frac{26}{x-3}$, where it is plain, that x must be such a number, that, when diminished by 3, it shall be a divisor of 26.

But the several divisors of 26 are 1, 2, 13, and 26, of which the only ones that will render the expression positive, are 1 and 2. Therefore $x=4$ or 5, and $y=-4-1+\frac{26}{4-3}=21 \mid$ or $-5-1+\frac{26}{5-3}=7$, which are all the answers in whole numbers that the question admits of.

3. Given $x^2=y^2+20y$, to find the values of x and y in whole positive numbers. Here, by the third form, $a=20$, and $b=0$; whence $y=\frac{400}{8n}+\frac{n-20}{2}=\frac{50}{n}+\frac{n}{2}-10$, and $x=10+y-n$, where it is plain, that n must be some even number which is a divisor of 50. But the only number of this kind, that will give positive results, is 2. $\therefore$ $y=\frac{50}{2}+1-10=16$, and $x=10+16-2=24$.

4. Given $x^2=5y^2-12y+64$, to find the values of x and y in whole positive numbers. Here, by the 4th form, $a=5$, $b=-12$, and $c=8$. Whence $y=\frac{-12-16n}{n^2-5}=\frac{16n+12}{5-n^2}$, and $x=8+ny$, where it is plain, that n must be some integer less than $\sqrt{5}$, which numbers are only 1 and 2.

$$\text{Therefore } y=\frac{12+16}{5-1}=7 \mid \text{ or } =\frac{12+32}{5-4}=44.$$

$$\text{And } x=8+1\times7=15 \mid =8+2\times44=96.$$

5. It is required to find two numbers such, that their product, added to their sum, shall be 79. Ans. $\begin{cases} 1, & 3, & 4, & 7, \\ 39, & 19, & 15, & 9. \end{cases}$

6. Given $x^2+xy=4x+3y+27$, to find the several values of x and y in whole numbers. Ans. $\begin{cases} x=4, & 5 \text{ and } 6. \\ y=27, & 11 \text{ and } 5. \end{cases}$

7. Given $x^2=y^2+100y+1000$, to find the least values of x and y in whole numbers. Ans. $x=70$ and $x=30$.

8. Given $x^2=50y^2+100y+100$, to find the values of x and y in whole numbers. Ans. $x=290$ and $y=40$.

In addition to the present article, it will here be proper to observe, that a more direct and general method of resolving indeterminate equations of the first degree, than that before given may be derived from the doctrine of continued fractions, as follows.

164. To determine, from the principles abovementioned, the values of x and y, in the equation $ax - by = \pm c$, which, when a and b are prime to each other, is always possible in whole Nos.

RULE. Find the quotients arising from dividing the coefficient of y by that of x, (or b by a,) as in determining their common measure, and set their corresponding converging fractions under them, according to the method made use of in Art. 165.

Then, if the last fraction but one, in this series, be denoted by $\frac{p}{q}$, we shall have, by multiplying each of its terms by c, $x=pc$, or $-pc$, and $y=qc$, or $-qc$, according as c is positive or negative; observing, when the values thus found, give a result with an opposite sign to that in the question, that they must be taken with their signs changed.

And if any equi-multiples ma, mb of the coefficients a, b be now taken, we shall have, in general terms, $x=bm\pm pc$, and $x=am\pm qc$; where, it is evident, by putting $m=-1, -2, -3$, &c. 0, 1, 2, 3, &c., and taking pc and qc as before, that an indefinite number of positive integral answers to the question may be obtained.

Note. From the rule above given, it is obvious, that when $c=1$, or $ax-by=\pm 1$, which is the form of the equation on which the solution of the general case depends, that taking $\frac{p}{q}=$ the same fractions as before, we shall then have $x=p$ or $-p$, and $y=q$ or $-q$. And, for the general value of the same quantities, $x=bm\pm p$, and $y = am \pm q$, where, taking p, q, and m, as above, it is plain that the number of answers, as in the former case, will be indefinite.*

1. Given $13x - 19y = 1$, to find integers for x and y.

* The first general solution of this problem, from prinriples similar to those above followed, is commonly ascribed to Bachet de Mesiriac, the editor of Diophantus; who gave both the rule and its investigation, in the third edition of his Mathematical Recreations, entitled *Problêmes Plaisans et délectables qui se font par les Nombres*, printed in 1624. But it is worthy of remark, that a rule of the same extent and import is given in the Hindu treatises on Algebra, by Brahmegupta, written in 696, and Bhascara, written in 1150.

Here

```
13)19(1
   13
    6)13(2
      12
       1)6(6
         6
         0
```

Whence Quotients...1, 2, 6, Therefore $x=p=3$
Converging fractions . $\frac{1}{0}$, $\frac{1}{1}$. $\frac{3}{2}$, $\frac{19}{13}$; and $y=q=2$; which numbers give, by substitution, $13\times3-19\times2=39-38=1$. And if, according to the rule, there be taken the general values $x=19m+3$, and $y=13m+2$, we shall have, by putting $m=0, 1, 2, 3, 4$, &c. respectively.

$\left\{\begin{array}{l} x=3,\ 22,\ 41,\ 60,\ 79,\ \&c. \\ y=2,\ 15,\ 28,\ 41,\ 54,\ \&c. \end{array}\right\}$ which values are all deduced from the first two, by adding successively the coefficient of y to the last value of x, and the coefficient of x to that of y.

2. Given $15x-17y=1$, to find the least integral values of x and y. Ans. $x=8$, and $y=7$.

3. Given $14x-17y=1$, to find the integral values of x and y.

```
14)17(1
   14
    3)14(4
      12
       2)3(1
         2
         1)2(2
```

Whence $\left\{\begin{array}{ll} \text{Quotients} & 1,\ 4,\ 1,\ 2, \\ \text{Converging fractions} & \frac{1}{0},\ \frac{1}{1},\ \frac{5}{4},\ \frac{6}{5},\ \frac{17}{14} \end{array}\right.$

Therefore $x=p=6$, and $y=q=5$; which numbers give, by substitution, $14\times6-17\times5=84-85=-1$, whence, as the result has here an opposite sign to that in the proposed equation, we must take $x=-p=-6$, and $y=-q=-5$, or change the signs of the above values; in which case we shall have $-14\times6+17\times5=-84+85=1$. And if, agreeably to the rule, there be taken the general values $x=17m-6$, and $y=14m-5$, we shall have, by putting $m=1, 2, 3, 4$ &c. as before, $\left\{\begin{array}{l} x=11,\ 28,\ 45,\ 62,\ 79,\ \&c. \\ y=\ 9,\ 23,\ 37,\ 51,\ 65,\ \&c. \end{array}\right\}$, which values, both in this and the following examples, are derived from each other, as above.

4. Given $24x-11y=-1$, to find the least integral values of and y. Ans. $x=5$, and $y=11$.

5. Given $13x-5y=-1$, to find the integral values of x and y.

```
13)5(0
    5)13(2
      10
       3)5(1
         3
         2)3(1
           2
           1)2(2
```

$\left\{\begin{array}{l} \text{Quotients}............0,\ 2,\ 1,\ 1,\ 2, \\ \text{Converging fractions}...\frac{1}{0},\ \frac{0}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{5},\ \frac{5}{13} \end{array}\right\}$

Therefore $x=-p=-2$, and $y=-q=-5$; which numbers give, by substitution, $-13\times2+5\times5=-26+25=-1$. And consequently, taking the general values $x=5m-2$, and $y=13m-5$, we shall have, by putting $m=1, 2, 3, 4$, &c. as usual,

$\left\{\begin{array}{l} x=3,\ \ 8,\ 13,\ 18,\ 23,\ \&c. \\ y=8,\ 21,\ 34,\ 47,\ 60,\ \&c. \end{array}\right\}$

6. Given $7x-12y=19$, to find the least integral values of x and y Ans. $x=13$, and $y=6$.

7. Given $18x-23y=4$, to find the integral values of x and y.

```
18)23(1
   18
    5)18(3
      15
       3)5(1
         3
         2)3(1
           2
           1)2(2
```

Quotients.............. 1, 3, 1, 1, 2
Converging fractions..... $\frac{1}{0}$, $\frac{1}{1}$, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{9}{7}$, $\frac{23}{18}$

Where p being $=9$, $q=7$, and $c=4$, we shall have, by the rule, $x=pc=9\times4=36$, and $y=qc=7\times4=28$; which numbers being substituted in the proposed equation, give $18\times36-23\times28=648-644=4$. And if these be taken, according to the rule, the general values, $x=23m+36$, and $y=18m+28$, we shall have, by making $m=-1$, 0, 2, 3, &c, as in the former instances,

$$\begin{cases} x=13,\ 36,\ 59,\ 82,\ 105,\ 128,\ \&c. \\ y=10,\ 28,\ 46,\ 64,\ 82,\ 100,\ \&c. \end{cases}$$

8. Given $25x-16y=-7$, to find the integral values of x and y. Here

```
25)16(0
16)25(1
 9)16(1
   7)9(1
    2)7(3
     1)2(2
```

Quotients...... 0, 1, 1, 1, 3, 2,
Conv. fractions.. $\frac{1}{0}$, $\frac{0}{1}$, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{7}{11}$, $\frac{16}{25}$

Where p being $=7$, $q=11$, and $c=-7$, we shall have, by the rule, $x=-pc=-49$, and $y=-qc=-77$, which numbers being substituted in the proposed equation, give $-25\times49+16\times77=-1225+1232=7$, whence, as the result has here an opposite sign to that in the question, we must take $x=+49$, and $y=+77$, in which case we shall have $25\times49-16\times77=1225-1232=-7$. And if there be taken, according to the rule, the general values $x=16m+49$, and $y=25m+77$, we shall have, by putting $m=-3$, -2, -1, 0, 1, 2, 3, &c. as before.

$$\begin{cases} x=1,\ 17,\ 33,\ 49,\ 65,\ 81,\ \&c. \\ y=2,\ 27,\ 52,\ 77,\ 102,\ 127,\ \&c. \end{cases}$$

9. Given $19x-117y=11$, to find the least integral values of x and y. Ans. $x=56$ and $y=9$.

10. Given $53x-27y=100$, to find the least integral values of x and y. Ans. $x=19$, and $y=41$.

11. Given $31x-21y=-23$, to find the least values of x and y; Ans. $x=4$, and $y=7$.

165. To determine, by the method employed in the last article, the integral values of x and y in the equation, $ax+by=c$, where a and b are supposed to be prime to each other, as before.

Rule. Find the quotients arising from dividing the coefficient of y by that of x, (or b by a), and set their corresponding converging fractions under them, as in the preceding problem; denoting the last but one in the series by $\frac{p}{q}$, as usual. Then, if any integral

number m can be taken, so that $ma > cq$, and $mb < cp$, we shall have, for the general values of the unknown quantities, $x = cp - mb$, and $y = ma - cq$. In which case, the question will admit of as many solutions as there are values of m that answer the above conditions. But if no such number can be found, the question is impossible.*

1. Given $3x + 5y = 26$, to find the values of x and y.

3)5(1
3
2)3(1
2
1)2(2
2
0

$\left\{\begin{array}{l}\text{Quotients} \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1,\ 1,\ 2, \\ \text{Converging fractions} \ldots\ldots\ldots \frac{1}{0},\ \frac{1}{1},\ \frac{2}{1},\ \frac{5}{2}\end{array}\right\}$

Therefore, since $p = 2$, $q = 1$, and $c = 26$, the general values of the unknown quantities, according to the rule, will be $x = 52 - 5m$, and $y = 3m - 26$. Where, in order to render $3m - 26$ positive, it is plain that m must be greater than $\frac{26}{3}$. Hence, taking $m = 9$, 10, &c. respectively, we shall have, by substitution, $\left\{\begin{array}{l} x = 7,\ 2, \\ y = 1,\ 4, \end{array}\right\}$ which are the only answers in whole numbers the question admits of.

2. Given $12x + 7y = 340$, to find the values of x and y.

12)7(0
7)12(1
5)7(1
2)5(2
1)2(2

$\left\{\begin{array}{l}\text{Quotients} \ldots\ldots\ldots\ldots 0,\ 1,\ 1,\ 2,\ 2, \\ \text{Converging fractions} \ldots \frac{1}{0},\ \frac{0}{1},\ \frac{1}{1},\ \frac{1}{2},\ \frac{3}{5},\ \frac{7}{12}\end{array}\right\}$

Therefore, since $p = 3$, $q = 5$, and $c = 340$, the general values of the unknown quantities will be $x = 1020 - 7m$, and $y = 12m - 1700$. Where, in order to render $12m - 1700$ positive, it is evident that m must be greater than $\frac{1700}{12}$, or 141.

Hence, taking $m = 142$, 143, 144, &c. respectively, we shall have $\left\{\begin{array}{l} x = 26,\ 19,\ 12,\ 5 \\ y = \ \ 4,\ 16,\ 28,\ 40 \end{array}\right|$ which are all the integral answers the question admits of.

3. Given $13x + 17y = 1600$, to find all the possible integral values of x and y.

---

* It may be here observed, that the number of solutions which any equation of this kind admits of, will always be equal to the difference of the integral parts of $\frac{cp}{b} - \frac{cq}{a}$, except when the greater of these two quantities is a whole number, in which case there will be one less. And if this difference be less than unity, the equation is impossible. Which circumstances, together with that mentioned in the preceding problem, are of great use in enabling us to judge of the possibility or impossibility of any indeterminate equation of the form $ax - bx = c$, or $ax + bx = c$, prior to its solution; and consequently, also, in proposing them so, that they may be within possible limits.

$$\begin{array}{l} 13)17(1 \\ \quad 13 \\ \hline \quad 4)13(3 \\ \quad\quad 12 \\ \hline \quad\quad 1)4(4 \\ \quad\quad\quad 4 \\ \hline \quad\quad\quad 0 \end{array}$$

$\left\{ \begin{array}{l} \text{Quotients} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1,\ 3,\ 4, \\ \text{Converging fractions} \ldots\ldots\ldots\ldots \frac{1}{0},\ \frac{1}{1},\ \frac{4}{3},\ \frac{17}{13} \end{array} \right\}$

Therefore, since $p = 4$, $q = 3$, and $c = 1600$, the general values of the unknown quantities will be $x = 6400 - 17m$, and $y = 13m - 4800$; where, in order that $13m - 4800$ may be positive, it is evident that m must be greater than $\frac{4800}{13}$, or 369. Hence, taking $m = 370, 371, 372,$ &c. respectively, we shall have, for the number of answers required,

$\left\{ \begin{array}{l} x = 110,\ 93,\ 76,\ 59,\ 42,\ 25,\ \ 8 \\ y = \ \ 10,\ 23,\ 36,\ 49,\ 62,\ 75,\ 88 \end{array} \right\}$ which are all found from the first two by taking 17 continually from each successive value of x and adding 13 to those of y.

4. Given $11x + 13y = 190$, to find all the possible values of x and y. Ans. only values $x = 9$, and $y = 7$.

5. Given $7x + 11y = 59$, to find the values of x and y.
Ans. Question impossible.

6. Given $13x + 14y = 200$, to find all the possible values of x and y. Ans. Only values $x = 10$, and $y = 5$.

7. Given $7x + 19y = 1921$, to find the least values of x and y. Ans. $x = 3$, and $y = 100$.

8. Given $21x + 5y = 20000$, to find the number of answers the question admits of. Ans. 190 different answers.

The doctrine of continued fractions is also peculiarly applicable to the solution of indeterminate equations of the second degree; the method of employing it, in cases of this kind, being as follows:

163. Determine, from the principles abovementioned the integral values of x and y in the equa $x^2 - ay^2 = \pm 1$, supposing a to be any positive whole number not a square; because in that case the difference of two integral squares would be $= 1$, which is impossible.

Rule. Find the quotients arising from determining the square root of a, according to the method laid down in the last article, and set their corresponding converging fractions under them, as usual. Then, if the given equation be of the form $x^2 - ay^2 = +1$, which is always possible, the numerator and denominator of the fraction that falls under the last figure of the first or second period of the abovementioned quotients, according to which of them occupies an even rank in those periods, (as one of them must always do,) will be the values of x and y respectively. And as this will be the case with every fraction falling, in a similar manner, under the last figure of any of the succeeding periods of quotients, it is plain that the question will admit of an indefinite number of answers; since these periods recur *ad infinitum*. Or if the fraction first found, which gives the least value of x and y, be

denoted by $\frac{p}{q}$, we shall have, for their general values,

$$x=\frac{(p+q\sqrt{a})^m+(p-q\sqrt{a})^m}{2},\ y=\frac{(p+q\sqrt{a})^m-(p-q\sqrt{a})^m}{2\sqrt{a}}$$

which values will always be integral, when m is taken equal to any whole number whatever.

Secondly, if the given equation be of the form $x^2-ay^2=-1$, and the last figure of the first period of quotients, found from the extraction of $\sqrt{a}$ as before, occupies an odd place in that period, the numerator and denominator of the fraction standing under it, will give the values of x and y. And if the fraction thus found be denoted by $\frac{p}{q}$, as in the former instance, we shall have for their general values in this case $x=\frac{(p+q\sqrt{a})^{2m+1}-(p-q\sqrt{a})^{2m+1}}{2}$; $y=\frac{(p+q\sqrt{a})^{2m+1}-(p-q\sqrt{a})^{2m+1}}{2\sqrt{a}}$; where, by taking $m=1, 2, 3$, &c. as before, the number of integral answers will evidently be indefinite. But if the last figure of the first period of the above-mentioned quotients occupies an even place in that period, the question is impossible.*

1. Given $x^2-11y^2=1$, to find the values of x and y in whole numbers. Here $\sqrt{11}=3+\frac{\sqrt{11}-3}{1}=3+\frac{2}{\sqrt{11}+3}=3+\frac{1}{\frac{\sqrt{11}+3}{2}}$

* The first solution of the equation above treated of, which is one of the most important problems in the indeterminate analysis, being that on which the general solution of every other equation of the second degree depends, is usually ascribed to *Lord Brounker*, who was led to the consideration of the question from its having been proposed by *Fermat*, as a challenge to the English mathematicians of that time.

Brounker's solution, which is very ingenious, is given by *Wallis* in his *Opera Math.* chap. 99, and nearly literally by *Euler*, in the 7th chapter of Vol. II. of his *Algebra*; but what will be considered as very remarkable, is, that one of the solutions of this problem given by the Hindu algebraist, *Bhascara*, in his *Vija Ganita*, is exactly the same as that of Lord Brounker; and that *Brahmegupta*, whose algebraical treatise, entitled *Cuttaca d'hyaya*, was written more than 1200 years ago, has given a solution of the general problem, which Mr. *Colebrooke* considers as perfectly complete, and free from all limitations except such as are fixed by the nature of the subject.

$$\frac{\sqrt{11}+3}{2}=3+\frac{\sqrt{11}-3}{2}=3+\frac{1}{\sqrt{11}+3}=3+\frac{1}{\frac{\sqrt{11}+3}{1}}$$

$$\frac{\sqrt{11}+3}{1}=6+\frac{\sqrt{11}-3}{1}=6+\frac{2}{\sqrt{11}+3}=6+\frac{1}{\frac{\sqrt{11}+3}{2}},$$

Where, the fractional part of the last result being the same as that of the first, we shall have $\left\{\begin{array}{l}\text{Quotients}\ldots\ldots\ldots 3,\ 3,\ 6,\ 3,\ 6,\ \&c.\\ \text{Converging frac'ns } \frac{1}{0},\ \frac{3}{1},\ \frac{10}{3},\ldots\ldots\&c.\end{array}\right\}$

And as the last figure of the periodic part of the quotients 3, 6, here occupies an even place in that period, and has the fraction $\frac{10}{3}$ standing under it, we shall have $x=10$, and $y=3$; which numbers, being substituted in the proposed equation, give $10^2-11\times3^2=100-99=1$. And if m be put $=2$, in the formulæ expressing their general values, we shall have

$$x=\frac{(10+3\sqrt{11})^2+(10-3\sqrt{11})^2}{2}=199;$$

$$y=\frac{(10+3\sqrt{11})^2-(10-3\sqrt{11})^2}{2\sqrt{11}}=60,$$ which numbers also, when substituted in the equation as before, give $199^2-11\times60^2=39601-39600=1$. And by taking $m=3$, 4, 5, &c. other values may be found at pleasure.*

2. Given $x^2-41y^2=-1$, to find the values of x and y.

Here $$\sqrt{41}=6+\frac{\sqrt{41}-6}{1}=6+\frac{5}{\sqrt{41}+6}=6+\frac{1}{\frac{\sqrt{41}+6}{5}}$$

$$\frac{\sqrt{41}+6}{5}=2+\frac{\sqrt{41}-4}{5}=2+\frac{5}{\sqrt{41}+4}=2+\frac{1}{\frac{\sqrt{41}+4}{5}}$$

$$\frac{\sqrt{41}+4}{5}=2+\frac{\sqrt{41}-6}{5}=2+\frac{1}{\sqrt{41}+6}=2+\frac{1}{\frac{\sqrt{41}+6}{1}}$$

$$\frac{\sqrt{41}+6}{1}=12+\frac{\sqrt{41}-6}{1}=12+\frac{5}{\sqrt{41}+6}=12+\frac{1}{\frac{\sqrt{41}+6}{5}}$$

---

* The method here used for the solution of questions of this kind, which is due to Lagrange, is more simple and complete than that of Euler; though even in this way the task of finding the values of x and y, when they are large numbers, is often very laborious.

Thus, the least values of these quantities that will solve the equation $x^2-211y^2=1$, are $x=278354373650$, and $y=19162705353$ from which example, and others of a still more complicated kind, that might be given, it appears how necessary it is to have an infallible method of proceeding in these cases; as we should be

Where the fractional part of the last result being the same as that of the first, we shall have

$$\left\{\begin{array}{llllllll} \text{Quotients} \ldots\ldots\ldots\ldots & 6, & 2, & 2, & 12, & 2, & 2, & 12, \text{ \&c.} \\ \text{Converging fractions} \ldots & \frac{1}{0}, & \frac{6}{1}, & \frac{13}{2}, & \frac{32}{5}, & \frac{397}{62}, & \frac{826}{129}, & \frac{2049}{320}, \text{\&c.} \end{array}\right\}$$

And as the last figure of the first period of these quotients, 2, 2 12, occupies an odd place, and has the fraction $\frac{32}{5}$ standing under it, we shall have $x = 32$, and $y = 5$, which numbers, being substituted in the proposed equation, give $32^2 - 41 \times 5^2 = 1024 - 1025 = -1$. And if m be put $= 1$, in the formulæ expressing the general values, we shall have

$$x = \frac{(32+5\sqrt{41})^3 + (32-5\sqrt{41})^3}{2} = 131168, \text{ and}$$

$$y = \frac{(32+5\sqrt{41})^3 - (32-5\sqrt{41})^3}{2\sqrt{41}} = 20485;$$ which numbers, when substituted as above, also give $131168^2 - 41 \times 20485^2 = -1$. And when $m = 2$, 3, 4, &c. other values may be found.

Also, if the proposed equation had been $x^2 - 41y^2 = +1$, we should have had, from the above process, $x = 2049$, and $y = 320$; these being the numerator and denominator of the fraction standing under the last figure of the second period, which here occupies an even place.

3. Given $x^2 - 13y^2 = 1$, to find the values of x and y.

Here, according to example 2 of the last article, we shall have, for $\sqrt{13}$, the following quotients and converging fractions:

$$\left\{\begin{array}{llllllllllll} 3, & 1, & 1, & 1, & 1, & 6 & 1, & 1, & 1, & 1, & 6, & \text{\&c.} \\ \frac{1}{0}, & \frac{3}{1}, & \frac{4}{1}, & \frac{7}{2}, & \frac{11}{3}, & \frac{18}{5}, & \frac{119}{33}, & \frac{137}{38}, & \frac{256}{71}, & \frac{393}{109}, & \frac{649}{180}, & \text{\&c.} \end{array}\right\}$$

where the last figure 6, of the first period, being in an odd place, we must take the fraction that stands under the 6 in the next period; which gives $x = 649$, and $y = 180$, for the least values of these quantities that satisfy the conditions of the question. And if the proposed equation had been $x^2 - 13y^2 = -1$, we should have had, for their least values, in this case, $x = 18$, and $y = 5$. And

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greatly deceived if, after having failed in trying a few moderately large numbers, we should thence conclude that the question was impossible. These circumstances induced Euler to form a table of the values of x and y, necessary for the solution of the equation $x^2 - ay^2 = 1$, for all the values of a from 1 to 100; which is given in Vol. II. of his Algebra, together with another of double the extent, by Lagrange. Legendre, in his *Essai* before mentioned, has also extended the same table to upwards of 1000, and has shown its application to the solution of the general case

$$x^2 - ay^2 = \pm b.$$

from these other values may be found, as in the examples before given.

4. Given $x^2 - 14y^2 = 1$, to find the least values of x and y.
Ans. $x = 15$, and $y = 4$.

5. Given $x^2 - 22y^2 = 1$, to find the least values of x and y.
Ans. $x = 197$, and $y = 42$.

6. Given $x^2 - 29y^2 = -1$, to find the least value of x and y.
Ans. $x = 70$ and $y = 13$.

7. Given $x^2 - 53y^2 = 1$, to find the least values of x and y.
Ans. $x = 66249$, $y = 9100$.

8. Given $x^2 - 113y^2 = -1$, to find the values of x and y
Ans. $x = 1204353$, and $y = 113296$.

9. Given $x^2 - 7y^2 = 1$, to find the first values of x and y.
Ans. $x = 3$, or 45, $y = 1$, or 17.

167. To determine the values of x and y in the equation $x^2 - ay^2 = \pm b$, when the absolute term b is less than $\sqrt{a}$.

Rule. Find the partial quotients arising from extracting the square root of a, after the manner before used, to the end, if necessary, of the first period, and set their corresponding converging fractions under them. Then, if the absolute term b be found in the denominator of any one of the complete quotients obtained from this development, the converging fraction standing under the partial quotient answering to that term, will give the solution of $x^2 - ay^2 = +b$, or $x^2 - ay^2 = -b$, according as the fraction occupies an even or an odd place in the period. But if the term b be not found in an odd place, the latter equation is impossible; and if it be not found in the denominator of any of the complete quotients, the proposed equation is impossible under either sign.*

Secondly, when the equation is possible, if the numerator and denominator of the fraction abovementioned be denoted by m and n respectively, and the values of p and q be found, by the last Article, in the equation $p^2 - aq^2 = +1$, or $p^2 - aq^2 = -1$, according as the substitution of m and n for x and y, gives a result with the same or a contrary sign to that proposed in the equation, we shall have, for new values of these quantities, $x = pm \pm aqn$, or $y = qm \pm pn$. And by substituting these values of x and y for m and n

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* The quotients arising from this mode of extracting the square root of any nonquadrate number being, in general, symmetrical, the absolute term b may be found several times in the same period; in which case we shall have as many solutions of the equation $x^2 - ay^2 = +b$, or $a^2 - ay^2 = -b$, as there are recurrences of this kind; and the same will likewise take place in all the other periods.

in the same formulæ, and so on, other values of them may be obtained at pleasure.*

1. Given $x^2-23y^2=2$, to find the values of x and y.

$$\text{Here } \sqrt{23}=4+\frac{\sqrt{23}-4}{1}=4+\frac{7}{\sqrt{23}+4}=4+\frac{1}{\frac{\sqrt{23}+4}{7}}$$

$$\frac{\sqrt{23}+4}{7}=1+\frac{\sqrt{23}-3}{7}=1+\frac{2}{\sqrt{23}+3}=1+\frac{1}{\frac{\sqrt{23}+3}{2}}$$

$$\frac{\sqrt{23}+3}{2}=3+\frac{\sqrt{23}-3}{2}=3+\frac{7}{\sqrt{23}+3}=3+\frac{1}{\frac{\sqrt{23}+3}{7}}$$

where, the absolute term 2 being found in the denominator of the second complete quotient, we shall have

$\left\{\begin{array}{l}\text{Partial quotients......4, 1, 3, \&c.}\\ \text{Converging fractions..}\frac{1}{0},\ \frac{4}{1},\ \frac{5}{1},\ \text{\&c.}\end{array}\right\}$ and consequently as the fraction $\frac{5}{1}$ answers to the partial quotient 3, in the last term, we shall have $x=m=5$, and $x=n=1$; which numbers being substituted in the proposed equation, give $5^2-23\times1=25-23=2$. Also if there be now taken the auxiliary equation $p^2-23q^2=1$, we shall have, by continuing the above process for $\sqrt{23}$, and taking the fraction which stands under the last quotient figure of the first period, $p=24$, and $q=5$; whence, according to the formulæ given in the latter part of the rule,

$$\left\{\begin{array}{l}x=pm\pm aqn\\ x=qm\pm\ pn\end{array}\right\}\text{ that is, }\left\{\begin{array}{l}x=5,\text{ or } 235,\\ y=1,\text{ or } 49,\end{array}\right.$$

And by putting these values for m and n in the same formulæ, other values of x and y may be found; and so on.

Given $x^2-13y^2=-3$, to find the values of x and y.

$$\text{Here } \sqrt{13}=3+\frac{\sqrt{13}-3}{1}=3+\frac{4}{\sqrt{13}+3}=3+\frac{1}{\frac{\sqrt{13}+3}{4}}$$

$$\frac{\sqrt{13}+3}{4}=1+\frac{\sqrt{13}-1}{4}=1+\frac{3}{\sqrt{13}+1}=1+\frac{1}{\frac{\sqrt{13}+1}{3}}$$

$$\frac{\sqrt{13}+1}{3}=1+\frac{\sqrt{13}-2}{3}=1+\frac{3}{\sqrt{13}+3}=1+\frac{1}{\frac{\sqrt{13}+2}{3}}$$

whence, the absolute term 3 being found in the denominator of the last of these complete quotients, we shall have

$\left\{\begin{array}{l}\text{Partial quotients........3, 1, 1, \&c.}\\ \text{Converging fractions ....}\frac{1}{0},\ \frac{3}{1},\ \frac{4}{1},\ \text{\&c.}\end{array}\right\}$ And consequently, as the fraction $\frac{4}{1}$ answers to the partial quotient 1 in that term,

* The general values of x and y in this Article may be otherwise obtained by determining those of p and q, according to the rules before given, and then substituting them for those letters in the above formulæ; but the method here followed is more convenient in practice.

we shall have $x = m = 4$, and $y = n = \pm 1$; which numbers being substituted in the proposed equation, give $4^2 - 13 \times 1 = 16 - 13 = +3$; and, as the result has here a different sign from that in the question, we must take the equation $p^2 - aq^2 = -1$, which being resolved by the method beforementioned, gives $p = 18$, and $q = 5$; whence, by substituting these values, together with those of m and n, in the common formulæ, we shall have

$$\left\{\begin{array}{l} x = pm \pm aqn \\ y = qm \pm pn \end{array}\right\} \text{ that is, } \left\{\begin{array}{l} x = 7, \text{ or } 137, \\ y = 2, \text{ or } 38, \end{array}\right\} \text{ for the values sought;}$$

which give, by substitution, $\left\{\begin{array}{c} 7^2 - 13 \times 2^2 = -3, \\ 137^2 - 19 \times 38^2 = -3, \end{array}\right\}$ and, by means of these values, others may be readily found.

We might here also proceed to the solution of the general case of these equations $x^2 - ay^2 = b$, where a and b are supposed to be any given integral numbers; but the artifices and train of reasoning required for this purpose would extend the present article to too great a length. We must therefore refer those who are desirous of farther information on this interesting subject, to *Lagrange's Additions*, at the end of *Euler's Algebra*, or to the *Essai sur le Theorie des Nombres* of *Legendre*, second edition. where they will find almost every branch of the Indeterminate Analysis treated with great perspicuity and elegance.

Of Continued Fractions.

168. A continued fraction is that which has for its denominator a whole number and a fraction; and so on, continually, or till the series terminates, by being broken of, after a certain number of terms.

Thus, $\dfrac{1}{2+\dfrac{1}{4+\dfrac{1}{6+\dfrac{1}{8\,\&c.}}}}$ Or $\dfrac{1}{a+\dfrac{1}{b+\dfrac{1}{c+\dfrac{1}{d\,\&c.}}}}$ are continued fractions of the kind that most usually occur.

These expressions arise in various ways, and are of great use in finding the approximate values of fractions and ratios, that are expressed in large numbers, as well as in the resolution of certain unlimited problems of the first and second degrees; in the latter of which the answers, in whole numbers, cannot be easily obtained by any other method. Thus, in order to represent the irreducible fraction, or ratio, $\frac{a}{b}$, by a continued fraction, let b be contained in a p times, with a remainder c; also, let c be contained in b, q times, with a remainder d; d in c, r times, with a remainder e; and so on, as in the following operation: Art. 48 p. 33.

Then we shall have, by the common method for the division of

numbers $\frac{a}{b}=p+\frac{c}{b}$, $\frac{b}{c}=q+\frac{d}{c}$, $\frac{c}{d}=r+\frac{e}{d}$, &c. where the integers p, q, r, &c. are called partial quotients: and each of these, with its remainder, or depending fraction, complete quotients.

And since, by taking the reciprocals of the second, third, &c. of these expressions, we have $\frac{c}{b}=\cfrac{1}{q+\frac{d}{c}}$, $\frac{d}{c}=\cfrac{1}{r+\frac{e}{d}}$ &c. $\Big\}$ If these be substituted for their equals in the former, there will arise

$$\frac{a}{b}=p+\frac{c}{b}=p+\cfrac{1}{q+\frac{d}{c}}=p+\cfrac{1}{q+\cfrac{1}{r+\frac{e}{d}}} \text{ \&c.}$$

Whence, by extending the number of terms, and generalizing the formulæ, we shall have

$$\frac{a}{b}=p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}}} \text{ \&c.} \qquad \text{Or, } \frac{a}{b}=\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}}}} \text{ \&c.}$$

according as the numerator is greater or less than the denominator; which expressions, in this case, will consist of a greater or less number of terms, according as the fraction $\frac{a}{b}$ is more or less complicated; but they will always terminate when $\frac{a}{b}$ is rational. Hence, from what has been here shown, it is obvious that any given irreducible fraction may be converted into a continued fraction, as follows;

Rule. Divide the greater of the two terms of the fraction by the less, and the last divisor continually by the last remainder, till nothing remains, as in finding their greatest common measure; then the successive quotients, thus found, will be the denominators of the several terms of the continued fraction, and their numerators are always unity, or 1.

1. Thus, if it be required to reduce $\frac{2431}{1051}$ to a continued fraction,

```
1051)2431(2
      329)1051(3
            64)329(5
                9)64( 7
                   1)9(9
```

where, since the numerator of the proposed fraction is greater than the denominator, the first quotient figure 2, will be an integer, and the rest of the quotients, taken in order, are 3, 5, 7 9. Whence, taking these for the denominators of the continued fractions sought, and 1 for each of their numerators, we shall have, according to the above rule,

$$\frac{2431}{1051}=2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9}}}}$$

for the continued fraction required.

2. And if $\frac{1096}{9119}$ is to be reduced to a continued fraction, we shall have, by the same rule,

$$\frac{1096}{9119}=\cfrac{1}{8+\cfrac{1}{3+\cfrac{1}{8+\cfrac{1}{6+\cfrac{1}{7.}}}}}$$

3. Also, if $\frac{421}{972}$ is to be reduced to a continued fraction, the result, found as above, will be

$$\frac{421}{972}=\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6.}}}}}$$

Any continued fraction may also be converted into a series of vulgar fractions, by finding the successive sums of its several terms, after the manner of reducing complete fractions to simple ones, in common arithmetic.

Thus, if it were required to reduce the following continued fraction,

$$\alpha+\cfrac{1}{\beta+\cfrac{1}{\gamma+\cfrac{1}{\delta}}}\ \&c.$$

to a series of common vulgar fractions, the operation will stand thus:

$$\alpha=\frac{\alpha}{1},\quad \alpha+\frac{1}{\beta}=\frac{\alpha\beta+1}{\beta},\quad \alpha=\cfrac{1}{\beta+\cfrac{1}{\gamma}}=\alpha+\frac{\gamma}{\beta\gamma+1}=\frac{\alpha\beta\gamma+\alpha+\gamma}{\beta\gamma+1}$$

$$\alpha+\cfrac{1}{\beta+\cfrac{1}{\gamma+\cfrac{1}{\delta}}}=\alpha+\cfrac{1}{\beta+\cfrac{\delta}{\gamma\delta+1}}=\alpha+\frac{\gamma\delta+1}{\beta\gamma\delta+\beta+\delta}$$

$$=\frac{\alpha\beta\gamma\delta+\alpha\beta+\alpha\delta+\gamma\delta+1}{\beta\gamma\delta+\beta+\delta}$$

Or, if the results thus obtained be placed one after another in a right line, with the quotients or denominators α, β, γ, δ, ε, &c. of the continued fraction put over them, the same expressions, making $\frac{1}{0}$ the first term, may be as follows;

$$\begin{array}{ccccc} \alpha & \beta & \gamma & \delta & \varepsilon \\ \frac{1}{0}, & \frac{\alpha}{1}, & \frac{\alpha\beta+1}{\beta}, & \frac{\alpha\beta\gamma+\gamma+\alpha}{\beta\gamma+1}, & \frac{\alpha\beta\gamma\delta+\gamma\delta+\alpha\beta+1}{\beta\gamma\delta+\delta+\beta}, \end{array}$$

&c. where it appears that the successive sums of the terms of the continued fraction, which, from their approaching continually nearer to its true value, are called converging fractions, may be obtained by adding the product of each numerator and the quotient standing over it, to the preceding numerator, for the next numerator, and following the same process for the denominator.

Hence, by means of the property here mentioned, we have the

following easy and commodious method of converting any given fractional expression into a series of converging fractions, without first reducing it to the continued form, and thence finding the successive sums of the terms; which is often a very embarrassing part of the process.

RULE. Find the quotients arising from dividing one term of the given fraction by the other, and the last divisor by the last remainder, &c. till nothing remains, as in a former rule, which will be the denominators of the continued fractions, and set them in a right line, beginning with 0, or the integer that first occurs, according as it is a proper or improper fraction.

Put $\frac{1}{0}$ under the first term of these quotients, in order to render the mode of proceeding more evident, and make the first quotient figure (0, or otherwise, as it may happen to be,) the numerator of the second fraction in the series, and 1 its denominator.

Then, the product of this numerator and the quotient standing over it, being added to the preceding numerator, will give the numerator of the fraction next following; and the product of the denominator and its corresponding quotient, added to the preceding denominator, will give its denominator; and so on, for the rest.

Where it is to be observed, that each of the converging fractions, thus obtained, will be in its lowest terms; and the last that occurs will be the same as the fraction first proposed.

1. Thus, if it be required to find the series of fractions converging towards the given fraction $\frac{64}{265}$, there will arise, by following the operation pointed out in the first part of the above rule,

265(64(0
0
64)265(4
256
9)64(7
63
1)9(9

And consequently, by proceeding according to the latter part of the rule, we shall have

Quotients..........0, 4, 7, 9,
Converging fractions. $\frac{1}{0}$, $\frac{0}{1}$, $\frac{1}{4}$, $\frac{7}{29}$, $\frac{64}{265}$

where $\frac{64}{265}$ is the same as the original fraction, or that first proposed.

2. Also, if it were required to find the series of fractions converging towards the given fraction $\frac{285}{127}$, there will arise

127)285(2
254
31)127(4
124
3)31(10
30
1)3(3

And consequently, by proceeding in the way before mentioned, we shall have

Quotients.........2, 4, 10, 3
Converging fract's.. $\frac{1}{0}$, $\frac{2}{1}$, $\frac{9}{4}$, $\frac{92}{41}$, $\frac{285}{127}$

where, as before, $\frac{285}{127}$ is the same as the given fraction.

Or if A, B, C, &c. be made to represent the numerators of the converging

fractions above given, taken in succession from $\frac{a}{1}$, A′, B′, C′, &c. their denominators, we shall likewise have, by the same rule,

$A=a$	$A'=1$	Or, $\frac{A}{A'}=\frac{a}{1}$; $\frac{B}{B'}=\frac{A\beta+1}{A'\beta}$;
$B=A\beta+1$	$B'=A'\beta$	
$C=B\gamma+A$	$C'=B'\gamma+A'$	$\frac{C}{C'}=\frac{B\gamma+A}{B'\gamma+A'}$; $\frac{D}{D'}=\frac{C\delta+B}{C'\delta+B'}$ &c.
$D=C\delta+B$	$D'=C'\delta+B'$	

where the law by which the terms may be continued is sufficiently evident.

Hence, in order to exemplify this latter method, let it be proposed, as one of the most useful cases relating to fractions of this kind, to express in small numbers the ratio of the circumference of the circls to its diameter; which, by stopping at the 5th place of decimals, is $\frac{3.14159}{1.00000}$.

Here, by following the process mentioned in the first part of the former rule, there will arise the 1st, 2d, 3d, and 4th quotient figures 3, 7, 15, and 1, respectively.

And consequently, since $a=3$, $\beta=7$, $\gamma=15$, $\delta=1$, &c. we shall have

$$A=a=3$$
$$B=A\beta+1=3\times7+1=22$$
$$C=B\gamma+A=22\times15+3=333$$
$$D=C\delta+B=333\times1+22=355$$
$$A'=1$$
$$B'=A'\beta=1\times7=7$$
$$C'=B'\gamma+A'=7\times15+1=106$$
$$D'=C'\delta+B'=106\times1+7=113$$

$$\text{Where } \frac{A}{A'}=\frac{3}{1},\ \frac{B}{B'}=\frac{22}{7},\ \frac{C}{C'}=\frac{333}{106},\ \frac{D}{D'}=\frac{355}{113},\ \&c.$$

as would have been the case if the results here obtained had been found by the rule before used. Whence the converging fractions required are $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, &c. which are alternately less and greater than the circumference divided by the diameter, and are each expressed in the least numbers possible; the second, $\frac{22}{7}$, being the ratio assigned by Archimedes, and the fourth, $\frac{355}{113}$, that given by Metius, which is much more accurate than the former.

And if a greater number of decimals be taken, and the operations be carried on to a farther extent, the converging fractions thus obtained will approach continually nearer to the true value of the circumference; but they will necessarily become more complicated the farther they are produced. A similar observation to that above made may also be applied to every case of the kind here treated of; in all of which the series of converging fractions $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, $\frac{D}{D'}$, &c. which are called principal fractions, will be

alternately greater and less than the total value of the continued, or original fraction; and each of them will express that value more accurately than any other fraction that can be conceived in more simple terms.

It may be farther remarked, that among continued fractions, those have been particularly distinguished in which the denominators, after a certain number of changes, are continually repeated in the same order; in which case, the continued fraction so formed is said to be periodic, and may then always be considered as the root of a quadratic equation, or a surd.

1. Thus, let $x = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p \text{ \&c.}}}}}$ Then, since the number of these fractions is unlimited, and all of them, after the first, return again in the same order, it follows that their sum is also $= x$.

Whence $x = \dfrac{1}{p+x}$, or by multiplication, $x^2+px=1$; and consequently, $x=-\frac{1}{2}p+\frac{1}{2}\sqrt{(p^2+4)}$, in which case the above continued fraction serves to determine the square root of the number $\sqrt{(p^2+4)}$; since we have

$$\tfrac{1}{2}\sqrt{(p^2+4)}=\tfrac{1}{2}p+x=\frac{p}{2} + \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p \text{ \&c.}}}}$$

And if p, in this last expression, be put $= 2$, we shall have

$\sqrt{2}=1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 \text{ \&c.}}}}$, a formula which has the advantage of showing the law by which the operation may be continued, more evidently than if the extraction had been performed in the usual way.*

2. Again, let us take the following continued fraction,

$x = \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{p + \cfrac{1}{q \text{ \&c.}}}}}$, in which the denominators recur period-

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* From the common method of extracting the roots of numbers, we know that $\sqrt{2}=1.41421356 \ldots$; but the bare inspection of this decimal leads to no rule for its farther extension; whereas, when expressed by a continued fraction, we see in what way it may be carried on to any degree of accuracy.

ically in pairs. Then we shall have $x = \frac{1}{p} + \frac{1}{q+x}$, from which there arises, by reduction, the quadratic equation $px^2 + pqx = q$; where $x = -\frac{q}{2} + \frac{1}{2p}\sqrt{(p^2q^2+4pq)}$, or $\frac{1}{2p}\sqrt{(p^2q^2+4pq)} = \frac{q}{2} + x$.

And if p, in this expression, be put $=2$, and $q=3$, we shall have $\sqrt{15} = 3+2\left\{\frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} \text{ \&c.}\right\}$, where the law of continuation is sufficiently obvious; and by substituting other numbers for p and q, various series of this kind may be obtained.

Also, if a continued fraction be irregular in some its first terms, or only becomes periodic at a certain distance from its commencement, it may still be resolved in a similar manner with the former. Thus, 3. Let $x = p + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} + \frac{1}{r} + \frac{1}{s}$ &c. Then, by making $y =$ that part of the fraction which is periodic, we shall have $x = p + \frac{1}{q+y}$, or $y = \frac{1-q(x-p)}{x-p}$. But by the nature of the proposed fraction, it is also evident that $y = \frac{1}{r} + \frac{1}{s} + y$. And consequently by reducing the expression to a more simple form, we shall have $ry^2 + rsy = s$, or $y^2 + sy = \frac{s}{r}$; whence, by substitution, there will arise the quadratic equation

$\left\{\frac{1-q(x-p)}{x-p}\right\}^2 + s\left\{\frac{1-q(x-p)}{x-p}\right\} = \frac{s}{r}$, from which the value of $x-p$, and thence that of x may be obtained, as below,

$$x = p + \frac{1}{q - \frac{s}{r} + \sqrt{\left(\frac{s^2}{4} + \frac{s}{r}\right)}}$$

4. In like manner, if there be taken the following continued fraction, $x = a + \frac{p}{q} + \frac{p}{q} + \frac{p}{q} + \frac{p}{q}$ &c. or $x - a = \frac{p}{q} + \frac{p}{q} + \frac{p}{q} + \frac{p}{b}$ &c. we shall obtain, by a substitution similar to that before used,

$x-a=\frac{p}{q+x-a}$, or $x=\frac{2a-q+\sqrt{(q^2+4p)}}{2}$. And consequently, by transposing $\frac{2a}{2}$, or a, to the other side of the equation, we shall have $\frac{1}{2}\sqrt{(q^2+4p)}-q^2=x-a$, or, $\frac{\sqrt{(q^2+4p)}-q}{2}=\cfrac{p}{q+\cfrac{p}{q+\cfrac{p}{q\text{ \&c.}}}}$

Or, by making $q=2a$, the expression for the simple radical will become $\sqrt{(a^2+p)}=a+\cfrac{p}{2a+\cfrac{p}{2a+\cfrac{p}{2a\text{ \&c.}}}}$ And in nearly the same way may any other expression of this kind be transformed to a quadratic, or a surd; to which they are always reducible, whether the periodic part consists of one, two, or more terms; or whether it commences in a regular or irregular manner.

A similar mode of solution may also be applied to continued surds, or expressions of the form $\sqrt{\{a+\sqrt{(a+\sqrt{(a+))}\}}}$, &c. the value of which, though apparently infinite, is always determinable by means of a certain equation; and, in some cases, it is a real integral or fractional quantity; for putting $x=\sqrt{\{a+\sqrt{(a+\sqrt{(a+))}\}}}$ &c. we shall have, by squaring each side of the equation, $x^2=a+\sqrt{\{a+\sqrt{(a+)}\}}$ &c. the latter part of which is evidently equal to the original surd. Whence $x^2=a+x$, or $x^2-x=a$; and consequently, $x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+a)}$, where, if a be now put equal to 2, the expression will become $\sqrt{\{2+\sqrt{(2+\sqrt{(2+))}\}}}$ &c. $=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+2)}=2$, or -1.

Again, let there be taken, as another instance of this kind, $x=\sqrt{\{a+\sqrt{(b+\sqrt{(a+\sqrt{(b+)))}\}}}}$ &c. Then, by squaring each side of the equation, as before, we shall have $x^2=a+\sqrt{\{b+\sqrt{(a+\sqrt{(b+))}\}}}$ &c., and $x^2-a=\sqrt{\{b+\sqrt{(a+\sqrt{(b+))}\}}}$ &c. And by again squaring each side of this last equation, $x^4-2ax^2+a^2=b+\sqrt{\{a+\sqrt{(b+)}\}}$ &c. or, $x^4-2ax^2-x=b-a^2$, which equation, when solved in the usual way, will evidently give the value of x.

To this we may add, that the square root of any quadratic number may be converted into a continued fraction, of the periodic kind, as follows.*

* It has been long known, that any given continued periodic fraction could be reduced to a quadratic equation, and thence to a simple surd; but *Lagrange* appears to have been the first who has proved the reverse of this proposition, by showing that the square root of any whole number, or the root of a quadratic equation, can always be expressed by a continued periodic fraction.—See his work entitled *De la Resolution des Equation Numeriques*, p. 65.

RULE. Find such an integral value a, of the given quantity x, that $x-a$ shall be less than unity, and make $x=\alpha+\frac{1}{x'}$. Find, also, such an integral value α' of x', that $x'-\alpha'$ shall be less than unity, and make $x'=\alpha'+\frac{1}{x''}$. Proceed in this manner with x'', by putting its greatest integral value equal to α'' ;, and so on.

Then, by successively substituting the value of x', x'', x''', &c. for their equals in the first of these expressions, we shall have

$$x=\alpha+\cfrac{1}{\alpha'+\cfrac{1}{\alpha''+\cfrac{1}{\alpha'''+\&c.}}}$$

for the continued fraction required. Where it is to be observed that the denominators α', α'', α''', &c. are the partial quotients before mentioned; and the quantities x', x'', x''', &c. resulting from the development of x, and of which the integers α', α'', α''', &c. form the greater part, are the complete quotients. And if the partial quotients, thus found, be placed in a right line, as usual, and their corresponding converging fractions be determined according to the method before used, they will give the approximate values of the root required.*

1, Thus, if it be required to convert the square root of 19 into a continued fraction, we shall have, by following the above rule,

$$x=\sqrt{19}=4+\frac{\sqrt{19}-4}{1}=4+\frac{3}{\sqrt{19}+4}=4+\frac{1}{\frac13(\sqrt{19}+4)}$$

$$x^{\mathrm{i}}=\frac{\sqrt{19}+4}{3}=2+\frac{\sqrt{19}-2}{3}=2+\frac{5}{\sqrt{19}+2}=2+\frac{1}{\frac15(\sqrt{19}+2)}$$

$$x^{\mathrm{ii}}=\frac{\sqrt{19}+2}{5}=1+\frac{\sqrt{19}-3}{5}=1+\frac{2}{\sqrt{19}+3}=1+\frac{1}{\frac12(\sqrt{19}+3)}$$

$$x^{\mathrm{iii}}=\frac{\sqrt{19}+3}{2}=3+\frac{\sqrt{19}-3}{2}=3+\frac{5}{\sqrt{19}+3}=3+\frac{1}{\frac15(\sqrt{19}+3)}$$

$$x^{\mathrm{iv}}=\frac{\sqrt{19}+3}{5}=1+\frac{\sqrt{19}-2}{5}=1+\frac{3}{\sqrt{19}+2}=1+\frac{1}{\frac13(\sqrt{19}+4)}$$

$$x^{\mathrm{v}}=\frac{\sqrt{19}+2}{3}=2+\frac{\sqrt{19}-4}{3}=2+\frac{1}{\sqrt{19}+4}=2+\frac{1}{1(\sqrt{19}+4)}$$

$$x^{\mathrm{vi}}=\frac{\sqrt{19}+4}{1}=8+\frac{\sqrt{19}+4}{1}=8+\frac{4}{\sqrt{19}+4}=8+\frac{1}{\frac14(\sqrt{19}+4)}$$

* It is not for the sake of the extraction of the square root, that this method has been devised, but on account of its application to indeterminate equations of the second degree; which admit of no other general method of solution; as was first shown by *Lagrange*, in the *Memoirs of Berlin*, for 1767 and 1768.

$$x^{\text{VII}}=\frac{\sqrt{19}+4}{3}=2+\frac{\sqrt{19}-2}{3}=2+\frac{5}{\sqrt{19}+2}=2+\frac{1}{\frac{1}{5}(\sqrt{19}+1)}$$

Where x^{VII} being the same as x^{1}, it is plain that, omitting the 4, which is the greatest integral part of $\sqrt{19}$,) the quotients 2, 1, 3, 1, 2, 8, already found, will always return again in the same order to infinity.

Whence $\sqrt{19}=4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{8}}}}}}$

And if it should be required to convert the square root of 19 into a series of converging fractions, without first reducing it to the continued form, they may be obtained in the way before used, from the integral parts of the above results only. Thus,

$$\left\{\begin{array}{l} \text{Quotients} \ldots\ldots\ldots\ldots 4,\ 2,\ 1,\ 3,\ 1,\ 2\ 8,\ 2,\ \text{\&c.} \\ \text{Converging fractions} \ldots \frac{1}{0},\ \frac{4}{1},\ \frac{9}{2},\ \frac{13}{3},\ \frac{48}{11},\ \frac{61}{14},\ \frac{170}{39},\ \frac{1421}{326}, \end{array}\right\}$$

each of which fractions expresses the square root of 19 nearer than any of the preceding ones; and it is manifest, by setting down the quotients of the next and following periods, that they may be continued at pleasure to any degree of accuracy required.

2. Also, if it were required to convert the square root of 13 into a series of converging fractions, we shall have

$$x=\sqrt{13}=3+\frac{\sqrt{13}-3}{1}=3+\frac{4}{\sqrt{13}+3}=3+\frac{1}{\frac{1}{4}(\sqrt{13}+3)}$$

$$x^{\text{I}}=\frac{\sqrt{13}+3}{4}=1+\frac{\sqrt{13}-1}{4}=1+\frac{3}{\sqrt{13}+1}=1+\frac{1}{\frac{1}{3}(\sqrt{13}+1)}$$

$$x^{\text{II}}=\frac{\sqrt{13}+1}{3}=1+\frac{\sqrt{13}-2}{3}=1+\frac{3}{\sqrt{13}+2}=1+\frac{1}{\frac{1}{3}(\sqrt{13}+3)}$$

$$x^{\text{III}}=\frac{\sqrt{13}+2}{3}=1+\frac{\sqrt{13}-1}{3}=1+\frac{4}{\sqrt{13}+1}=1+\frac{1}{\frac{1}{4}(\sqrt{13}+1)}$$

$$x^{\text{IV}}=\frac{\sqrt{13}+1}{4}=1+\frac{\sqrt{13}-3}{4}=1+\frac{1}{\sqrt{13}+3}=1+\frac{1}{1(\sqrt{13}+3)}$$

$$x^{\text{V}}=\frac{\sqrt{13}+3}{1}=6+\frac{\sqrt{13}-3}{1}=6+\frac{4}{\sqrt{13}+3}=6+\frac{1}{\frac{1}{4}(\sqrt{13}+3)}$$

$$x^{\text{VI}}=\frac{\sqrt{13}+3}{4}=1+\frac{\sqrt{13}-1}{4}=1+\frac{3}{\sqrt{13}+1}=1+\frac{1}{\frac{1}{3}(\sqrt{13}+1)}$$

where x^{VI} being the same as x^{1}, the integral parts of the process, 1, 1, 1, 1, 6, as in the former example, will return again continually, in the same order.

Whence, for the remaining part of the operation, we shall have

$$\left\{\begin{array}{l} \text{Quotients} \ldots\ldots\ldots\ldots 3,\ 1,\ 1,\ 1,\ 1,\ 6,\ 1\ \text{\&c.} \\ \text{Converging fractions} \ldots \frac{1}{0},\ \frac{3}{1},\ \frac{4}{1},\ \frac{7}{2},\ \frac{11}{3},\ \frac{18}{5},\ \frac{119}{33}\ \text{\&c.} \end{array}\right\} \text{each of}$$

which will be the nearest square root of 13 that can be expressed in small numbers.

170. The operation made use of above for converting the square root of 19, and other nonquadrate numbers, into converging fractions, may be rendered much more simple by observing the following law of the formation of the successive quotients, viz. let $\frac{\sqrt{19}+m}{n}=u+$ &c. and $\frac{\sqrt{19}+m'}{n'}=u'+$ &c. represent any two consecutive fractions in the examples there given, where m, m', m'' &c. are the numbers that are to be added to the root; n, n', n'' the divisors; and u, u', u'', &c. the integral parts of the respective quotients.

Then, from the obvious nature of the operation, we shall have in this case $m'=nu-m$, and $n'=\frac{19-m'^2}{n}$; so that each value of m', n', and u, may be readily deduced from those of m, n, and u in the preceding fraction. Hence the operation above referred to will stand, by means of this law, as follows:

$\frac{\sqrt{19}+0}{1}=4+$ &c.	$1\times4-0=4$	$\frac{19-4^2}{1}=3$
$\frac{\sqrt{19}+4}{3}=2+$ &c.	$3\times2-4=2$	$\frac{19-2^2}{3}=5$
$\frac{\sqrt{19}+2}{5}=1+$ &c.	$5\times1-2=3$	$\frac{19-3^2}{5}=2$
$\frac{\sqrt{19}+3}{2}=3+$ &c.	$2\times3-3=3$	$\frac{19-3^2}{2}=5$
$\frac{\sqrt{19}+3}{5}=1+$ &c.	$5\times1-3=2$	$\frac{19-2^2}{5}=3$
$\frac{\sqrt{19}+2}{3}=2+$ &c.	$3\times2-2=4$	$\frac{19-4^2}{3}=1$
$\frac{\sqrt{19}+4}{1}=8+$ &c.	$1\times8-4=4$	$\frac{19-3^2}{1}=3$

Where it will be found, by continuing the process, that the period of quotients 2, 1, 3, 1, 2, 8, will recur again in the same order, ad infinitum. Hence, placing these quotients as in the former part of the work, and following a similar mode of operation, we shall have the following series of converging fractions:

$$\left\{\begin{matrix} 4, & 2, & 1, & 3, & 1, & 2, & 8, & \\ \frac{1}{0}, & \frac{4}{1}, & \frac{9}{2}, & \frac{13}{3}, & \frac{48}{11}, & \frac{61}{14}, & \frac{170}{39}, & \text{\&c.} \end{matrix}\right\}$$ each of which expresses $\sqrt{19}$ more nearly than any preceding one; and it is evident that they may be continued at pleasure to any degree of accuracy required. In the same manner also may the square root of any nonquadrate

number N, be extracted, by supposing a to be the greatest integer contained in $\sqrt{N}$, m, m', m'', &c. the numbers to be added to $\sqrt{N}$, n, n', n'', &c. the divisors, and u, u', u'', &c. the integral parts of the several fractions, as before; viz.

$\dfrac{\sqrt{N}+0}{1}=a+\&c.$	$1\times a-0=m$	$\dfrac{N-m^2}{1}=n$
$\dfrac{\sqrt{N}+m}{n}=u+\&c.$	$n\times u-m=m'$	$\dfrac{N-m'^2}{n}=n'$
$\dfrac{\sqrt{N}+m'}{n'}=u'+\&c.$	$n'\times u'-m'=m''$	$\dfrac{N-m''^2}{n'}=n''$
$\dfrac{\sqrt{N}+m''}{n''}=u''+\&c.$	$n''\times u''-m''=m''$	$\dfrac{N-m'''^2}{n''}=n'''$

Where, by continuing the extraction, as in the former case, we shall always arrive at a fraction equal to $\dfrac{\sqrt{N}+m}{n}$, or the second in the series; after which the quotients will constantly recur again in the same order. It is also evident from the practical example before given, and may be demonstrated generally, that the last quotient of every complete period of quotients, in this mode of extracting the square root of any number, is always equal to twice the greatest integer contained in $\sqrt{N}$.*

The Differential Method of Series.

171. The Differential Method is the method of finding the successive differences of the terms of a series, and thence any intermediate term, or the sum of the whole series.

Problem I. To find the first term of any order of differences.

Let a, b, c, d, e, &c. represent any series; then if the successive differences of the terms be taken, these differences will form a new series, which is called the first order of differences; in like manner, if the successive differences of the terms of this last series be taken, a new series, called the 2d order of differences, will be obtained.

1st order of differences,

	$b-a,\quad c-b,$	$d-c,$	$e-d$	&c.
	$b-a$	$c-b$	$d-c$	
2nd order	$c-2b+a,$	$d-2c+b$	$e-2d+c,$	&c.
		$c-2b+a$	$d-2c+b$	
3rd order.............		$d-3c+3b-a,$	$e-3d+3c-b,$	&c.

* The reader who may wish to see a full account of this subject is referred to the *Theory of Numbers*, by Legendre, so often mentioned; the *Memoirs of the Academy of Berlin*, an. 1767 and

Now, since in the first order the first term in any difference is the same, except the sign, as the second in the succeeding difference, in subtracting any difference from the succeeding, the first term of the former must be placed under the second term of the latter, and consequently the same must take place in every succeeding order.

Hence the coefficients of the several terms, composing either of the differences belonging to any order, are respectively the same as the coefficients of the terms in the expanded binomial, being generated in exactly the same way, the terms that are subtracted being in reality added with contrary signs. Therefore, the first difference of the nth order is $a-nb+\frac{n.(n-1)}{2}c-\frac{n.(n-1)(n-2)}{2.3}d$ $+$ &c. when n is an even number. and

$$-a+nb-\frac{n.(n-1)}{2}c+\frac{n.(n-1)(n-2)}{2.3}d-\text{\&c.}$$

when n is an odd number.

1. Required the first term of the fourth order of differences of the series 1, 8, 27, 64, 125, &c. Here a, b, c, d, e. &c. $=1$, 8, 27, 64, 125, &c. and $n=4$. $\therefore a-nb+\frac{n(n-1)}{2}c-\frac{n(n-1)(n-2)}{2.3}d\ldots.+\frac{n(n-1)(n-2)(n-3)}{2.3.4}e=a-4b+6c-4d+e=1-32+162-256+125=0$; hence the first term of the fourth order is 0.

2. Required the first term of the 5th order of differences of the series 1, 3, 3^2, 3^3, 3^4, &c. Here a, b, c, d, e, &c. $=1$, 3, 9, 27 81, &c. and $n=5$, $\therefore -a+nb-\frac{n(n-1)}{2}c+\frac{n(n-1)(n-2)}{2.3}d\ldots-\frac{n(n-1)(n-2)(n-3)}{2.3.4}e+\text{\&c.}=-a+5b-10c+10d-5e+f=-1+15-90+270-405+243=32=$ the first term of the fifth order of differences.

3. Required the first term of the third order of differences of the series 1, 2^3, 3^3, 4^3, &c. Ans. 6.

4. Required the first term of the fourth order of differences of the series 1, 6, 20, 50, 105, &c. Ans. 2.

Problem II. To find the nth term of the series a, b, c, d, e, &c.

1768; and to the second volume of *Euler's Algebra*, English edition; where he will find the whole doctrine of Continued Fractions, and every part of the Indeterminate Analysis, amply developed.

Let d', d'', d''', d^{iv}, &c. represent the first term in the first, second, third, fourth, &c. order of differences respectively, then in the general expressions for the first term of the nth order, we shall have, by making n successively equal to 1, 2, 3, &c., and transposing,

	Or, by substitution,
$b = a + d'$,	$b = a + d'$,
$c = -a + 2b + d''$	$c = a + 2d' + d''$
$d = a - 3b + 3c - d'''$,	$d = a + 3d' + 3d'' + d'''$
$e = -a + 4b - 6c + 4d + d^{iv}$,	$e = a + 4d' + 6d'' + 4d''' + d^{iv}$
&c. = &c.	

where the coefficients of a, d', d'', d''', &c. in the $(n+1)$th term of the series a, b, c, d, &c. are the same as the coefficients of the terms of a binomial raised to the nth power; that is, the $(n+1)$th term is $a + nd' + \frac{n(n-1)}{2}d'' + \frac{n(n-1)(n-2)}{2.3}d''' +$ &c.; and $\therefore$ nth term is $a + (n-1)d' + \frac{(n-1)(n-2)}{2}d'' + \frac{(n-1)(n-2)(n-3)}{2.3}d''' +$ &c.

Cor. The above series will terminate only when one of the quantities d', d'', d''', &c. becomes 0; hence, when the differences do not at length vanish, the nth term cannot be accurately found, but must be approximated to.

1. Required the tenth term of the series 1, 4, 8, 13, 19, &c.

$$\left.\begin{array}{l} 1,\ 4,\ 8,\ 13,\ 19, \\ \quad 3,\ 4,\ 5,\ 6, \\ \quad\ 1,\ 1,\ 1, \\ \qquad 0. \end{array}\right\}$$

Here the first terms of the differences are 3, 1, and 0; that is, $d' = 3$, $d'' = 1$, and $d''' = 0$, also $a = 1$, and $n = 10$;

$\therefore a + (n-1)d' + \frac{(n-1)(n-2)}{2}d'' = 1 + 27 + 36 = 64$, which is the tenth term required.

2. Required the twelfth term in the series 1, 2^3, 3^3, 4^3, 5^3, &c.

$$\left.\begin{array}{l} 1,\ 8,\ 27,\ 64,\ 125, \\ \quad 7,\ 19,\ 37,\ 61, \\ \quad\ 12,\ 18,\ 24, \\ \qquad 6,\ 6, \\ \qquad 0. \end{array}\right\}$$

Here $d' = 7$, $d'' = 12$, $d''' = 6$, $d^{iv} = 0$, also $a = 1$, and $n = 12$; $a + (n-1)d' + \frac{(n-1)(n-2)}{2}d'' +$ &c. $= 1 + 77 + 660 + 990 = 1728$, the twelfth term.

3. Required the 20th term of the series 1, 3, 5, 7, &c.
4. Required the 20th term of the series 1, 3, 6, 10, 15, &c.
5. Required the 15th term of the series 1, 2^2, 3^2, 4^2, &c.

Ans. 39. Ans. 210. Ans. 225.

Problem III. To find the sum of n terms of a series.

If in the series a, b, c, d, &c. $a = 0$, then, by the last problem, the expression for the $(n+1)$th term will be

$$nb+\frac{n(n-1)}{2}d''+\frac{n(n-1)(n-2)}{2.3}d'''+\&c.$$

Now, if we put $b=\alpha$, $c=\alpha+\beta$, $d=\alpha+\beta+\gamma$, &c. then the $n(+1)$th term of the series a, b, c, d, &c., or the nth term of the series b, c, d, &c. will be n terms of the series $\alpha+\beta+\gamma+\&c.$, and d'' in the former series is the same as d' in this; hence the sum of n terms of the series $\alpha, \beta, \gamma, \delta$, &c. is

$$na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2.2}d''+\&c.$$

1. Required the sum of n terms of the series 1, 3, 5, 7, &c.

$$\left.\begin{array}{l}1, 3, 5, 7,\\ 2, 2, 2,\\ 0, 0.\end{array}\right\}\begin{array}{l}\text{Here } d'=2, \text{ and } d''=0;\ \text{also } \alpha=1;\\ \therefore na+\dfrac{n(n-1)}{2}d'=n^2=\text{sum of } n \text{ terms.}\end{array}$$

2. Required the sum of n terms of the series $1, 2^2, 3^2, 4^2, 5^2$, &c.

$$\left.\begin{array}{l}1,\ 4,\ 9,\ 16,\ 25,\\ \quad 3,\ 5,\ 7,\ 9,\\ \quad\quad 2,\ 2,\ 2,\\ \quad\quad\quad 0, 0.\end{array}\right\}\text{Here } d'=3,\ d''=2, \text{ and } d'''=0;\ \text{also } \alpha=1$$

$$\therefore na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2.3}d''=\frac{3n^2-3n}{2}\ldots.$$

$$+\frac{n^3-3n^2+2n}{3}=\frac{n(n+1)(2n+1)}{6}=\text{sum of } n \text{ terms.}$$

3. Required the sum of n terms of the series 1, 2, 3, 4, 5, &c.
4. Find the sum of 12 terms of the series 1, 3, 7, 12, 18.
5. Required tue sum of n terms of the series 1, 3, 6, 10, 15, &c.

Answers, $\dfrac{n^2+n}{2}$, 430, and $\dfrac{n(n+1)(n+2)}{6}$.

6. Required the sum of n terms of the series $1, 2^3, 3^3, 4^3$, &c.
7. Required the sum of n terms of the series $1, 2^4, 3^4, 4^4$, &c.

Answers. $\dfrac{n^2(n+1)^2}{4}$, and $\dfrac{n^5}{5}+\dfrac{n^4}{2}+\dfrac{n^3}{3}-\dfrac{n}{30}$.

On the Summation of the Infinite Series.

172. An Infinite Series is a progression of quantities proceeding onwards without termination, but usually according to some regular law, discoverable from a few of the leading terms.

173. A converging series is a series whose successive terms decrease or become less and less, as the series

$\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}+\dfrac{1}{x^4}$, x being a whole number.

174. A diverging series is one whose successive terms increase or become greater and greater. Such is the series

$$\frac{1}{1+2}=1-2+4-8+16-\&c.$$

175 A neutral series is one whose terms are all equal, but have signs alternately $+$ and $-$, as the series

$$\frac{1}{1+1}=1-1+1-1+1-\&c.$$

176, An ascending series is one in which the powers of the unknown quantity ascend, as in the series $a+bx+cx^2+dx^3$.

177. A descending series is one in which the powers of the unknown quantity descend, as in the series $a+bx^{-1}+cx^{-2}+dx^{-3}$.

178. The summation of series is the finding a finite expression equivalent to the series.

179. As different series are often governed by very different laws, the methods of finding the sum which are applicable to one class of series will not apply universally; a great variety of useful series may be summed by help of the following considerations.

180. As $\frac{q}{n}-\frac{q}{n+p}=\frac{pq}{n(n+p)}$; $\therefore \frac{q}{n(n+p)}=\ldots\frac{1}{p}\left\{\frac{q}{n}-\frac{q}{n+p}\right\}$

$\therefore$ any fraction of the form $\frac{q}{n(n+p)}$ is equal to $\frac{1}{p}$th the diff. between the two fractions $\frac{q}{n}$ and $\frac{q}{n+p}$ hence, if this diff. be known, the value of $\frac{q}{n(n+p)}$ will be known, whether $\frac{q}{n}$ and $\frac{q}{n+p}$ be known or not; and it $\therefore$ follows, that if there be any series of frac. each having the form $\frac{q}{n(n+p)}$, the sum of the series will be $=$ to $\frac{1}{p}$th the diff. between a series of frac. of the form $\frac{q}{n}$, and another of the form $\frac{q}{n+p}$, and if this difference can be obtained, the sum of the proposed series may be readily found, whatever be the values of p, q, and n.

1. Required the sum of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$ &c. continued to infinity.

Here $q=1$, and $p=1$, also $n=1$, 2, 3, &c. successively;

$$\therefore \left\{\begin{array}{l} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\&c.\text{ ad inf.} \\ -(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\&c.\text{ ad inf.})\end{array}\right\}=1=\text{sum.}$$

2. Required the sum of the above series to n terms.

$$\left\{\begin{array}{l} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots\frac{1}{n} \\ -(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots\frac{1}{n}+\frac{1}{n+1}) \end{array}\right\}=1-\frac{1}{n+1}=\frac{n}{n+1}.$$

3. Required the sum of the series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+$ &c. ad infinitum. Here $p=2$,

$$\left\{\begin{array}{l} 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\text{\&c. ad inf.} \\ -(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\text{\&c. ad inf.}) \end{array}\right\}=1,\ \therefore \frac{1}{p}=\frac{1}{2}=\text{sum}.$$

4. Required the sum of the above series to n terms.

$$\left\{\begin{array}{l} 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\frac{1}{2n-1} \\ -(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\frac{1}{2n-1}+\frac{1}{2n+1}) \end{array}\right\}=1-\frac{1}{2n+1}\cdots$$

$=\frac{2n}{2n+1}$, and $\frac{1}{p}$ of this sum is $\frac{n}{2n+1}=$ sum.

5. Required the sum of the series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+$&c to infinity. Here $p=3$.

$$\left\{\begin{array}{l} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{\&c.} \\ -(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\text{\&c.}) \end{array}\right\}=1+\frac{1}{2}+\frac{1}{3}=1\tfrac{5}{6},$$ and $\frac{1}{p}$th of this is $\frac{11}{18}=$ sum.

6. Required the sum of n terms of the above series.

$$\left\{\begin{array}{l} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\cdots\frac{1}{n} \\ -(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\cdots\frac{1}{n+3}) \end{array}\right\}=1+\frac{1}{2}+\frac{1}{3}-$$

$$\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)=\frac{n}{n+1}+\frac{n}{2n+4}+\frac{n}{3n+9};$$

$$\therefore \frac{n}{3n+3}+\frac{n}{6n+12}+\frac{n}{9n+27}=\text{sum}.$$

7. Required the sum of the series $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+$ &c. Here $p=2$, and $q=2$, 3, 4, &c. successively.

$$\left\{\begin{array}{l} \frac{2}{3}-\frac{3}{5}+\frac{4}{7}-\frac{5}{9}+\text{\&c.} \\ -(\frac{2}{5}-\frac{3}{7}+\frac{4}{9}-\text{\&c.}) \end{array}\right\}=\frac{2}{3}-1+1-1+1-1+\text{\&c.}=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$$

and $\frac{1}{p}$ of this is $\frac{1}{12}=$ sum.

8. Required the sum of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$&c. ad infin-

itum. This series is evidently the same as the following, viz: $1+\frac{1}{2}+\frac{1}{2.5}+\frac{1}{2.5}+$&c. $\div 2$, it becomes $\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+$ &c., whose sum is 1 (Ex. 1st; $\therefore$ the sum of the series is 2.

9. Required the sum of the series $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+$ &c. ad infinitum. This series is the same as $\frac{1}{4}\left(\frac{1}{3.2}+\frac{1}{6.3}+\frac{1}{9.4}+\right.$ &c. $=\frac{1}{12}\left(\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\text{\&c.}\right)=$ (Ex. 1) $\frac{1}{12}$; also the sum to n terms is $\frac{n}{12(n+1)}$.

10. Required the sum of the series $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+$ &c. ad infinitum. Ans. $\frac{3}{4}$.

11. Find the sum of the series $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-$&c. ad infinitum. Ans. $\frac{1}{4}$.

12. Find the sum of the series $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+$ &c. ad infinitum. Ans. $\frac{1}{12}$.

13. Find the sum of the above series to n terms.

Ans. $\frac{1}{12}-\frac{1}{4(4n+3)}$

14. Required the sum of the series $\frac{4}{1.5}+\frac{4}{5.9}+\frac{4}{9.13}+\frac{4}{13.17}$ +&c. ad infinitum. Ans. 1

181, since $\frac{q}{n(n+p)}-\frac{q}{(n+p)\,(n+2p)}=\dots\frac{2pq}{n(n+p)(n+2p)}$;

$\therefore \frac{q}{n(n+p)(n+2p)}=\dots\frac{1}{2p}\left\{\frac{q}{n(n+p)}-\frac{q}{n(n+p)\,(n+2p)}\right\}$

hence the sum of any series of fractions, each of which is of the form $\frac{q}{n(n+p)(n+2p)}$ is equal to $\frac{1}{2p}$ the difference between one series, whose terms are of the form

$\frac{q}{n(n+p)}$, and another, whose terms are of the form

$\frac{q}{(n+p)(n+2p)}$.

1. Required the sum of the series $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+$

&c. ad infinitum. Here $p=1$, and $q=4, 5, 6$, &c. successively;

$$\therefore \left\{\begin{array}{l} \frac{4}{1.2}+\frac{5}{2.3}+\frac{6}{3.4}+\text{\&c.} \\ \quad -\left(\frac{4}{2.3}+\frac{5}{3.4}+\text{\&c.}\right) \end{array}\right\} = \frac{4}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\text{\&c.} =$$

(Art. 180, Ex. 1), $2\frac{1}{2}$ and $\frac{1}{2p}$ of this is $1\frac{1}{4}=$ sum.

2. Required the sum of $\frac{3}{5.8.11}+\frac{9}{8.11.14}+\frac{15}{11.14.17}+$&c. ad infinitum. Here $p=3$,

$$\left\{\begin{array}{l} \frac{3}{5.8}+\frac{9}{8.11}+\frac{15}{11.14}+\text{\&c.} \\ \quad -\left(\frac{3}{8.11}+\frac{9}{11.14}+\text{\&c.}\right) \end{array}\right\} = \frac{3}{5.8}+\frac{6}{8.11}+\frac{6}{11.14}+\text{\&c.}$$

$$\frac{3}{5.8}+\frac{1}{3}\left\{\begin{array}{l} \frac{3}{8}+\frac{6}{11}+\frac{6}{14}+\text{\&c.} \\ \quad -\left(\frac{6}{11}+\frac{6}{14}+\text{\&c.}\right) \end{array}\right\} = \frac{3}{5.8}+\frac{1}{8}=\frac{13}{40},$$ and $\frac{1}{2p}$ of this is $\frac{13}{240}=$ sum.

3. Sum the series $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+$&c. ad infinitum.

Ans. $\frac{1}{8}$.

4. Find sum of the series $\frac{1}{1.3.5}+\frac{4}{3.5.7}+\frac{7}{5.7.9}+\frac{10}{7.9.11}+$&c ad infinitum.

Ans. $\frac{5}{24}$.

5. Required the sum of the series

$$\frac{a}{n(n+p)(n+2p)}+\frac{a+b}{(n+p)(n+2p)(n+3p)}+\ldots\ldots$$

$$\frac{a+2b}{(n+2p)(n+3p)(n+4p)}$$ &c. ad infinitum. Ans. $\frac{pa+bn}{2p^2n(n+p)}$.

(182.) Likewise, since

$$\frac{q}{n(n+p)(n+2p)}-\frac{q}{(n+p)(n+2p)(n+3p)}=\ldots$$

$$\frac{3pq}{n(n+p)(n+2p)(n+3p)}, \therefore \frac{q}{n(n+p)(n+2p)(n+3p)}=$$

$$\frac{1}{3p}\left\{\frac{q}{n(n+p)(n+2p)}-\frac{q}{(n+p)(n+2p)(n+3p)}\right\}$$ $\therefore$ any series of fra. of the form $\frac{q}{n(n+p)(n+2p)(n+3p)}$ is $=$ to $\frac{1}{3p}$ the diff. between a series of the form $\frac{q}{n(n+p)(n+2p)}$, and another of the

form $\dfrac{q}{(n+p)(m+2p)(n+3p)}$.

1. Required the sum of the series $\dfrac{1}{1.2.3.4}+\dfrac{1}{2.3.4.5}+\dfrac{1}{3.4.5.6}$ +&c. ad infinitum. Here $p=1$,

$$\left.\begin{array}{l}\dfrac{1}{1.2.3}+\dfrac{1}{2.3.4}+\dfrac{1}{3.4.5}+\&c.\\ \quad -(\dfrac{1}{2.3.4}+\dfrac{1}{3.4.5}+\&c.\end{array}\right\}=\frac{1}{1.2.3}=\tfrac{1}{6};\ \therefore \frac{1}{3p}(\tfrac{1}{6})=\tfrac{1}{18}.$$

2. Required the sum of the series $\dfrac{1}{1.3.5.7}+\dfrac{2}{3.5.7.9}+\dfrac{3}{5.7.9.11}$ +&c. ad infinitum. Here $p=2$,

$$\left.\begin{array}{l}\dfrac{1}{1.3.5}+\dfrac{2}{3.5.7}+\dfrac{3}{5.7.9}+\\ \quad -(\dfrac{1}{3.5.7}+\dfrac{2}{5.7.9}+).\end{array}\right\}=\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+=\tfrac{1}{12};$$

$\therefore \dfrac{1}{3p}(\tfrac{1}{12})=\tfrac{1}{72}=$ sum.

3. Sum the series $\dfrac{2}{3.6.9.12}+\dfrac{5}{6.9.12.15}+\dfrac{8}{9.12.15.18}$ + &c. ad infinitum. Ans. [illegible].

4. Sum the series $\dfrac{6^2}{1.2.3.4}+\dfrac{7^2}{2.3.4.5}+\dfrac{8^2}{3.4.5.6}$+ ad infinitum. Ans. [illegible].

(183.) In a similar manner, it may be shown that the sum of any series of fractions of the form

$\dfrac{q}{n(n+p)(n+2p)\ldots\ldots(n+mp)}$ is $=$ to $\dfrac{1}{mp}$ the diff. between a series of the form $\dfrac{q}{n(n+p)(n+2p)\ldots\ldots\{n+(m-1)p\}}$, and another of the form $\dfrac{q}{(n+p)(n+2p)(\ldots\ldots(n+mp)}$.

184. Again, since

$$\frac{a(a+b)(a+2b)\ldots\ldots(a+pb)}{n(n+b)\ \ldots\ldots\ \{n+(p-1)b\}}-\frac{a(a+b)(a+2b)\ \ldots\ldots\ \{a+(p+1)b\}}{n(n+b)\ldots\ldots(n+pb)}$$

$$=\frac{a(n-a-b)(a+b)(a+2b)\ldots\ldots(a+pb)}{n(n+b)(n+2b)\ldots\ldots(n+pb)}.$$

$$\therefore \frac{a(a+b)(a+2b)\ \ldots\ldots\ (a+pb)}{n(n+b)(n+2b)(\ldots\ldots(n+pb)}=$$

$$\frac{1}{n-a-b}\left\{\frac{a(a+b)\dots\dots(a+pb)}{n(n+b)\dots\dots\{n+(p-1)b\}}-\right.$$
$$\left.\frac{a(a+b)\dots\dots\{a+(p+1)b\}}{n(n+b)\dots\dots(n+pb)}\right\}$$ $\therefore$ any series of fractions of the form $\frac{a(a+b)\dots\dots(a+pb)}{n(n+b)\dots\dots(n+pb)}$ is $=$ to $\frac{1}{n-a-b}$ the difference of a series of the form $\frac{a(a+b)\dots\dots(a+pb)}{n(n+b)\dots\dots\{n+(p-1)b\}}$, and another of the form

$$\frac{a(a+b)\dots\{a+(p+1)b\}}{n(n+b)\dots\dots(n+pb)}.$$

1. Sum the series $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\dots\frac{1.3.5.7}{2.4.6.8}+$ &c. to r terms. Here $a=1$, $b=2$, and $n=2$,

$$\left\{\begin{array}{l}1+\frac{1.3}{2}+\frac{1.3.5}{2.4}+\dots\frac{1.3.5.7\dots(2r-1)}{2.4.6\dots(2r-2)} \\ -\left(\frac{1.3}{2}+\frac{1.3.5}{2.4}+\dots\dots\dots\frac{1.3.5.7\dots(2r+1)}{2.4.6\dots2r}\right)\end{array}\right\}=$$

$-\frac{1.3.5.7\dots(2r+1)}{2.4.6\dots2r}$, and $\frac{1}{n-a-b}$ of this is $\frac{1.3.5.7\dots(2r+1)}{2.4.6\dots2r}-1=$ sum of r terms; when r is infinite, this expression is evidently infinite also.

2. Required the sum of the series

$$\frac{a}{n}+\frac{a(a+b)}{n(n+b)}+\frac{a(a+b)(a+2b)}{n(n+b)(n+2b)}+\text{\&c. to } r \text{ terms.}$$

$$\left\{\begin{array}{l}a+\frac{a(a+b)}{n}+\dots\frac{a(a+b)\dots\{a+(r-1)b\}}{n(n+b)\dots\{n+(r-2)b\}} \\ -\left(\frac{a(a+b)}{n}\dots\dots\dots\dots\frac{a(a+b)\dots(a+rb)}{n(n+b)\dots\{n+(r-1)b\}}\right)\end{array}\right\}$$

$$=a-\frac{a(a+b)(a+2b\dots(a+rb)}{n.(n+b)\dots\{n+(r-1(b\}};$$

$$\therefore \text{sum}=\frac{a}{n-a-b}-\frac{a(a+b)(a+2b)\dots(a+rb)}{(n-a-b)n(n+b)\dots\{n+(r-1)b\}}.$$

If r be infinite, then this expression for the sum will become finite only in particular cases: Thus, if $n=a+2b$, the second fraction in the above expression will be $\frac{a(a+b)}{b\{a(r+1)b\}}$, which evidently vanishes when r is infinite, in which case the sum is $\frac{a}{n-a-b}$; and the same would, of course, be the case if n were greater than $a+2b$.

But if n were equal to $a+b$, then the said fraction would become $\frac{a(a+b)(a+2b)\ .\ .\ .\ (a+rb)}{(a+b)(a+2b)\ .\ .\ .\ (a+rb)}=a$, and the sum $\frac{a}{n-n}-a=\frac{a}{0}-a$, an expression of no definite signification.

3. Find the sum of r terms of the series $\frac{2}{3}+\frac{2.4}{3.5}+\frac{2.4.6}{3.5.7}+\frac{2.4.6.8}{3.5.7.9}$. Ans. $\frac{2.4.6.8\ .\ .\ .\ (2r+2)}{3.5.7.9\ .\ .\ .\ (2r+1)}-2$.

4. Required the sum of the series $\frac{2}{5.6}+\frac{2.3}{5.6.7}+\frac{2.3.4}{5.6.7.8}+$ &c. ad infinitum. Ans. $\frac{2}{15}$.

185. As every summable infinite series may be supposed to arise from the expansion of some fractional expression, the value of the series may be often obtained by first assuming it equal to a fraction whose denominator is such, that when the series is multiplied by it, the product may be finite, which product being equal to the numerator of the assumed fraction, determines its value, as in the examples following.

1. Required the sum of the infinite series $x+x^2+x^3+$ &c.

Assume the series equal to $\frac{z}{1-x}$; then

$$\begin{array}{l} x+x^2+x^3+\&c. \\ 1-x \\ \hline x+x^2+x^3+\&c. \\ \quad -x^2-x^3-\&c. \\ \hline z=x \quad * \quad * \quad * \end{array}$$

that is, $x+x^2+x^3+\&c.=\frac{x}{1-x}$.

If $x=\frac{1}{2}$, then $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\&c.=\frac{\frac{1}{2}}{\frac{1}{2}}=1$.

If $x=\frac{1}{3}$, then $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\&c.=\frac{\frac{1}{3}}{\frac{2}{3}}=\frac{1}{2}$.

2. Required the sum of the infinite series $x-x^2+x^3-x^4+$ &c. Assume the series $=\frac{z}{1+x}$

then
$$\begin{array}{l} x-x^2+x^3-x^4+\&c. \\ 1+x \\ \hline x-x^2+x^3-x^4+\&c. \\ \qquad x^2-x^3+x^4-\&c. \\ \hline z=x \quad 0 \quad 0 \quad 0 \end{array}$$

that is, $x-x^2+x^3-x^4+\&c.=\frac{x}{1+x}$.

If $x=\frac{1}{2}$, then $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\&c.=\frac{\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{3}$.

If $x=1$, then $1-1+1-1+1-\&c.=\frac{1}{1+1}=\frac{1}{2}$.

If $x=2$, then $2-4+8-16+\&c.=\frac{2}{1+2}=\frac{2}{3}$.

3. Required the sum of the infinite series $x+2x^2+3x^3+3x^3+\&c.$ Assume the series $\frac{z}{(1-x)^2}=\frac{z}{1-2x+x^2}$; then

$$\begin{array}{l} x+2x^2+3x^3+\&c. \\ 1-2x+x^2 \\ \hline x+2x^2+3x^3+\&c. \\ \quad -2x^2-4x^3-\&c. \\ \qquad\qquad x^3+\&c. \\ \hline \end{array}$$

$x=x$ * * *

that is, $x+2x^2+3x^3+$ &c. $=\dfrac{x}{(1-x)^2}$.

If $x=\frac{1}{2}$, then $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+$ &c. $=\dfrac{\frac{1}{2}}{\frac{1}{4}}=2$.

If $x=\frac{1}{3}$, then $\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+$ &c $=\dfrac{\frac{1}{3}}{\left|\frac{2}{3}\right|^2}=\frac{3}{4}$.

4. Find the sum of the infinite series $x+4x^2+9x^3+16x^4+$.

Assume the sum $=\dfrac{z}{(1-x)^3}$;

then $(1-x)^3\times(x+4x^2+9x^3+\&c.)=x+x^2$;

$$\therefore x+4x^2+9x^3+\&c. =\frac{x+x^2}{(1-x)^3}=\frac{x(1+x)}{(1-x)^3}.$$

If $x=\frac{1}{2}$, then $\frac{1}{2}+\frac{4}{4}+\frac{9}{8}+\frac{16}{16}+$ &c. $=6$.

1. Required the sum of the series $\dfrac{3}{1.2.2}+\dfrac{1}{2.3.2^2}+\dfrac{5}{3.4.2^3}+$ &c. ad infinitum.

$$\left\{\begin{array}{l}\dfrac{3}{1.2}+\dfrac{4}{2.2^2}+\dfrac{5}{3.2^3} \\ \qquad -\left(\dfrac{3}{2.2}+\dfrac{4}{3.2^2}+\&c.\right)\end{array}\right\}\begin{array}{l}=\dfrac{3}{1.2}-\left(\dfrac{2}{2.2^2}+\dfrac{3}{3.2^3}+\&c.\right. \\ =\dfrac{3}{1.2}-\left(\dfrac{1}{2^2}+\dfrac{1}{2^3}+\dfrac{1}{2^4}+\&c.\right.\end{array}$$

$=\frac{3}{2}-\frac{1}{2}=1=$ sum.

2. Find the sum of the series $\dfrac{5}{1.2.3.2^2}+\dfrac{6}{2.3.4.3^3}+\dfrac{7}{3.4.5.2^4}$ +&c. ad infinitum. Ans. $\frac{1}{4}$.

3. Required the sum of the series $x+3x^2+6x^3+10x^4+$ &c. ad infinitum. Ans. $\dfrac{x}{(1-x)^3}$.

4. Required the sum of n terms of the series
$\dfrac{1}{4.8}-\dfrac{1}{6.10}+\dfrac{1}{8.12}-$ &c. Ans. $\dfrac{n}{16(1+n)}-\dfrac{n}{12(3+2n)}$.

5. Find the sum of the series $\dfrac{1}{8.18}+\dfrac{1}{10.21}+\dfrac{1}{12.24}+$ &c. ad infinitum. Ans. $\frac{3}{80}$.

6. Find the sum of the series $\dfrac{10.18}{2.4.9.12}+\dfrac{12.21}{4.6.12.15}+\dfrac{14.24}{6.8.15.18}$ +&c. ad infinitum. Ans. $\frac{11}{32}$.

Diophantine Analysis.

186. Diophantine analysis is that part of algebra which relates to the finding particular rational values for general expressions

under a surd form; the principal methods of effecting which are comprehended in the following problems:

Problem I. To find such values of x as will render rational the expression $\sqrt{(ax^2+bx+c)}$. Before we can give any direct investigation of this problem, it will be necessary to consider the nature of the known quantities a, b, c, because there are several cases in which the thing here proposed to be done becomes impossible, and that solely on account of these known quantities.

CASE I. *When $a=0$, or when the expression is of the form* $\sqrt{(bx+c)}$. Put $\sqrt{(bx+c)}=p$, or $bx+c=p^2$, then $x=\frac{p^2-c}{b}$; consequently, whatever value be given to p, there must necessarily result a corresponding value of x that will render the proposed equation rational, and equal to p.

1. Find a number such, that if it be multiplied by 5, and the product increased by 2, the result shall be a square.

Put $5x+2=p^2$, then $x=\frac{p^2-2}{5}$; if $p=2$, then $x=\frac{2}{5}$; and by assuming other values for p, different values of x may be had.

2. Find two numbers, whose difference shall be equal to a given number a, and the difference of whose squares shall be also a square.

Let x be one number, then $a+x$ is the other, and we have to make $(a+x)^2-x^2$, or a^2+2ax, a square. Put $a^2+2ax=p^2$, then $x=\frac{p^2-a^2}{2a}$, where the value of p may be any number.

CASE II. *When $a=0$, or when the expression is of the form* $\sqrt{(ax^2+bx)}$. Put $\sqrt{(ax^2+bx)}=px$, or $ax^2+bx=p^2x^2$, then $ax+b=p^2x$; whence $x=\frac{b}{p^2-a}$, and whatever value may be given to p in this expression, there will result a value of x that will make the proposed expression rational.

1. Find a number such, that if its half be added to double its square, the result shall be a square.

Let x be the number, then we must have $2x^2+\frac{1}{2}x=$ a square, which denote by p^2x^2, then $2x+\frac{1}{2}=p^2x$, or $2x-p^2x=-\frac{1}{2}$; $\therefore x=\frac{\frac{1}{2}}{p^2-2}$, p being any number whatever: If p be taken $=2$, then $x=\frac{1}{4}$.

2. Find two numbers, whose sum shall be equal to a given number a, and whose product shall be a square.

Let x be one number, then $a-x$ is the other, and we have to make $ax-x^2$ a square: Put $ax-x^2=p^2x^2$, then $a-x=p^2x$, whence

$x=\frac{a}{p^2+1}$, p being any number whatever.

Otherwise. Assume $ax^2+bx=y^2$. Now we may take $\left.\begin{array}{l} y=r(ax+b) \\ y=sx \end{array}\right\}$, where r and s must be so assumed that their product may be 1; which is the case generally when

$r=\frac{m}{n}$, and $s=\frac{n}{m}$. Equating the values of x, we get $rax+rb=sx$; $\therefore x=\frac{br}{s-ar}=\frac{bm^2}{n^2-am^2}$. The substitution of $\frac{m}{n}$ for r, and $\frac{n}{m}$ for s, is easily made; we have only to put m^2 in the place of r, and n^2 in the place of s, and when the coefficients are not affected with r or s, we must put the product mn after the coefficient; this will be easily understood from the nature of algebraic fractions.

CASE III. *When c is a square, or when the expressions are of the form* $\sqrt{(ax^2+bx+c^2)}$. Put $\sqrt{(ax^2+bx+c^2)}=px+c$, then $ax^2+bx+c^2=p^2x^2+2cpx+c^2$, or $ax^2+bx=p^2x^2+2cpx$, $\therefore ax+b=p^2x+2cp$, whence $x=\frac{2cp-b}{a-p^2}$.

1. Find two numbers, whose sum shall be 16, and such, that the sum of their squares shall be a square.

Let x be one number, then $16-x$ is the other, and we have to make $x^2+(16-x)^2$, or $2x^2-32x+256$, a square, $=(px-16)^2$ $=p^2x^2-32px+256$, and we then have $2x^2-32x=p^2x^2-32px$, or $2x-32=p^2x-32p$, whence $x=\frac{32(p-1)}{p^2-2}$. If we take $p=3$, we shall $x=9\frac{1}{7}$, $\therefore$ the two numbers are $9\frac{1}{7}$, and $6\frac{6}{7}$.

Otherwise. Put $ax^2+bx+c^2=y^2$. Then $ax^2+bx=y^2-c^2$.

$$\text{Assume } \left\{\begin{array}{l} y+c=r(ax+b) \\ y-c=sx \end{array}\right\}.$$

whence $rax+rb-c=sx+c$, and $x=\frac{2c-br}{ar-s}=\frac{2cmn-bm^2}{am^2-n^2}$.

CASE IV. *When a is a square, or when the expression is of the form* $\sqrt{(a^2x^2+bx+c)}$. Put $\sqrt{(a^2x^2+bx+c)}=ax+p$, or

$$a^2x^2+bx+c=a^2x^2+2pax+p^2,$$

then $bx+c=2pax+p^2$, $\therefore x=\frac{c-p^2}{2pa-b}$.

1. Find a number such, that if it be increased by 2 and 5 separately, the product of the sums shall be a square.

Let x be the number, then we have to make $(x+2)(x+5)$, or $x^2+7x+10$, a square, which denote by $(x-p)^2$, then $x^2+7x+10$

$=x^2-2px+p^2$, or $7x+10=-2px+p^2$, $\therefore x=\frac{p^2-10}{7+2p}$. If we take $p=4$, we shall have $x=\frac{2}{5}$.

Otherwise. For $a^2x^2+bx+c=\square$ *, put $x=\frac{1}{z}$, and it becomes, by rejecting the denominator z^2, $a^2+bz+cz^2=\square$; which is of the same form as above only a is in the place of c: hence, changing a for c, we have $z=\frac{2amn-bm^2}{cm^2-n^2}$. $\therefore x=\frac{1}{z}=\frac{cm^2-n^2}{2amn-bm^2}$.

CASE V. *When neither a, nor c, are squares, but when b^2-4ac is a square.* In this case it will first be necessary to show that the expression ax^2+bx+c will always be resolvable into two possible factors: For if we put $x^2+\frac{b}{a}x+\frac{c}{a}=0$, solve the equation, or find the two values of x in it, as $x=k$, and $x=k'$, then $x-k$, and $x-k'$, will obviously be the two factors of $x^2+\frac{b}{a}x+\frac{c}{a}$; and $\therefore$ $a(x-k)(x-k')$ will be equal to the proposed expression: Now the values of x in the above equation are

$x=-\frac{b}{2a}+\frac{\sqrt{(b^2-4ac)}}{2a}$, and $x=-\frac{b}{2a}-\frac{\sqrt{(b^2-4ac)}}{2a}$, or putting $b^2-4ac=d^2$, the values of x are $\frac{d-b}{2a}$, and $-\frac{b+d}{2a}$, and $\therefore$

$$(ax+\frac{b-d}{2})(x+\frac{b+d}{2a})=ax^2+bx+c;$$

we see therefore that the proposed expression under those conditions is always resolvable into two factors.

Let there be then $\sqrt{(ax^2+bx+c)}=\sqrt{\{(fx+g)(hx+k)\}}$, which put equal to $p(fx+g)$, then $(fx+g)(hx+k)=p^2(fx+g)^2$; or $(hx+k)=p^2(fx+g)$; whence $x=\frac{p^2g-k}{h-p^2f}$.

In addition to the above, we will here show the method of finding the factors of the formula ax^2+bx+c, when b^2-4ac is a square. Assume

$$ax^2+bx+c=(mx+p)(nx+q)=mnx^2+(mq+np)x+pq.$$

Hence we have by equating the coefficients, $a=mn$, $b=mq+np$, $\therefore c=pq$, we have also, $ac=mnpq$, and by subtracting 4 times this, from the square of the equation $b=mq+np$, and extracting the root, we have $mq-np=\sqrt{(b^2-4ac)}$; hence b^2-4ac must be a square: put it $=f^2$, then $mq-np=f$.

Hence by addition, subtraction, &c.

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* This symbol is used to denote the words *a square*

$mq=\frac{b+f}{2}$, $np=\frac{b-f}{2}$, or $q=\frac{b+f}{2m}$, $p=\frac{b-f}{2n}$.

Thus to find the factors of the formula, $6x^2+13x+6$. Here $m=3$, $n=3$, $f=\sqrt{(169-144)}=5$, $\therefore p=2$, $q=3$.

Hence the factors are $3x+2$, and $2x+3$. Taking the above general expressions for the factors of the formula ax^2+bx+c, we shall find the value of x to be $\frac{(b+f)nr^2-(b-f)ms^2}{2mn(ms^2-nr^2)}$; where r and s may be any numbers and m and n are factors of a, the coefficients of x.

1. Find such a value of x as will render the expression $6x^2+13x+6$ a square.

Here $a=6$, $b=13$, and $c=6$, and as this expression evidently does not belong to any of the preceding cases, it will be proper to try whether b^2-4ac is a square, which it is found to be, viz. 25; we are certain, therefore, that the expression may be represented by two factors, which are readily found to be $2x+3$, and $3x+2$.

Put therefore $6x^2+13x+6$, or $(2x+3)(3x+2)=\{p(2x+3)\}^2$, and it follows that $3x+2=p^2(2x+3)$, whence $x=\frac{3p^2-2}{3-2p^2}$.

If we take $p=1$, then $x=1$, and the expression is equal to 25.

Otherwise. Assume $ax^2+bx+c=(fx+g)(hx+k)=y^2$.

Now take $\left.\begin{array}{l} y=r(fx+g) \\ y=s(hx+k) \end{array}\right\}$; $\therefore rfx+rg=shx+sk$;

and therefore $x=\frac{gr-ks}{hs-fr}=\frac{gm^2-kn^2}{hn^2-fm^2}$.

CASE VI. *When the proposed expression can be divided into two parts, one of which is a square, and the other the product of two factors.*

This is the last case in which any general method of proceeding can be pointed out, and may often be serviceable when the expression does not come under either of the preceding cases: It is, however, sometimes troublesome to find whether the proposed expression can be decomposed as this case requires, or not; but if it be ascertained that it can, the expression $\sqrt{(ax^2+bx+c)}$ may be put under the form $\sqrt{\{(dx+e)^2+(fx+g)(hx+k)\}}$, and if we equate this with $(dx+e)+p(fx+g)$, there will result

$$(dx+e)^2+(fx+g)(hx+k)$$
$$=(dx+e)^2+2p(dx+e)(fx+g)+p^2(fx+g)^2, \text{ or}$$
$$hx+k=2p(dx+e)+p^2(fx+g);\ x=\frac{p(2e+pg)-k}{h-p(2d+pf)}.$$

1. Find a value of x such, that $2x^2+8x+7$ shall be a square.

This expression, after a few trials, is found to be equivalent to

$(x+2)^2+(x+1)(x+3)$, which being equated with
$\{(x+2)-p(x+1)\}^2=(x+2)^2-2p(x+2)(x+1)+p^2(x+1)^2$
there results $x+3=-2p(x+2)+p^2(x+1)$; whence $x=\frac{p^2-4p-3}{1+2p-p^2}$. If we take $p=3$, we shall have $x=3$, and $2x^2+8x+7=49$.

2. Find a value of x such that $15x^2+13x+6$ may be a square. Otherwise. Put, as before, $ax^2+bx+c=(dx+e)^2+(fx+g)(hx+k)=y^2$. Then $y^2-(dx+e)^2=(fx+g)(hx+k)$;

$$\text{take } \left.\begin{aligned} y+dx+e&=r(fx+g)\\ y-dx-e&=s(hx+k)\end{aligned}\right\}$$

$$\text{whence } rfx+rg-dx-e=shx+sk+dx+e;$$

$$\text{and } x=\frac{gr-ks-2e}{hs-fr+2d}=\frac{gm^2-kn^2-2emn}{hn^2-fm^2+2dmn}.$$

In the above values of x, m and n may be any numbers whatever.

The third and fourth cases may be solved after another manner, more general; putting $x=\frac{y}{z}$, and rejecting the denom. which is a square, the formula ax^2+bx+c, becomes $ay^2+byz+c^2z^2=\square$; assume this $=(cz+mx)(cz+ny)=c(r+ms)^2\times c(r+ns)^2$; hence we have $ay^2+byz+c^2z^2=c^2z^2+c(m+n)yz+mny^2$.

Equating coefficients,

$$b=c(m+n),\ mn=a, \text{ or } cm^2=bm-mnc=bm-ac;$$

now $cz+my=c(r-ms)^2=cr^2+2crms+cm^2s^2=cr^2+2crms+bms^2-acs^2$, (since $cm^2=bm-ac$); hence, we have

$$z=r^2-as^2, \text{ and } y=2crs+bs^2.$$

Cor. 1. If a, b, and c are each 1, then the formula becomes y^2+yz+z^2, and $z=r^2-s^2$, $y=2rs+s^2$.

Cor 2. If a and c are each 1, and $b=0$, then the formula becomes y^2+z^2, and $y=2rs$, $z=r^2-s^2$.

182. We have now given all the cases in which methods have been discovered to render the expression $\sqrt{(ax^2+bx+c)}$ rational; but as it may have rational values in other cases, it is of importance to be able to determine them.

Now this can only be done when one satisfactory value is already known, which value must therefore be found by trial; this being obtained, other values may be readily deduced.

183. Suppose the expression $\sqrt{(ax^2+bx+c)}$ is found to become rational when $x=r$, and that the value of the expression in this case is s; then $ar^2+br+c=s^2$: Put $x=y+r$, and we have, by substitution, $ax^2+bx+c=a(y+r)^2+b(y+r)+c$
$=ay^2+(2ar+b)y+ar^2+br+c$; and as this form comes under

Case 3, the value of y, in order that this last expression may be a square, can be found, and thence that of $x=y+r$.

1. Find such values of x that will render the expression $\sqrt{(10+8x-2x^2)}$ rational.

This expression is found to become rational when $x=1$.

Put $\therefore$ $x=1+y$, and we have, by substitution, $10+8x-2x^2=16+4y-2y^2$, which must be a square; denote it by $(4-py)^2=16-8py+p^2y^2$, and we shall have

$$16+4y-2y^2=16-8py+p^2y^2,$$

or $4-2y=-8p+p^2y$; whence $y=\dfrac{8p+4}{p^2+2}$: If we take $p=1$, then $y=4$, and $\therefore$ $x=5$, and the value of the proposed expression is 0.

2. Find values of x that will make the expression $\sqrt{(5x^2+12x+8)}$ rational.

3. Find a number such, that if three times itself be taken from three times its square, the remainder increased by 3 shall be a square.

Problem II. To find such values of x as will render rational the expression $\sqrt{(ax^3+bx^2+cx+d}$. There are but two cases in which a direct solution can be given to this problem: these are the following:

CASE 1. *When the two last terms are absent, or when the expression is of the form* $\sqrt{(ax^3+bx^2)}$. Put $\sqrt{(ax^3+bx^2)}=px$, or $ax^3+bx^2=p^2x^2$, then $ax+b=p^2$; whence $x=\dfrac{p^2-b}{a}$.

1. Find a number such, that if three times its cube be added to twice its square, the sum shall be a square. Here we must make $3x^3+2x^2$ a square; let p^2y^2 be the square, then $3x+2=p^2$, $\therefore$ $x=\dfrac{p^2-3}{2}$. If we take $p=3$, we have $x=3$, the number.

CASE II. *When the last term is a square, or the expression is of the form* $\sqrt{(ax^3+bx^2+cx+d^2)}$. Put $\sqrt{(ax^3+bx^2+cx+d^2)}=\dfrac{c}{2d}x+d$;* then $ax^3+bx^2+cx+d^2=\dfrac{c^2}{4d^2}x^2+cx+d^2$,

or $ax^3+b^2=\dfrac{c^2}{4d^2}x^2$; $\therefore$ $ax+b=\dfrac{c^2}{4d^2}$; whenc $x=\dfrac{c^2-4bd^2}{4ad^2}$.

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* The expression is assumed equal to $\dfrac{c}{2d}x+d$, in order that the two last terms in its square may be the same as the corresponding terms in the proposed expression.

This solution gives only one value of x, but from this, other values, when possible, may be obtained by the method next following.

When the 2d and 3d terms are absent, this method evidently fails.

1. Find such a value of x as will make the expression $3x^3-5x^2+6x+4$ a square. Put $3x^3+5x^2+6x+4=(\frac{3}{2}x+2)^2=\frac{9}{4}x^2+6x+4$, then $3x^3-5x^2=\frac{9}{4}x^2$, or $3x-5=\frac{9}{4}$; whence $x=\frac{29}{12}$, which value being substituted in the proposed expression, makes it equal to $(\frac{45}{8})^2$.

To these two cases may be added, as in the last problem, a third, by which other values may be had from one being previously known.

(4.) Suppose it is already known that the expression $\sqrt{(ax^3+bx^2+cx+d)}$ becomes rational when $x=r$, and that the value of the expression then becomes $=s$; that is, let $ar^3+br^2+cr+d=s^2$; then, as in Art. (3), put $x=y+r$, and we have

$$\left.\begin{array}{r} ay^3+3ary^2+3ar^2y+ar^3=ax^3 \\ by^2+2bry+br^2=bx^2 \\ cy+cr=cx \\ d=d \\ \hline ay^3+\quad b'y^2+\quad c'y+\quad s^2=\Box, \end{array}\right\}$$

b', c', and s^2, representing the sums of the quantities under which they are respectively placed, therefore the value of y may be determined by last case.

The expression $\sqrt{(x^3-x^2+2x+1)}$ is found to become rational when $x=2$; it is required to find another value of x that will answer. Put $x=y+2$, then $x^3-x^2+2x+1=y^3+3y^2+8y+9$; assume this last expression equal to $(\frac{4}{3}y+3)^2$, or $\frac{16}{9}y^2+8y+9$;

$$\text{then } y^3+3y^2=\tfrac{16}{9}y^2, \text{ or } y+3=\tfrac{16}{9}$$

$$\text{whence } y=-\tfrac{11}{9}, \text{ and } \therefore x=3+y=\tfrac{7}{9}.$$

Scholium. There are many cases in the preceding problem in which the unknown quantity admits of only one rational value, and many more in which the expression is impossible. If any expression can be divided into factors, one of which is a square, this square may be rejected, and the remaining factors only used; thus, if the expression ax^3+bx^2, or $x^2(ax+b)$, is to be made a square, it will be only necessary to make $ax+b$ a square; also, in in the expression x^3-x^2-x+1, which is equal to $(1-x)^2(1+x)$, it will only be necessary to make $1+x$ a square, in order that the whole expression may be a square.

Problem III. To find such values of x as will render rational the expression $\sqrt{(ax^4+bx^3+cx^2+dx+e)}$. In this problem there are three cases in which a direct solution can be obtained.

CASE I. *When both the first and last terms are complete squares, or when the expression is of the form* $\sqrt{(a^2x^4+bx^3+cx^2+dx+e^2.)}$

Put $a^2x^4+bx^3+cx^2+dx+e^2=(ax^2+mx+e)^2=$ $a^2x^4+2amx^3+(m^2+2ae)x^2+2mex+e^2$; then, in order that the three first terms in each side of this equation may destroy each other, we must take $b=2am$, or $m=\frac{b}{2a}$, and there will result $cx^2+dx=$

$$(m^2+2ae)x^2+2mex;$$

whence $x=\frac{d-2me}{m^2+2ae-c}$, or substituting for m its equal $\frac{b}{2a}$ we have $x=\frac{4a(ad-be)}{b^2+4a^2(2ae-c)}$; or, since e is found in the proposed expression only in its second power, it may be taken either positively or negatively; hence we get another value of x, viz. $x=\frac{4a(ad+be)}{b^2-4a^2(2ae+c)}$. Or this case of the problem may be solved differently by making $d=2me$, when m will be equal to $\frac{d}{2e}$, instead of $\frac{b}{2a}$, and we shall have $bx^3+cx^2=2amx^3+(m^2+2ae)x^2$; whence $x=\frac{m^2+2ae-c}{b-2am}$; or, substituting for m its equal $\frac{d}{2e}$, we have $x=\frac{d^2+4e^2(2ae-c)}{4e(be-ad)}$; or $x=\frac{d^2-4e^2(2ae+c)}{4e(be+ad)}$ this last value being obtained from supposing e negative, as before: Hence, by employing these two methods, four solutions may be obtained; it must be observed, however, that they all fail when b and d are both 0.

1. It is required to find such a value of x, that in the expression $x^4-6x^3+4x^2-24x+16$ may be a square. Put, according to the first of the above methods, $x^4-6x^3+4x^2-24x+16=(x^2-3x-4)^2$ $=x^4-6x^3+x^2+24x+16$, and there results $4x^2-24x=x^2+24x$, or $4x-24=x+24$; $x=\frac{48}{3}=16$.

If, according to the second method, we put the expression equal to $(x^2+3x-4)^2=x^4+6x^3+x^2-24x+16$, we have $6x^3+x^2=-6x^3+4x^2$, whence $x=\frac{1}{4}$.

Case II. *When the first term only is a square, or when the expression is of the form* $\sqrt{(a^2x^4+bx^3+cx^2+dx+e.)}$ Put $a^2x^4+bx^3+cx^2+dx+e=(ax^2+mx+n)^2=a^2x^4+2amx^3+(m^2+2an)x^2+2mnx+n^2$; then, in order that the 3 first terms in this equation may destroy each other, we must make

$$\left.\begin{array}{l} b=2am \\ c=m^2+2an \end{array}\right\} \text{ whence } \left\{\begin{array}{l} m=\frac{b}{2a} \\ n=\frac{c-m^2}{2a}=\frac{4a^2c-b^2}{8a^3}, \text{ we have} \therefore dx \end{array}\right.$$

$+e=2mnx+n^2$; whence $x=\dfrac{n^2-e}{d-2mn}$, or, substituting for m and n their values as deduced above, we have

$x=\dfrac{(4a^2c-b^2)^2-64a^6e}{8a^2\{8a^4d-b(4a^2c-b)^2\}}$; when b and d are both 0, this formula fails, the same as in the last case.

1. Required a value of x such, that the expression

$$4x^4+4x^3+4x^2+2x-6 \text{ may become a square.}$$

Here $m=1$, and $n=\frac{3}{4}$, therefore

$$4x^4+4x^3+4x^2+2x-6=(2x^2+x+\tfrac{3}{4})^2=4x^4+4x^3+4x^2+\tfrac{3}{2}x+\tfrac{9}{16}$$

and we have $2x-6=\frac{3}{2}x+\frac{9}{16}$;

whence $x=\frac{105}{8}=13\frac{1}{8}$.

CASE III. *When the last term only is a square, or when the expression is of the form* $\sqrt{(ax^4+bx^3+cx^2+dx+e^2)}$.

Put $ax^4+bx^3+cx^2+dx+e^2=(mx^2+nx+e)^2=$

$$m^2x^4+2mnx^3+(n^2+2me)x^2+2nex+e^2;$$

then, in order that the three last terms on each side of this equation may destroy each other, we must make

$$\left.\begin{array}{l} d=2ne \\ c=n^2+2me \end{array}\right\} \text{ whence } \left\{\begin{array}{l} n=\dfrac{d}{2e} \\ m=\dfrac{c-n^2}{2e}=\dfrac{4ce^2-d^2}{8e^3} \end{array}\right.$$

, and then we shall have $ax^4+bx^3=m^2x^4+2mnx^3$, or $ax+b=m^2x+2mn$; whence $x=\dfrac{2mn-b}{a-m^2}$, or substituting for m and n, their values as deduced above, we have $x=\dfrac{8e^2\{d(4ce^2-d^2)-8be^4\}}{64ae^6-(4ce-d^2)^2}$, which formula fails under the same circumstances as those of the preceding cases.

The first case of this problem is evidently included in each of the two last cases, and therefore either of the two formulæ last deduced is also applicable to the first case.

1. Find such a value of x as will make the expression $5x^4-4x^3+3x^2-2x+1$ a square. Here $m=1$, and $n=-1$, therefore put

$$5x^4-4x^3+3x^2-2x+1=(x^2-x+1)^2=$$
$$x^4-2x^3+3x^2-2x+1,$$

and we have $5x^4-4x^3=x^4-2x^3$,

or $5x-4=x-2$;

whence $x=\frac{2}{4}=\frac{1}{2}$.

When the proposed expression does not come under either of the above cases, then, as in the preceding problems, one satisfactory value of the unknown quantity must be discovered by trial, after which, other values, when possible, may be obtained; but in

this, as well as in the preceding problems, there are many expressions in which the unknown quantity admits of only one value, and, in a great many instances, the value is impossible. We now proceed to show how to find other values from having one value already given.

185. Suppose it is already known that the expression $\sqrt{(ax^4+bx^3+cx^2+dx+e)}$ becomes rational when $x=r$, and that we have $ar^4+br^3+cr^2+dr+e=s^2$. Assume $y+r=x$, and we have

$$\left.\begin{aligned} ay^4+4ary^3+6ar^2y^2+4ar^3y+ar^4 &= ax^4 \\ by^3+3bry^2+3br^2y+br^3 &= bx^3 \\ cy^2+2cry+cr^2 &= cx^2 \\ dy+dr &= dx \\ e &= e \\ \hline ay^4+b'y^3+c'y^2+d'x+s^2 &= \square \end{aligned}\right\}$$

the terms in the last line representing the sums of the quantities under which they are respectively placed.

Hence the expression is reduced to a form in which the preceding case will apply, and therefore the value of y, and thence that of x, may be determined.

As formulæ of the form ax^4+b often occur in the solution of Diophantine Problems, it demands some further notice here.

Supposing p to be a value of x, that renders ax^4+b a square, so that $ap^4+b=q^2$, to find a general expression for other values:

Assume $x=p+y$; then because $ap^4+b=q^2$, by substitution ax^4+b becomes $q^2+4ap^3y+6ap^2y^2+4apy^3+ax^4=\square$; which formula falls under case 3, and by comparing each term with the general expression, we have $q=e$, $4ap^3=d$, $6ap^2=c$, $4ap=b$, and $a=a$; whence we have

$$n=\frac{d}{2e}=\frac{4ap^3}{2q}=\frac{2ap^3}{q},\ m=\frac{4ce^2-d^2}{8e^3}=\frac{24ap^2q^2-16a^2p^6}{8q^3}=$$

$$\frac{ap^2(3q^2-2ap^4)}{q^2}=\frac{ap^2(q^2+2b)}{q^3},$$ because $ap^4=q^2-b$. And

$$y=\frac{b-2mn}{m^2-a}=\frac{4apq^6-4a^2p^5q^2(q^2+2b)}{a^2p^4(q^2+2b)^2-aq^6}=\frac{4pq^6-(q^2-b)(q^2+2b)4pq^2}{(q^2-b)(q^2+2b)^2-q^6}$$

(dividing by a, and substituting q^2-b for ap^4)

$$=p\times\frac{8bq^2-4q^4}{3q^4-4b^2};$$ hence we have $$x=p+y=p\times\frac{8bq^2-q^4-4b^2}{3q^4-3b^2},$$

This simple value of x shows that the most complex expressions may often be reduced, by skilful management; the reduction was first given by Euler. This formula may often be applied to difficult questions, where expressions of the form ax^4+b often occur.

1. Find such values of x, that the expression $3x^4+2x^3-5x^2+7x-3$ may be a square. It appears, upon trial, that if 1 be substituted for x, that the expression will become a square, viz. 4.

Put therefore $x=y+1$, and we have

$$3x^4+2x^3-5x^2+7x-3=3y^4+14y^3+19y^2+15y+4,$$

which must be made a square ; therefore, according to the last case, denote this square by $(\frac{79}{64}y^2+\frac{15}{4}y+2)^2=\frac{6241}{4096}y^4+\frac{1185}{128}y^3+19y^2+15x+4$, we shall then have $3y^4+14y^2=\frac{6241}{4096}y^4+\frac{1185}{128}y^3$, or $3y+14=\frac{6241}{4096}y+\frac{1185}{128}$; whence $y=\frac{19424}{6047}$, and, consequently, $x=\frac{25471}{6047}$.

Problem IV. To find such values of x as will render rational the expression $\sqrt[3]{(ax^3+bx^2+cx+d.)}$ In this problem there are likewise only three cases in which a direct solution can be obtained ; these are as follows.

CASE I. *When both first and last terms are cubes, or when the expression is of the form* $\sqrt[3]{(a^3x^3+bx^2+cx+d^3.)}$

Put $a^3x^3+bx^2+cx+d^3=(ax+d)^3=a^3x^3+3a^2dx^2+3ad^2x+d^3$, and we have $bx^2+cx=3a^2dx^2+3ad^2x$, or $bx+c=3a^2dx+3ad^2$;

$$\text{whence } x=\frac{3ad^2-c}{b-3a^2d}.$$

1. Find a value of x such, that the expression

$$x^3+9x^2+4x+8 \text{ may be a cube.}$$

Put $x^3+9x^2+4x+8=(x+2)^3=x^3+6x^2+12x+8$, and we shall then have $9x^2+4x=6x^2+12x$, whence $x=\frac{8}{3}=2\frac{2}{3}$.

CASE II. *When the first term only is a cube, or when the expression is of the form* $\sqrt[3]{(a^3x^3+bx^2+cx+d)}$.

Put $a^3x^3+bx^2+cx+d=(ax+m)^3=a^3x^3+3a^2mx^2+3am^2x+m^3$, and make $3a^2m=b$, or $m=\frac{b}{3a^2}$, and we shall then have $cx+d=3am^2x+m^3$; whence $x=\frac{m^3-d}{c-3am^2}$; or subs. for m its $=$, $\frac{b}{3a^2}$ we have $x=\frac{b^3-27da^6}{(3ca^3-b^2)9a^3}$.

1. Find a value of x that will make the expression

$$8x^3-4x^2+2x-12 \text{ a cube.}$$

Put $8x^3-4x^2+2x-12=(2x-\frac{1}{3})^3=8x^3-4x^2+\frac{2}{3}x+\frac{1}{27}$, and we get $2x-12=\frac{2}{3}x+\frac{1}{27}$; whence $x=\frac{325}{36}$.

CASE III. *When the last term only is a cube, or when the expression is of the form* $\sqrt[3]{(ax^3+bx^2+cx+d^3)}$.

Put $ax^3+bx^2+cx+d^3=(mx+d)^3=m^3x^3+3m^2dx^2+3md^2x+d^3$, and make $c=3md^2$, or $m=\frac{c}{3d^2}$, and there results $ax^3+bx^2=m^3x^3+3m^2dx^2$, or $ax+b=m^3x+3m^2d$; whence $x=\frac{3m^2d-b}{a-m^3}$, or subs.

for m its equal $\frac{c}{3d^2}$, we have $x = \frac{(c^3 - 3bd^3)\,9d^2}{27ad^3 - c^3}$.

1. Required such a value of x that will make the expression $2x^3+3x^2-4x+8$ a cube. Put $2x^3+3x^2-4x+8 = (-\frac{1}{3}x+2)^3 = -\frac{1}{27}x^3 + \frac{2}{3}x^2 - 4x + 8$, and we have $2x^3 + 3x^2 = -\frac{1}{27}x^3 + \frac{2}{3}x^2$ or $2x + 3 = -\frac{1}{27}x + \frac{2}{3}$; whence $x = -\frac{63}{55}$.

These two last cases are evidently applicable to those forms belonging to case 1st, and, therefore, when the first and last terms are both cubes, three solutions may be obtained, one from each case; it must however be observed, that they all fail when b and c are both 0. Having now given all the cases in which a direct solution of the problem can be obtained, it remains to show, as in the preceding problems, how, from having a particular solution, others may be derived from it.

(6.) This expression $\sqrt[3]{(ax^3 + bx^2 + bx + d)}$ becomes rational when $x = r$, and that then $ar^3+br^2+cr+d=s^3$. Assume $y + r = x$, and we have

$$\left.\begin{array}{rrrrl}
ay^3 + & 3ary^2 + & 3ar^2y + & ar^3 & = ax^3 \\
 & by^2 + & 2bry + & br^2 & = bx^2 \\
 & & cy + & cr & = cx \\
 & & & d & = d \\
\hline
ay^3 + & b'y^2 + & c'y + & s^3 & = \text{a cube.}
\end{array}\right\}$$

The expression is therefore reduced to a form which is resolvable by last case.

1. It is required to find such values for x, that the expression $2x^3-4x^2+6x+4$ may be a cube. It appears upon trial, that $x=1$ is a satisfactory value; put then $x=y+1$, and the expression becomes $2y^3+2y^2+4y+8$, which put equal to $(\frac{1}{3}x+2)^3 = \frac{1}{27}y^3+\frac{2}{3}y^2+4y+8$, and there results $2y^3+2y^2=\frac{1}{27}y^3+\frac{2}{3}y^2$, or $2y+2=\frac{1}{27}y+\frac{2}{3}$; whence $y=-\frac{36}{53}$, and, consequently, $x=\frac{17}{53}$.

On Double and Triple Equalities.

195. In the preceding problems, the object of our investigations has been to find rational values for expressions under a surd form, and our inquiries have been directed to each expression separately. Questions, however, often occur in the diophantine analysis, that requires us to find values for the unknown quantity, or quantities, that shall not only render a single expression, a square, cube, &c. but that shall also, at the same time, fulfil similar conditions in one or more other expressions, containing the same unknown quantity or quantities. In the case where two expressions are concerned, it is called a *double equality*, and when there are three expressions, a *triple equality*, &c. The following methods of resolving these equalities will be of service to the student in ordinary cases; but in those where the methods here given are

found to be insufficient, he must be guided by his own penetration and ingenuity, since no general method of proceeding, that shall be suitable to every case that may occur, can be given.

Problem I. To resolve the double equality $ax+b=\Box$, and $cx+d=\Box$. Put $ax+b=p^2$, and $cx+d=q^2$, then equating the two values of x, which the equations furnish, we have $\frac{p^2-b}{a}=\frac{q^2-d}{c}$, or $cp^2-cb=aq^2-ad$; therefore $c^2p^2=caq^2-cad+c^2b$; and, consequently, q must be such a value that the expression $caq^2-cad+c^2b$ may become a square, which value may be ascertained by one or the other of the preceding methods, and thence the value of x may be determined.

Problem II. To resolve the double equality, or

$\left\{\begin{array}{l} ax^2+bx=\Box, \\ cx^2+dx=\Box, \end{array}\right\}$ Put $x=\frac{1}{y}$, then if each equality be multiplied by y^2, there will result the double equality $a+by=\Box$, and $c+dy=\Box$; which belongs to the preceding problem. Or put $ax^2+bx=p^2x^2$, then $ax+b=p^2x$, and consequently,

$$x=\frac{b}{p^2-a}, \text{ and } \therefore cx^2+dx=c\left(\frac{b}{p^2-a}\right)^2+d\left(\frac{b}{p^2-a}\right)=\Box;$$

or multiplying by the square $(p^2-a)^2$, it becomes $cb^2-abd+dbp^2=\Box$; whence p may be determined, and thence x.

Problem III. Resolve the double equality

$\left.\begin{array}{l} ax^2+bx+c=\Box, \\ dx^2+ex+f=\Box. \end{array}\right\}$ Here it will be necessary first to resolve the equality by Problem 1, Art. (1), and to substitute the value x so deduced in the second equality $dx^2+ex+f=\Box$, which will in consequence, rise to the fourth power, and therefore its solution will belong to Prob. 3, p. 397.

Problem IV. To resolve the triple equality

$\left\{\begin{array}{l} ax+by=\Box, \\ cx+dy=\Box; \\ ex+fy=\Box: \end{array}\right\}$ Put $\left\{\begin{array}{l} ax+by=t^2 \\ cx+dy=u^2 \\ ex+fx=s^2 \end{array}\right\}$ then, by expunging y from the two first equations, we have $x=\frac{dt^2-bu^2}{ad-bc}$; and by expunging x from the same equations, we hove $y=\frac{au^2-ct^2}{ad-bc}$; therefore, by substituting for x and y, in the third equation, their respective values here exhibited, we shall have $\frac{af-be}{ad-bc}u^2+\frac{cf-de}{ad-bc}t^2=\Box$; or putting $u=tz$, and dividing the expression by the

square t^2, there arises the equality $\frac{af-be}{ad-bc}z^2-\frac{cf-de}{ad-bc}=\square$; from which the values of z may be determined. Having then found the values of z, we shall have, from the above values of x and y, observing to write tz for u, the following results: viz. $x=\frac{d-bz^2}{ad-bc}t^2$ and $y=\frac{az^2-c}{ad-bc}t^2$, where t may be any value whatever.

189. The above are the most general methods hitherto discovered for the resolution of double and triple equalities; we may now therefore proceed to show the practical application of the foregoing parts of the present chapter to the solution of diophantine questions; but, as has been already said, the student must expect to meet with cases in which the mode of proceeding must be left, in a great measure, for his own judgment and penetration to suggest. Indeed, the subject on which we are now treating has exercised the ingenuity of some of the most eminent mathematicians of Europe; but Euler and Lagrange have been the most successful in combating the difficulties with which it is attended. The performances of the former are contained in the second volume of his Algebra, which with the additions of Lagrange, forms the most complete body of information on the diophantine analysis extant, and it to this work, chiefly, that the attention of the student is directed. In the following solutions it will frequently be observed that much depends upon the nature and relation of the assumptions made at the commencement, as a little artifice and ingenuity here will often enable us readily to satisfy one or two conditions of the question, when those that remain may be obtained by one or other of the known methods already given.

1. Find three numbers such, that if to the square of each the product of the other two be added, the results shall be all squares.

If x, y, and z denote the numbers, then $x^2+yz=\square$, $y^2+xz=\square$, $z^2+xy=\square$. Assume $x=mz$, $y=nz$, then the expressions become $m^2+n=\square$, $n^2+m=\square$, $1+mn=\square$; now the first and second expressions are evidently squares, if $m+n=\frac{1}{4}$, $\therefore$ $m=\frac{1}{4}-n$; by substitution, $1+mn=1+\frac{1}{4}n-n^2=\square=(1-cn)^2$;

$$\text{whence } n=\frac{2c+\frac{1}{4}}{c^2+1}=\frac{8c+1}{4(c^2+1)}, \text{ and } m=\tfrac{1}{4}-n=\frac{c(c-8)}{4(c^2+1)}$$

$\therefore$ $x=mz=\frac{c(c-8)}{4(c^2+1)}z$, $y=nz=\frac{8c+1}{4(c^2+1)}z$; hence if z be taken $=4(c^2+1)$, then $x=c(c-8)$, $y=8c+1$; where c may be any number greater than 8: taking $c=9$, we get $z=328$, $y=73$, $x=9$.

Let y, $x-y$ and $4x$ denote the numbers. Then it only remains to make $16x^2+xy-y^2=\square=(4x-ry)^2=16x^2-8rxy+r^2y^2$.

Hence, by reduction $(8r+1)x=(r^2+1)y$; and consequently $x=r^2+1$; $y=8r+1$, and the numbers are $8r+1$, $r(r-8)$ and $4(r^2+1)$ whence r may be taken at pleasure. provided it be greater than 8. Let $r=9$, and the numbers are 73,9 and 328.

2. Find two numbers such, that their sum, the sum of their squares, and the sum of their cubes, may be all squares.

Let ax and bx denote the numbers; then $(a+b)x=\square=n^2$, $a^2+b^2=\square$, $(a^3+b^3)x=\square$, whence $x=\dfrac{n^2}{a+b}$; and by substituting this value of x in the third formula, we have $\dfrac{a^3+b^3}{a+b}n^2=\square$, or $(a^2-ab+b^2)n^2=\square$, or $a^2-ab+b^2=\square$. Put $\left.\begin{array}{r}a^2+b^2=A^2\\ a^2-ab+b^2=B^2\end{array}\right\}$; by subtraction $ab=A^2-B^2$; now take $A+B=2a$, $A-B=\frac{1}{2}b$, whence we get $A=a+\frac{1}{4}b$, and $a^2+b^2=A^2=a^2+\frac{1}{2}ab+\frac{1}{16}b^2$, or $16b=8a+b$, or $15b=8a$: hence $a=15$, $b=8$; $\therefore x=\dfrac{n^2}{a+b}=\dfrac{n^2}{23}=23$, (when $n=23$); $\left\{\begin{array}{l}ax=345,\\ cx=184,\end{array}\right.$ numbers.

Let $3x$ and $4x$ denote the numbers. Then it will only remain to satisfy the last condition $(3x)^3+(4x)^3=\square$, or $27x^3+64x^3=91x^3=\square=a^2x^2$, or $x=\dfrac{a^2}{91}$; if $a=91$, then $x=91$, and $\left\{\begin{array}{l}3x=273,\\ 4x=364.\end{array}\right.$

Let $8x$, and $15x$, denote the numbers; then we have $(8x)^3+(15x)^3=512x^3+3375x^3=\square$, or $(13x)^2\times 23x=3887x^3=\square$, or $x=23$, and $8x=184$, $15x=345$. Ans.

Let $(r^2-s^2)x$ and $2rsx$. Then $(r^2-s^2)^3x+8r^3s^3=\square=t^2$. $\therefore x=\dfrac{t^2}{(r^2-s^2)^3+8r^3s^3}$ take $r=4$, $s=1$. Then $x=\dfrac{t^2}{13^2\times 23}$. Again take $t=13\times 23$, then $x=23$, and the numbers are 345 and 184.

3. Find two numbers such, that not only each of them, but also their sum and their difference, when increased by unity, shall be all square numbers?

Let x^2+2x and x^2-2x denote the numbers. Then it remains to make $2x^2+1$ and $4x+1$ squares. Put $2x^2+1=m^2$ and $4x+1=n^2$. Then by subtraction $2x^2-4x=m^2-n^2$. Now take $m+n=x-2$ and $m-n=2x$. Then $m=(3x-2)\div 2$. Hence we have $2x^2+1=(9x^2-12x+4)\div 4$; or $8x^2+4=9x^2-12x+4$; $x=12$. And the numbers are 168 and 120.

By the question, $x+1$, $y+1$, $x-y$, $x+y+1$, must be all squares. Assume $x+1=a^2+b^2$ and $y=2ab$, which assumption fulfils the third and fourth conditions; to make a^2+b^2 a square, assume $a=m^2-n^2$, $b=2mn$, and it only remains to make $y+1=2ab+1=4mn(m^2-n^2)+1=\square$, which is accomplished by taking $m^2-n^2=1+mn$, or $m^2-mn=n^2+1$; from which $m=\frac{1}{2}n\pm$

$\frac{1}{2}\sqrt{(5n^2+4)}$; hence $5n^2+4=\square=(2-nr)^2$, $\therefore n=\frac{4r}{r^2-5}$. If $r=2$, $n=-8$ or 8; if $r=3$, $n=3$, taking $n=3$, we have $m=\frac{3}{2}\pm\frac{7}{2}=5$ or -2; hence $a=-5$ or 16, $b=-12$ or 30, $x=a^2+b^2-1=168$ or 1155, $y=2ab=120$ or 960, and innumerable other answers might be obtained.

4. Find three cubes such, that if unity be subtracted from each, the sum of the remainders shall be a square.

To make $a^3+b^3+c^3-3$ a square, assume $a=n-b$, and it becomes $n^3-3n^2b+3nb^2+c^3-3$; and since n may be any number, let $n=3$, and it is $9b^2-27b+24+c^3$, which is easily made a square. Take $c=2$, then $(\frac{1}{15})^3$, $(\frac{28}{15})^3$, and 2^3 Ans.

5. Find two numbers such, that their sum shall be a square, their difference a cube, and the sum of their squares a cube.

Let $x+y$ and x be the numbers; then the 2d and 3d conditions $y=$cube, and $2x^2+2xy+y^2=$cube. Put $x=yz$; then $2y^2z^2+2y^2z+y^2=$ cube, or since y is a cube, we may reject y^2, and then $2z^2+2z+1=$ cube $=(1+\frac{2}{3}z)^3=1+2z+\frac{4}{3}z^2+\frac{8}{27}z^3$; and $\therefore 54=36+8z$, and $z=\frac{9}{4}$; hence $x=\frac{9}{4}y$, and since $y=$ cube, let $y=(2r)^3=8r^3$; then $x=18r^3$; and by the first condition, $2x+y=44r^3=\square$, or by dividing by $4r^2$, $11r=\square$; whence $r=11$; consequently $x=18r^3=18\times1331=23958$, and $x+y=26r^3=26\times1331=34606$.

Let x, and y, denote the numbers, then $x+y=a^2$, $x-y=b^3$, then $x=\frac{a^2+b^3}{2}$, $y=\frac{a^2-b^3}{2}$; and $x^2+y^2=\frac{a^4+b^6}{2}=$ cube, or $a^4+b^6=2\times$cube, put $a=nb$, $\therefore n^4b^4+b^6=2\times$ cube, or $n^4b+b^3=2\times$cube. put $b=cn$, then $cn^5+c^3n^3=2\times$cube, or, $cn^2+c^3=2\times$ cube, again put $n=c+e$, $\therefore cn^2+e^3=2c^3+2c^2e+ce^2=2\times$cube $=2(c+f)^3$, $\therefore 2c^2e+ce^2=6c^2f+6cf^2+2f^3$, put $2c^2e=6c^2f$, $\therefore e=3f$, the preceding equation then becomes $9cf^2=6cf^2+2f^3$, or $3c=2f$, therefore c, and f, are any numbers in the ratio of 2 to 3. Take $c=2$, and $f=3$, then $e=3f=9$, $n=c+e=11$, $b=cn=22$, $a=nb=242$, and $x=34606$, $y=23958$.

6. Find 3 numbers such that the product of every two plus the sum of the same 2 numbers, may make a square. Let x, y, and $x+y+p$ denote the numbers then put $xy+x+y=a^2$. (1) Also $x^2+xy+xp+x+x+y+p=$ (by (1)) $=x^2+(p+1)x+a^2+p=b^2$, (2) & Similarly $y^2+(p+1)y+a^2+p=c^2$, (3) (2) & (3) become squares by making $p=7$, and $a=3$, then by (1) I have $xy+x+y=9$ or $x=\frac{9-y}{y+1}$ .Put $y=1$, then $x=4$, and $z=12$. and 4, 1, 12, as required.

7. Find 3 rational right angle triangles having equal areas.

Let m^2-n^2, $2mn$, be the sides of the first m^2-q^2, and $2mq$ those of the second, and p^2-m^2, $2mp$, those of the third. Now because the areas are equal I have $(m^2-n^2)n=(m^2-q^2)q$. Hence I have $m^2=n^2+nq+q^2$; also for the same reason $(m^2-n^2)n=(p^2-m^2)p$. $\therefore m^2=p^2-np+n^2=n^2+nq+q^2$ $\therefore n=p-q$, hence $m^2=n^2+nq+q^2=p^2-pq+q^2=(r-p)^2=r^2-2rp+p^2$, $\therefore p=\frac{r^2-q^2}{2r-q}$. Let $q=1$, $r=3$, then $p=\frac{8}{5}$, $m=\frac{7}{5}$. Hence $m^2-n^2=\frac{40}{25}$, $2mn=\frac{42}{25}$ are the sides of the first triangle & $\frac{58}{25}=$ its hypothenuse. Also I have $m^2-q^2=\frac{24}{25}$, $2mq=\frac{70}{25}$ for the legs of the second and $\frac{74}{25}=$ its hypothenuse, & $p^2-m^2=\frac{15}{25}$, $2mp=\frac{112}{25}$ are the legs of another or 3d and $\frac{113}{25}=$ its hypothenuse or by rejecting the common divisor 25, I may take

$$\left.\begin{array}{rrr} 40, & 42, & 58 \\ 24, & 70, & 74 \\ 15, & 112, & 113 \end{array}\right\} \text{the sides and hypothenuse of each triangle as required.}$$

8. Find a cube number which added to the sum of its divisors shall be a square. Let $(2x+1)^3$ be the number, which I shall suppose $2x+1$, is a prime number; then the divisors are 1, $2x+1$, and $(2x+1)^2$, per question $2x+2+(2x+1)^2+(2x+1)^3=4(1+3x+4x^2+2x^3)=\square$, or $1+3x+4x^2+2x^3=\square$, the least value of which will make this a square, and at the same time make $(2x+1)$ a prime is $x=3$, which gives $1+3x+4x^2+2x^3=\square=100$, and $2x+1=7$, and $(2x+1)^3=343$, is the number required.

9. Find a square such that being added to the sum of its divisors, the sum may be a $\square$. Let $(2x+1)^4$ denote the number in which $(2x+1)$ is a prime; then the divisors are $1, 2x+1, (2x+1)^2$ and $(2x+1)^3$, whose sum is $=4(1+3x+4x^2+2x^3)$; hence $(2x+1)^4+4(1+3x+4x^2+2x^3)=\square$; and the least value of x, which makes this a square and $(2x+1)$, a prime is $x=1$; $\therefore 2x+1=3$ and $(2x+1)^4=81=$ the number sought.

10. Find a square number such that the sum of its divisors shall be a square. Let $(2x+1)^2$ be the number; then its divisors are 1, $2x+1$, whose sum $=2x+2=\square$. Let $x=1$, then $2x+2=4=2^2$, and $(2x+1)^2=9=$ the numbers sought. Again, let $(2x+1)^4=$ the number sought; then the sum of its divisors $=4(1+3x+4x^2+2x^3)$, which becomes a square when $x=3$; $\therefore 2x+1=7$ and $(2x+1)^4=7^4=2401$, the number sought.

11. Find a $\square$ number such, that the divisors of its divisors being subtracted from it the remainder shall be an nth. power. Let a^2 be the number, where a is supposed to be a prime. Then its divisors are 1, and a; $\therefore a^2-a-1=$ an nth power, which is satisfied by making $a=2$, and $a^2=4=$ the number sought for, its divisors are 1, 2, and $4-2-1=1=1^n$, as required.

12. Find two square numbers such, that each added to the sum

of its divisors shall make the same number. Let a^2b^2, c^2d^2, be the numbers, where a, b, c, and d are each supposed to be prime numbers. Then the sum of the first and its divisors $=(1+a+a^2)\times(1+b+b^2)$, and the sum of the 2d and its divisors $=(1+c+c^2)(1+d+d^2)$; hence, per question, $(1+a+a^2)(1+b+b^2)=(1+c+c^2)(1+d+d^2.)$ Suppose $a=2$, then $1+b+b^2=\frac{(1+c+c^2)(1+d+d^2)}{7}$ $=$ an integer, or $(1+c+c^2)(1+d+d^2)$ must be divisible by 7, and it is evident that if it is divisible by 7, one of its factors must be divisible by 7. Now the least value which c can have to render $1+c+c^2$ divisible by 7, is 11, which gives $1+c+c^2=133$; this, divided by 7, gives 19. Hence $1+b+b^2=19+19d+19d^2$, or $1+4b+4b^2=(1+2b)^2=73+76d+76d^2=\square$. Put $2d+1=x$, then $4d^2+4d+1=x^2$, and $4d^2+4d=x^2-1$; therefore $76d^2+76d=19x^2-19$, and $73+76d+76d^2=19x^2+54=\square$. Put $x=3y$, then $19y^2+6=\square$, where y is supposed to be prime to 6. Put $19y^2+6=z^2$; this becomes a square when $y=1$, which gives $19+6=25=5^2$; $\therefore z^2-19y^2=6=5^2-19$. Suppose that $19p^2+1=q^2$, or $q^2-19p^2=1$, then I have $z^2-19y^2=(5^2-19)\times(q^2-19p^2)$ (since $q^2-19p^2=1$), or $z^2-19y^2=5^2q^2-19q^2-19^2\cdot5^2p^2+19^2p^2$); put $z=5x+19p$, then by substitution and reduction, $y^2=(q+5p)^2$ or $y=q+5p$. Hence I have only to solve the equation $q^2-19p^2=1$. (This is easily solved by a table at the end of Barlow's Theory of Numbers, $q=170$, $p=39$; where it is evident that p and q may be taken either positively or negatively, since their squares are both $+$ in the equation; $\therefore$ put $p=39$, $q=-170$, then $y=25$, and $x=3y=75$, and $2d+1=x=75$; $d=37=$ a prime. But $54+19x^2=(1+2b)^2=(1+2b)^2=327^2$; $\therefore 1+2b=327$, or $b=163$, which is a prime, as it ought to be. Hence $a^2b^2=2^2\times163^2=326^2$, and $c^2d^2=11^2\times37^2=407^2$; $\therefore 326^2=106276$, and $407^2=165649$, are the numbers sought, which will be found to answer the question. See page 376.

13. Find three triangles such, that the sum of the squares of the hypothenuses of two of them may be to the sum of the $\square$ of the base and perpendicular of the other, as 5 is to 1.

Let the first two triangles be $m(x^2-y^2)$, $m.2xy$, $m(x^2+y^2)$, and $n(x^2-y^2)$, $n.2xy$, $n(x^2+y^2)$; and let the third be x^2-y^2, $2xy$, x^2+y^2. Then $(m^2+n^2)(x^2+y^2)^2=5(x^2+y^2)^2$; $m^2+n^2=5=4+1$. By the formula page 454, $m=\frac{2r^2+2rs-2s^2}{r^2+s^2}$, $n=\frac{r^2+4rs-s^2}{r^2+s^2}$. Letting $x=2$, $y=1$, $s=1$, $r=3$, we have as triangles 8.8, 6.6, 11; 1.2, 1.6, 2; and 3, 4, 5, or if we multiply each term by 5. we have

as an answer in whole numbers, 44, 33, 55; 6, 8, 10; and 15, 20, 25.

14. Find two whole numbers such, that their difference shall be a square and the sum of their squares a cube. Ans. 75, 100.

Let $4x^2$ and $3x^2$ denote the numbers; then $4x^2-3x^2=x^2=\square$, and it remains to make $(4x^2)^2+(3x^2)^2=16x^4+9x^4=25x^4=x^3a^3$, and $x=\frac{1}{25}a^3$. Let $a=5$, then $x=\frac{125}{25}=5$, and the numbers are $4x^2=100$, $3x^2=75$.

15. Find two squares whose sums shall be a biquadrate.

Let x^2, y^2, be the squares. Put $x^2+y^2=a^4$, or $x^2=a^4-y^2=(a^2-my)^2$, $y=\frac{2a^2m}{m^2+1}$, and $x=\frac{a^2(1-m^2)}{1+m^2}$. Let $m=2$, $a=5$, then $y=400$, $x=225$.

To make $x^2+y^2+z^2=\square=(x+a)^2$, $x=\frac{y^2+z^2-a}{2a}$. Let $y=4$, $z=3$, $a=3$; then $x=\frac{8}{3}$, $y^2=16$, $z=9$, $x^2=\frac{64}{9}$, or by $\times$ by $9=3^2$, I have or can take $y^2=144$, $z^2=81$, $x^2=64$. Otherwise, let $4x^4$, $4x^2y^2$, y^4 denote the numbers; then $4x^4+4x^2y^2+y^4=(2x^2+y^2)^2$, and the question is answered. Let $x=2$, $y=3$, then $4x^4=64$, and $4y^2x^2=144$, $y^4=81$, as above.

16. Find two different isoceles triangles such, that their perimeters and areas shall be both expressed by the same nubers.

Let $2x=$ the base, and $y=$ one of the equal sides of one of the triangles, and $2v=$ base, and $z=$ the other $=$ sides of the other triangle. Then I have $x\sqrt{(y^2-x^2)}=v\sqrt{(z^2-v^2)}$ and $2x+2y=2v+2z$, or $x^2(y^2-x^2)=v^2(z^2-v^2)$, and $y+x=z+v$. Divide the first by the second and $x^2(y-x)=v^2(z-v)$, or $y=\frac{v^2}{x^2}(z-v)+x$. But $y=z+v-x$. Hence $\frac{v^2}{x^2}(z-v)+x=z+v-x$; or, $z=v+\frac{2x^2}{v+x}$, and $y=\frac{x^2-vx+2v^2}{v+x}$, and by multiplying by $v+x$, I have expressed in whole numbers, viz. $2x(v+x,)$ and $x^2+vx+2v^2+2v(v+x)$, and $v^2+vx+2x^2$, in which x and v may be taken at pleasure. Let $v=1$, $x=3$, then I have 24 and 14, for the base and side of one of them, and 8 and 22 for another; and as they are both divisible by 2, $\therefore$ 12, 7, and 4, and 11.

17. Find three numbers in arithmetical progression such, that the sum of every two of them may be a square.

Let the numbers be denoted by $2x^2+2ay$, $2x^2$, and $2x^2-2ay$, then $4x^2+2ay=\square$; $4x^2-2ay=\square$. Put $4x^2=a^2+y^2$, put $a=r^2-s^2$, $y=2rs$, then $2x=r^2+s^2$, or $x=\frac{1}{2}(r^2+s^2)$, and then the numbers are $\frac{1}{2}(r^2+s^2)^2+4rs(r^2-s^2)$, $\frac{1}{2}(r^2+s^2)^2$, and $\frac{1}{2}(r^2+s^2)^2-4rs(r^2-s^2)$; take $r=9$, and $s=1$, then 482, 3362, and 6242.

18. Find 3 numbers such, that the sum of all three, and also the sum of any two of them, shall be a cube.

Let $x+y+z=a^3$, $x+y=b^3$, $x+z=c^3$, and $y+z=d^3$, to find x, y and z in positive integers. By adding the 3 first equations and dividing by 2, I have $x+y+z=a^3=\frac{b^3+c^3+d^3}{2}$; $\therefore 2a^3-b^3-c^3=d^3=$ a cube. Put $a=e^3$, $b=e^3+6$, $c=e^3-6$; then $2a^3-b^3-c^3=-(6e)^3=d^3$, or $d=-6e$, if $e=3$, $a=27$, $b=33$, $c=21$, & $d=-18$, or by rejecting the factor 3, I may take 9, 11, 7, and -6, for the roots of the cubes. But as one of these are negative, I will now proceed to find positive roots by the aid of the roots found. Let $v+9=a$, $mv+11=b$, $nv+7=c$, $pv-6=d$, then $2(v+9)^3-(nv+7)^3-(pv-6)^3=(mv+11)^3$, or $(2-n^3-p^3)v^3+(18-7n^2+6p^2)\times 3v^2+(162-49n-36p)\times 3v=m^3v^3+33m^2v^2+363mv$. Put $m=\frac{162-49n-36p}{121}$; then $v=\frac{33m^2-3(18-7n^2+6p^2)}{2-n^3-p^3-m^3}$; If $n=1, p=1$, then $m=\frac{7}{11}$ and $v=\frac{50094}{343}$, whence a, b, and c are easily found, and then x, y and z.

19. Find 3 whole numbers such, that the product of all 3, divided by the sum of all 3 numbers, the quotient shall be a square, and the square of each number added to the former square, each result shall be a square.

Let the three numbers be denoted by x, y and z, then I have to make $\frac{xyz}{x+y+z}=\square$, $x^2+\frac{xyz}{x+y+z}=\square$, $y^2+\frac{xyz}{x+y+z}=\square$, and $z^2+\frac{xyz}{x+y+z}=\square$, first put $\frac{xyz}{x+y+z}=a^2$, and then $x=\frac{a^2(y+z)}{yz-a^2}$ (1) equation; then $x^2+a^2=\square=(a+mx)^2=a^2+2amx+m^2x^2$; then I find $x=\frac{2am}{1-m^2}$, and $y^2+a^2=(a+ny)^2=a^2+2any+n^2y^2$; $\therefore y=\frac{2an}{1-n^2}$, and $z^2+a^2=(a+pz)^2=a^2+2apz+p^2z^2$, and $z=\frac{2ap}{1-p^2}$, substituted in the first equation, give by reduction the equation $m^2+m\left(\frac{p+n}{1-np}-\frac{1-np}{n+p}\right)=1$, the second equation. Now, by completing the square, &c. I have $m=\frac{1-np}{n+p}$; here x, y and z are easily found. Let $n=\frac{1}{2}$, $p=\frac{2}{3}$, then $m=\frac{4}{7}$, $x=\frac{56a}{33}$, $y=\frac{4a}{3}$ and $z=\frac{12a}{5}$. Put $a=33\times 3\times 5=495$, $x=280$, $y=220$, $z=396$, then $x=840$, $y=660$, $z=1188$; then $\frac{xyz}{x+y+z}=a^2=495^2$, and $x^2+a^2=975^2$, $y^2+a^2=825^2$, and $z^2+a^2=1287^2$, as required.

20. Find x, y and z in rational numbers such that $x+y+z=d^4$, $x+y=b^2$, $y+z=c^2$, $y+z=a^6$. By the three last I have $x+y+z=d^4=\frac{1}{2}(b^2+c^2+a^6)$ or $(b^2+c^2)=2d^4-a^6$, (1). This shows that d and a are to be so assumed that $2d^4-a^6=$ the sum of two square. Again, by the second $x=b^2-y$, and by the third $z=c^2-y$, then by the 4th $x+z=a^6=b^2+c^2-2y$, or $y=\dfrac{b^2+c^2-a^6}{2}$, and $x=\dfrac{b^2+a^6-c^2}{2}$, $z=\dfrac{c^2+a^6-b^2}{2}$; now $2d^4-a^6$ becomes the sum of two square by taking $d=3$, and $a=2$, which gives $2d^4-a^6=98=49+49$. Let $b=7\times\dfrac{2mn+m^2-n^2}{m^2+n^2}$, and $c=7\times\dfrac{2mn-(m^2+n^2)}{m^2+n^2}$; then $b^2+c^2=98$, as it ought to do. Let $m=4$, $n=3$, then $b=7\times\frac{31}{25}$, $c=7\times\frac{17}{25}$. Hence $y=17$, $x=336\times(\frac{7}{25})^2+32=\frac{36464}{625}$, and $z=32-336\times(\frac{7}{25})^2=\frac{3536}{625}$; then these values of y, x, and z answer all the conditions of the question. Should integral numbers be required, they will be found by multiplying the above values of x, y and z by 5^{12}, and then taking the product for y, x and z, respectively. Hence

$y=4150390625$, $x=14243750000$, $z=1381250000$, and $x+y+z=375^4$, $x+z=50^6$, $x+y=135625^2$, $y+z=74375^2$.

21. Find three whole numbers such, that their sum and their 3 differences shall be all squares. Ans. 5, 14, and 30.

Let x, y, and z, denote the numbers; then $x-y$, $x-z$, $y-z$, and $x+y+z$, are to be squares. Put $x-y=m^2$, $y-z=n^2$, then $x-z=m^2+n^2=\square$, and $x+y+z=3y+m^2-n^2=\square=s^2$; whence $y=(s^2-m^2+n^2)\div 3$, when s, m, and n, may be taken at pleasure, provided m^2+n^2 be a square; let $m=4$, $n=3$, and $s=7$, then $y=14$, $x=m+y=30$, $z=y-n^2=5$. Hence the numbers are 5, 14, and 30. In the same way, we may find other sets of numbers that will answer the conditions, as 10, 19, 35; 22, 31, 47; 29, 38, 54, &c. &c. either of these being much less than those given by Bonnycastle.

22. Find three numbers in continued geometrical progression such, that their three differences shall be all sqares.

Let the numbers be denoted by x, xz^2, and xz^4, then $x(z^4-z^2)=\square$, $x(z^4-1)=\square$, $x(z^2+1)=\square$, or $x(z^2+1)=\square$, and $x(z^2+1)=\square=a^2$, and $x(z^2+1)\times(z^2-1)=\square$, $(z^2+1)a^2=\square$, $z=\dfrac{m^2-n^2}{2mn}$, $\left(\dfrac{m^4-2m^2n^2+n^4-4m^2n^2}{4m^2n^2}\right)x=a^2$. Or better thus, which is a more simple expression: x, xz^2, and xz^4, denote the number, then $x(z^2-1)=\square=a^2$, $(z^2+1)a^2=\square$, or $z^2+1=\square$. Put $z=\dfrac{m^2-n^2}{2mn}$, then $x\left\{\left(\dfrac{m^2-n^2}{2mn}\right)^2-1\right\}=a^2$, and then I have $x=$

$\frac{a^2}{(\frac{m^2-n^2}{2mn})^2-1}$, or, $x=\frac{4m^2n^2a^2}{(m^2-n)^2-4m^2n^2}$ $x=\frac{9a^2}{7}=63=\frac{a^2}{(\frac{4}{3})^2-1}$

$m=3$, $n=1$, $a=7$, $x=63$, $z=\frac{m^2-n^2}{2mn}=\frac{8}{6}=\frac{4}{3}$, and $xz^2=63\times\frac{16}{9}$ $=63\times7=112$; and $xz^4=63\times(\frac{16}{9})^2=\frac{112\times16}{9}=\frac{1792}{9}$, and the numbers are 63, 112, $\frac{1792}{9}$. Multiplied by 9, I have 567, 1008, 1792, as required.

22. Find two such rational fractions, that the cube of either being added to the square of the other, shall make the same sum; and furthermore, that their sum and the sum of their squares may both be square numbers. Let the two fractions be denoted by $\frac{a^2x}{x+y}$, and $\frac{a^2y}{x+y}$; for the sum of these is $\frac{a^2(x+y)}{x+y}=a^2=\square$. And the sum of their squares is $\frac{a^4(x^2+y^2)}{(x+y)^2}$, which will evidently be a $\square$, when x^2+y^2 is a square.

Again, by the question, the cube of either of the fractions being added to the square of the other, must make the same sum; that is, $\frac{a^6x^3}{(x+y)^3}+\frac{a^4y^2}{(x+y)^2}=\frac{a^6y^3}{(x+y)^3}+\frac{a^4x^2}{(x+y)^2}$; which easily reduces to $a^2(x^3-y^3)=(x+y)(x^2-y^2)$, whence $a^2=$ $\frac{(x+x)(x^2-y^2)}{x^3-y^3}=\frac{(x+y)^2}{x^2+xy+y^2}$; and $\therefore$ $x^2+xy+y^2=\square$. The question has, $\therefore$, been reduced to the finding of such values of x and y as will make x^2+y^2 and x^2+xy+y^2 both rational squares, viz. $x=2415$, and $y=1768$ are such values as will make the expressions x^2+y^2 and x^2+xy+y^2 both rational $\square$; whence, and from what has been deduced, see page 464, question 90.

$a^2=\frac{(x+y)^2}{x^2+xy+y^2}=(\frac{4183}{3637})^2$; and hence $\frac{a^2x}{x+y}=\frac{2415\times4183}{3637^2}=$ $\frac{10101945}{13227769}$; and $\frac{a^2y}{x+y}=\frac{1768\times4183}{3637^2}=\frac{7395544}{13227769}$ are two fractions that will answer the question.

23. Find three numbers in geometrical progression such, that if any one of them be increased by its root, the sum shall be a $\square$.

Let the numbers be denoted by x^4, x^2y^2, and y^4; then $x^4+x^2=$ $\square$, or $x^2+1=\square$, $x^2y^2+xy=\square$, and $y^4+y^2=\square$, or $y^2+1=\square$.

Put $x^2y^2+xy=a^2x^2y^2$; then $xy=\frac{1}{a^2-1}\div\frac{2x}{2a}=$ gives $\frac{2a(a^2-1)}{2a}$ take $y=\frac{a^2-1}{2a}$, and $x=2a$; $\therefore$ y^2+1 is satisfied $4a^2+1=\square=$

$(2a-m)^2$; $a=\frac{m^2-1}{4m}$; if $m=5$, $a=\frac{6}{5}$, $x=\frac{12}{5}$, and $y=\frac{11}{60}$.

But as these are heavy numbers, let me try a different notation. Let x^2, $4x^2$, and $16x^2$ be the numbers; then $x^2+x=\square$, $4x^2+2x=\square$, and $16x^2+4x=\square$, or $4x^2+x=\square$. Put $x^2+x=(rx)^2=r^2x^2$, $x=\frac{1}{r^2-1}$; substitute in the 2d and 3d, I have $\frac{4}{r^4-2r^2+1}+\frac{2}{r^2-1}$, or $2r^2+2=\square$, $\frac{4}{r^4-2r^2+1}+\frac{1}{r^2-1}$, or $r^2+3=\square$. Put $r^2+3=(r-s)^2=r^2-2rs+s^2$; then $r=\frac{s^2-3}{2s}$, and by substituting in $2r^2+2=2s^4-4s^2+18=\square$. Put $s=1+a$, then $2s^4-4s^2+18=16+8a^2+8a^3+2a^4=\square=(4+a^2)^2=16+8a^2+a^4$. Hence $8+2a=a$ and $a=-8$, and $s=1+a=-7$, $r=\frac{s^2-3}{2s}=-\frac{23}{7}$, $x=\frac{1}{r^2-1}=\frac{1}{\frac{480}{49}}=\frac{49}{489}$, and the numbers are $(\frac{49}{480})^2$, $(\frac{98}{480})^2$, and $(\frac{196}{480})^2$, Ans.

25. Find two numbers such, that if either of them be added to the $\square$ of the other, the sum shall be a square number. Let $x-y$ and $4xy$ denote the numbers; then $(x-y)^2+4xy=x^2+2xy+y^2$, and $16x^2y^2+x-y=\square=(4xy+a)^2=16x^2y^2+8axy+a^2$, and $y=\frac{x-a^2}{8ax+1}$; if $x=2$, $a=1$, then $y=\frac{1}{17}$, and the numbers are $\frac{33}{17}$, and $\frac{8}{17}$; if $x=\frac{1}{2}$ and $a=\frac{1}{2}$, then the numbers are $\frac{5}{12}$, $\frac{2}{12}$; also $\frac{1}{12}$ and $\frac{1}{6}$. Otherwise, let x, and $2x+1$ denote the numbers; then the $\square$ of the first and second is $=x^2+2x+1$, and the first added to the $\square$ of the second $=4x^2+5x+1=\square=(2x-a)^2=4x^2-4ax+a^2$, $x=\frac{a^2-1}{4a+5}$, if $a=3$, then $\frac{8}{17}$, and $\frac{33}{17}$ are the numbers; if $a=2$, then $\frac{3}{13}$ and $\frac{19}{13}$. Otherwise, x^2 and y^2 denote the numbers; then $x^2+y=(x-ay)^2=x^2-2axy+a^2y^2$, and $y^2+x=(2a-y)^2=4a^2-4ay+y^2$, $x=\frac{a^2y-1}{2a}$, and in the second $x=4a^2-4ay$. Put these values of $x=$ to each other, then $\frac{a^2y-1}{2a}=4a^2-4ay$, and $y=\frac{8a^3+1}{9a^2}$ and $x+y=\frac{4a^4+8a^3-4a+1}{9a^2}=\square$, as it is $=(\frac{2a+2a-1}{3a})^2$; therefore a may be taken at pleasure, provided it be greater than unity, $x=\frac{1}{6}$, $y=\frac{1}{12}$.

25. Find three squares in arithmetical progression such, that if any one of them be increased by its root, the sum shall be a $\square$! Let x^2, $25x^2$, and $49x^2$ denote the numbers; $x^2+x=\square$, $25x^2+$

$5x=\square$, and $49x^2+7x=\square$; then put $x^2+x=n^2x^2$, or $x+1=n^2x$, or $x=\frac{1}{n^2-1}$. Now $25x^2+5x=\frac{25}{(n^2-1)^2}+\frac{5}{n^2-1}=\square$, or $20+5n^2=\square$, $49x^2+7x=\frac{49}{(n^2-1)^2}+\frac{7}{n^2-1}=\square$, or $42+7n^2=\square$. Put $n=1+a$; then $20+5n^2=25+10a+5a^2=\square=(5+ar)^2=25+10ar+a^2r^2$, or $a=\frac{10r-10}{5-r^2}$; thence $n=1+a=\frac{10r-r^2-5}{5-r^2}$, and the last expression to be made a square is $42+7n^2=42+\frac{7r^4-140r^3+770r^2-700r+175}{(5-r^2)^2}$, or $49r^4-140r^3+350r^2-700r+1225=\square=(7r^2-10r+s)^2=49r^4-140r^3+r^2(14s+100)-2ars+s^2$. Hence I have $14s+100=350$, or $s=\frac{125}{7}$, and $-20rs+s^2=-700r+1225$, and $r=\frac{1225-s^2}{700-20s}=\frac{1225-\frac{15625}{49}}{700-\frac{2500}{7}}=\frac{44400}{16800}=\frac{37}{14}$; $n=\frac{10r-r^2-s}{5-r^2}=\frac{2831}{389}$; consequently $x=\frac{1}{n^2-1}=\frac{151321}{7863240}$, and the numbers are as follows: $(\frac{151321}{7863240})^2$, $(\frac{756605}{7863240})^2$ and $(\frac{1059247}{7863240})^2$.

Let $x^2+ax=\square$, $x^2+bx=\square$, and $x^2+cx=\square$. Put $x^2+ax=(x-p)^2=x^2-2px+p^2$, $x=\frac{p^2}{2p+a}$, and by substituting in the second and third, I have

$$\left(\frac{b^2p^2}{2p+a}\right)^2+\left(\frac{bp^2}{2p+a}\right)=\square \text{ or, } \quad p^2+2pb+ab=\square=m^2$$
$$\left(\frac{c^2p^2}{2p+a}\right)^2+\left(\frac{cp^2}{2p+a}\right)=\square \text{ or, } \quad p^2+2pc+ac=\square=n^2$$

By subtraction I have $m^2-n^2=2p(b-c)+a(b-c)$
$m+n=(2p+a)$, $m-n=b-c$, $m=p+\frac{a+b-c}{2}$; then $p^2+2pb+ab=m^2=p^2+p(a+b-c)+\frac{1}{4}(a+b-c)^2$, and $p=\frac{ab-\frac{1}{4}(a+b-c)^2}{a-(b+c)}$ if $x^2+\frac{2}{121}x$, $x^2+\frac{6}{121}x$, $x^2+\frac{9}{121}x$; if $a=\frac{2}{121}$, $b=\frac{6}{121}$, $c=\frac{9}{121}$, $p=\frac{-47}{13\times 484}=\frac{-47}{6292}$, and then $x=\frac{47^2}{(-94+104)\times 6292}=\frac{2209}{62920}$ $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{4}$, $p=\frac{7}{16}$, $x=\frac{49}{(14+16)\times 16}=\frac{49}{480}$. If $a=1$, $b=\frac{1}{5}$, $c=\frac{1}{7}$, then $p=\frac{389}{3220}$, $x=\frac{389^2}{2442\times 3220}=(\frac{151321}{7863240})$;
If $a=\frac{1}{5}$, $b=\frac{1}{7}$, $c=\frac{1}{35}$, and then $p=\frac{19}{140}$, $x=\frac{19^2}{66\times 140}=\frac{361}{9240}$ and the numbers are $(\frac{361}{1848})^2$, $(\frac{361}{1320})^2$, and $(\frac{361}{264})^2$.

Make $9x^2-3x=\square$, $16x^2-4x=\square$, and $25x^2-5x\ \square$.

To make $x^2-ax=\square$, $x^2-bx=\square$, and $x^2-cx=\square$.

If $a=\frac{1}{3}$, $b=\frac{1}{4}$, $c=\frac{1}{5}$, then $x=\frac{450241}{1313760}$; and then I have

Ans. $3x=\frac{450241}{437920}$, $4x=\frac{450241}{328440}$, and $5x=\frac{450241}{262752}$.

27. Find two triangular numbers whose sum is a triangular number.

Let x and y denote the sides of the triangle; then $\frac{x^2+x}{2}$, $\frac{y^2+y}{2}$ are the numbs. Assume $x^2+x+y^2+y=(m+x)^2+m+x$, then $x=\frac{y^2+y-m^2-m}{2m}$. Put $y=pm$, then $x=\frac{m(p+1)+1}{2}\times(p-1)$. Put $p-1=2q$; $\therefore p+1=2q+2$. Hence $x=2m(q+1)q+q$, $y=pm=m(2q+1)$; $\therefore$ let $m=1$, $q=1$, then $x=5$, $y=3$, $\frac{1}{2}(x^2+x)=15$, $\frac{1}{2}(y^2+y)=6$, and $15+6=21=\frac{1}{2}(6^2+6)$, which is a triangular, as required.

28. To find two pentagonal numbers whose sum is a pentagonal number.

Let x, y be the sides of the pentagons; then the pentagons are expressed by $\frac{3x^2-x}{2}$, and $\frac{3y^2-y}{2}$. Assume, then, $3x^2-x+3y^2-y=3(m+x)^2-m-x$, or $x=\frac{3y^2-y-3m^2+m}{6m}$. Put $y=pm$, then $x=\frac{3m(p+1)-1}{6}\times(p-1)$; $\therefore$ put $p-1=6q$, and $p+1=6q+2$. Hence $x=\{6m(3q+1)-1\}q$, and $y=pm=m(1+6q)$. Let $m=1$, $q=1$, then $x=23$, and $y=pm=m(1+6q)=7$. But $\frac{3(23^2)-23}{2}=782$, and $\frac{3(7^2)-7}{2}=70$. Hence 782 and 70 are pentagular numbs. But $782+70=852=\frac{3(24^2)-24}{2}$, and is therefore a pentagular number.

29. Find two $n$thgonal numbers such that their sum may be an $n$thgonal number.

Let x and y denote their sides; then the numbers are $\frac{(n-2)x^2-(n-4)x}{2}$, and $\frac{(n-2)y^2-(n-4)y}{2}$; that is, assume $(n-2)x^2-(n-4)x+(n-2)y^2-(n-4)y=(n-2)\times(x+m)^2-(n-4)(x+m.)$. Then $x=\frac{(n-2)y^2-(n-4)y-(n-2)m^2+(n-4)m}{2m(n-2)}$

Put $y=pm$, then I shall have $x=\frac{(n-2)\times(p+1)m-(n-4)}{2(n-2)}\times$

$(p-1)$. Put $p-1=2(n-2)q'$, and $p+1=2\{1+(n-2)q\}$; then $x=\{2m(n-2)+(n-2)^2q\}-(n-4)q$, & $y=pm=m(2q(n-2)+1)$ $=m\{2q(n-2)+1\}$, as required.

30. Find two $n$thgonal numbers whose difference is an $n$thgonal number. This question is easily answered from the last one; but I shall proceed with the solution independent of it.

Let x and y denote the sides; then, as before, the numbers are $\frac{n(-2)x^2-(n-4)x}{2}$, $\frac{(n-2)y^2-(n-4)y}{2}$. Let. $(n-2)x^2-(n-4)x$ $-\{(n-2)y^2-(n-4)y\}=(n-2)\times(x-m')^2-(n-4)\times(x-m')$; then $x=\frac{(n-2)m'^2+(n-4)m'+(n-2)y^2-(n-4)y}{2m'(n-2)}$. Put $p'm'$ $=y$, then $x=\frac{(n-2)m'\times(p'^2+1)}{2(n-2)}-\frac{(n-4)(p'-1)}{2(n-2)}$. Put $m'=$ $2m''$, and $p'-1=2)n-2)q^2$; then I have $x=m''\{(2q'(n-2)+1)^2$ $+1\}-(n-4)q'$, and $y=p'm'=2p'm''=2m''\{1+2q'(n-2)\}$

31. Find two triangular numbers such, that their sum and difference may be each triangular numbers.

Let x and y denote the sides of the triangular; then by putting $x=q\{2m(q+1)q+1\}$, and $y=m(2q+1)$, the first condition is answered, (as appears by question 27, page 415,) and by the second condition, $\frac{x^2+x}{2}-\frac{y^2+y}{2}=\frac{(4q^4+8q^3-4q-1)m^2+(4q^3+6q^2-1)m}{2}$ $+\frac{q^2+q}{2}$, or $\frac{(4q^4+8q^3-4q-1)m^2+(4q^3+6q^2-1)m+q^2+q)}{2}=$ $\frac{(q+mp)^2+(q+mp)}{2}$. Hence, it plainly appears that

$m=\frac{(2q+1)p-)4q^3+6x^2-1)}{4q^4+8q^3-4q-1-p^2}$. Let $q=2$, then $m=\frac{5p-55}{119-p^2}$, and by taking $p=-11$, $m=55$. Hence $x=662$, $y=275$; then $\frac{x^2+x}{2}=219453$, and $\frac{y^2+y}{2}=37950$, are two triangular numbers which answer the conditions of the question, for their sum is $=$ $\frac{717^2+717}{2}$, which is a triangle, and their difference $=\frac{603^2-603}{2}$, which is also a triangular number; for any number of the form m^2-n is also of the form m^2+n. Since $4(m^2-m)+1=(2m-1)^2$ assume $(2m-1)^2=(2n+1)^2$, or $2m-2=2n$; $\therefore n=m-1$, and $m^2-m=n^2+n$, as stated above. Hence, by putting $m=603$, I have $n=m-1=602$; $\therefore$ the difference of the triangles $=$ $\frac{602^2+602}{2}$, and is therefore a triangle, as required. Again, let

$q=1$, then $m=\frac{3p-9}{7-p^2}$. Put $p=3$, then $m=9$; $\therefore 27=y$, and $x=37$, and the triangles are 703, 378. Also $703+378=\frac{46^2+46}{2}$, and $703-378=\frac{26^2-26}{2}=\frac{25^2+25}{2}$. Let $q=1$, as before. But $p=-1$, then $m=-2$; $\therefore y=-6$, $x=-7$; or, since $6^2-6=5^2+5$, and $7^2-7=6^2+6$, I put $y=5$, $x=6$, and the triangles are 21 and 15, as required; for $21+15=\frac{8^2+8}{2}$, and $21-15=6=\frac{3^2+3}{2}$. Ans.

32. Show that any triangular number, when multiplied by 8, and the product increased by 1, gives a $\square$ number; also, that any pentagonal number $\times$ by 24, and the product increased by 1, is also a $\square$ number. Let x denote the side of the triangle; then $\frac{x^2+x}{2}$ is the triangle; then $8\frac{(x^2+x)}{2}+1=4x^2+4x+1=(2x+1)^2=\square$. Again, let x denote the side of the pentagon; then $\frac{3x^2-x}{2}$, is the pentagon. But $24(\frac{3x^2-x}{2})+1=36x^2-12x+1=(6x-1)^2=\square$, and it may also be remarked, that any hexagonal number, when multiplied by 8, and the product increased by 1, is a square number. For let x denote the side of the hexagon, then $\frac{4x^2-2x}{2}=$ the hexagon; but $8(\frac{4x^2-2x)}{2}+1=16x^2-8x+1=\square=(4x-1)^2$.

33. To show that any triangular number is also of the form of a hexagonal number when the side is an odd number, let y denote the side of the triangular number; then $\frac{1}{2}(y^2+y)$ is the triangular. Also, let x denote the side of an equal hexagonal number, then $\frac{4x^2-2x}{2}=$ hexagon; and by putting these values $=$ to each other, I have $\frac{4x^2-2x}{2}=\frac{y^2+y}{2}$. Multiply by 8, and add 1 to both sides, and there results $16x^2-4x+1=(4x-1)^2=(2y+1)^2$, or $4x=2y+2$, and $x=\frac{y+1}{2}=$ an integer, if y is an odd number, and therefore the equation $\frac{4x^2-2x}{2}=\frac{y^2+y}{2}$ can be satisfied in integers when y is an odd number. Cor. Any hexagonal number is a triangular number; for, by the equation $4x=2y+2$, I have $2y=4x-2$, or $y=2x-1$; $\therefore$ as x is an integer, y is an integer, or y can be found in integers so as to satisfy the equation $\frac{4x^2-2x}{2}=\frac{y^2+y}{2}$, when x is supposed to be given, (page 417, ques-

tion 33,) I can find two hexagonal numbers whose sum and difference are triangular numbers, for 27 and 37 were found for the sides of two triangular numbers whose sum and difference are triangular numbers. By putting p and q for the sides of the hexagon, I have, by the formula, $x=\frac{y+1}{2}$. By putting 37 for y, and 27 for x, $p=19$, and by putting q for x, and 27 for y, I have $q=14$; $\therefore$ 19 and 14 are the sides of two hexagons, whose sum and difference are triangles, numbers as required.

34. Find two $n$thgonal numbers such, that their sum or difference shall be $n$thgonal numbers.

Let x and y denote the sides of the $n$thgonal numbers; then the numbers are $\frac{(n-2)x^2-(n-4)x}{2}$, and $\frac{(n-2)y^2-(n-4)y}{2}$. Assume $(n-2)x^2-(n-4)x+(n-2)y^2-(n-4)y=(n-2)(x+m)^2-(n-4)(x+m)$; then $x=\frac{(n-2(y^2-(n-4)y-(n-2)m^2+(n-4)m}{2m(n-2)}$. Put $y=pm$, and $p-1=2(n-2)p$, or $p=1+2(n-2)q$; then I have have $x=\{2m(n-2)+(n-2)^2q)-(n-4)\}q$, (1). ; $y=\{2q(n-2)+1\}m$, (2). Put $n-2=n'$, $n-4=n''$; then $x=\{2m(n'+n'^2q)-n''\}q$, and $y=(2qn'+1)m$; assume $n'(2m\{n'+n'^2q\}-n'')^2q^2-n''\{2m(n'+n'^2q)-n'')\}q-n'(2qn'+1)^2m^2+n''(2qn'+1)m=n'(mt-n''q)^2-n''(mt-n''q)$; hence I have $m=\frac{n''}{n'}\times\frac{(4n'q^2+2q).(n'+n'^2q)-(2n'q+1)(t+1)}{4q^2(n'+n'^2q)^2-(2qn'+1)^2-t^2}$, (3.) Assuming q and t in (3), m is found. and then x and y become known by (1) and (2). This is on the supposition that x and y may be fractions. But in particular cases, if integers and positive numbers are required, then such numbers are to be assumed for q and t as will give x and y integers, and positive. If $n=4$, then $n''=0$, $m=0$. In this case, the $n$thgonal numbers are square ; and it is evident that no two square numbers are by the process such, that their sum or difference shall both be square. Indeed, if $x^2+y^2=\square$, and $x^2-y^2=\square$, then their product $x^4-y^4=\square$, which Euler has proved to be impossible.

Cor. If $n=3$, then the $n$thgonal numbers are triangular, and (3) become $m=\frac{2(q+1)t-(4q^3+6q^2-1)}{4q^4+8q^3-4q-t^2-1}$, (4). Also $x=q\{2m(q+1)+1\}$, and $y=m(2q+1)$, (5). If $q=2$ $t=-11$, the triangles are 219253, and 37950, which answer the conditions. Also, if $q=1$, $t=-3$, then the triangular 703 ; 378. which also answers; and if $q=1$, and $t=-1$, the triangles are 21, and 15, Ans.

35. Find two right-angled triangles with a common hypothenuse such, that the difference between the squares of the greater and less legs of each, respectively, shall be squares.

Let a denote the common hypothenuse b and f, and c and e the two legs of each; then $a^2=b^2+f^2=c^2+e^2$, and $\{b^2-c^2=d^2=\square$, and $e^2-f^2=\square=d^2\}$. Hence I have to make $a^2-b^2=\square, a^2-c^2=\square$, and $b^2-f^2=\square$. Assume $a^2=(p^2+q^2)(r^2+s^2)$, $b=pr\pm qs$, $c=ps\pm qr$; then it remains to make squares of the expressions $(p^2+q^2)(r^2+s^2)$, and $(p^2-q^2)(r^2-s^2)$. Assume $r=pm-qn$, $s=pn+qm$, then m^2+n^2 must be a square. Take $m=g^2-l^2$, $n=2gl$; now $\frac{r+s}{p-q}=mt+n$, $\frac{r-s}{p+q}=\frac{m}{t}-n$, (putting $t=\frac{p+q}{p-q}$). Again, $mt+n=g^2-l^2+2gln=\square=(g-lv)^2=g^2-2glv+l^2v^2$; $\therefore g=r^2+1$, $l=2(v+t)$, and $mt-nt^2=4t^4+8t^3v-t^2(t^2-1)^2+4t(t+v)(t+1)=4t^4+8t^3v+t^2(4+6v-v^4)+4tv(v^2+1)=\square=(2t^2+2tv-w)^2=4t^4+8t^3v+t^2(4v^2+w)-4vwt+w^2$. Hence, I have $3+6w^2-v^4=4v^2-4v$, from which $w=\frac{v^4-2v^2-3}{4}=\frac{(v^2+1)(v^2-3)}{4}$. Again, $4tv^2(v+1)=-4vtw+w^2$. Hence $t=\frac{a^2}{4v(w+v+1)}=\frac{(v^2-3)^2}{16v}=\frac{r+s}{r-s}$. Therefore $p=(v^2-3)^2+16v$, and $g=(v^2-3)^2-16v$, and $r=v^2+1)$, $s=2(v^2+w)$, or $r=8v$, $s=v+9$; then if $v=1, r=5, s=-3, g=4, l=5$, or $r=5, s=-3$, $p=173, q=-165$; consequently, by substitution in the given formula, I have $a=1394$, $b=370$ or 1360, $e=1344$ or 306, and the sides, by dividing by 2, become $\left\{\begin{matrix} 697, & 680, & 153, \\ 697, & 672, & 185. \end{matrix}\right\}$ Ans.

36. Find four numbers such, that the product of any three, increased by unity, shall be a square.

Let a, b, c, and d, denote the numbers; then I have to make $abc+1=\square$, $acd+1=\square$, and $abd+1=\square$, $bcd+1=\square$; then, from the first $abc+1=\square=(na+1)^2=n^2a^2-2an+1$. Hence I find $a=\frac{bc+2n}{n^2}$. Then, by substitution,

$$\left.\begin{matrix} \frac{b^2cd+2bdn}{n^2}+1=\square \\ \frac{bc^2d+2cdn}{n^2}+1=\square \end{matrix}\right\} \begin{matrix} \text{or, } n^2+2bdn+b^2cd=\square=m^2, \\ \text{or, } n^2+2cdn+bc^2d=\square=n^2, \text{ and} \end{matrix}$$

by subtraction $\overline{2n(bd-cd)+bc(bd-cd)=m^2-n^2}$;

$m+n=2m+bc$, and $m-n=bd-cd$;

then $m=n+\frac{bc+bd-cd}{2}$, I have $n^2+2bdn+b^2cd=m^2=$

$n^2 + n(bc + bd - cd) + \frac{1}{4}(bc + bd - cd)^2 =$
$(n + \frac{bc+bd-cd}{2})^2$; then $n = \frac{b^2c^2d - \frac{1}{4}(bc+bd-cd)^2}{bc-(b+c)d}$. Let $b = \frac{1}{2}$,
$c=2$, $d=3$; then I find $n = \frac{25}{16} \times \frac{2}{13} = \frac{25}{104}$, $a = \frac{77}{52} \times \frac{104^2}{25^2} =$
$\frac{77 \times 208}{625} = \frac{16016}{625}$. Again $abc+1 = \square = 1\text{st}$, $abd+1 = \square = 2\text{d}$,
$acd+1 = \square = 3\text{d}$, and $bcd+1 = \square = 4\text{th}$.

Let $ad = b+c+2p$, and $bc + 1 = p^2$; then the 2d and 3d are square. Now $bc = p^2-1$; $\therefore b = p-1$, $c = p-1$, $ad = 4p$, and $a = \frac{4p}{d}$; by substitution in the first & third, I find, $\frac{4p(p^2-1)}{d} + 1 = \square$, and $(p^2-1)d+1 = \square$. The latter is a $\square$ when $d=1$. Hence $4p^3 - 4p + 1 = \square$, which is the case when p is 1 or 6. Then $a=24$, $b=7$, $c=5$, and $d=1$.

Otherways To transform into squares the 4 following formulæ;

$$xyz + 1,\ xyv + 1,\ xzx + 1,\ \text{and}\ yzv + 1.$$

Put $xy = b^2-c^2$, $z = d^2-e^2$, $v = f^2-g^2$, Then the first and 2nd formulæ are squares, if (A), $be - cd = \pm 1$, (B), $bg - cf = \pm 1$; whence, by substitution in the third and fourth formulæ,

$$\frac{(b^2-c^2)(d^2-c^2)(f^2-g^2)}{y} + 1 = \square,\ (d^2 - e^2)(f^2 - g^2)y+1 = \square.$$

by the theory of continued fractions, if we form a series of fractions, by adding the numerators and denominators of the succeeding fractions to the preceding ones, we will satisfy the equations (A) and (B) thus: $\frac{2}{1}$, $\frac{1}{1}$, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{7}{5}$, $\frac{11}{8}$, $\frac{18}{13}$. &c., where any 3 contiguous ones may be taken for b, c, d, e, f, and g. Thus, take $\frac{b}{c} = \frac{3}{2}$, $\frac{d}{e} = \frac{4}{3}$, $\frac{f}{g} = \frac{7}{5}$; then $\frac{840}{y} + 1 = \square$, and $168y+1 = \square$.
Hence $y=1$, $z = d^2-c^2 = 7$, $v = 24$, $x = \frac{5}{1} = 5$, as before.

37. Find three numbers whose sum shall be a biquadrate, and the sum of the 1 and 3d a square cube, and the sums of the 1st and 2d, and 2d and 3d, squares.

Let $4x$, x^2-4x, $2x+1$; then $(x + 1)^2$ must be the 4th power; or $x+1 = \square$. Also $6x+1 = \square$, $x = \frac{r^2-1}{6}$, $x+1 = \frac{r^2+5}{6}$, or $6r^2 + 30 = \square$. Put $r = 1+s$, $36+12s+6s^2 = 36-12ts+t^2s^2$; therefore $s = 12.\frac{t+1}{t^2-6}$, if $t=3$, $s=16$, & $r=17$, $x=48$; 192, 2112, & 97.

Let $192x^4$, $2112x^4$, $97x^4$, Nos. then $289x^4 =$ square cube, or $17x^2 =$ cube; $\therefore x=17$, and $192 \cdot 17^4$, $2112 \cdot 17^4$, $197 \cdot 17^4$.

See page 446, for more diophantine questions and solutions.

Miscellaneous Questions.

1. What two numbers are those, whose sum is 19, and whose difference multiplied by the greater is 60 ?

Let x, and $19-x$ be the numbers ; $\therefore 2x^2-19x=60$, and by case 1, $x^2-\frac{19}{2}x+(\frac{19}{4})^2=30+\frac{361}{16}=\frac{841}{16}$, or $x=\pm\frac{29}{4}+\frac{19}{4}=12$, or $-\frac{5}{2}$.

2. If the square of a certain number be taken from 40, and the square root of this difference be increased by 10, and the sum multiplied by 2, and the product divided by the number itself, the quotient will be 4. Required the number. Ans. 6.

Let $x=$ the number ; $\therefore \frac{2\{\sqrt{(40-x^2)}+10\}}{x}=4$, & $\sqrt{(40-x^2)}=2x-10$; $\therefore 40-x^2=4x^2-40x+100$, or $x^2-8x=-12$.

3. There is a field in the form of a rectangular parallelogram, whose length exceeds the breadth by 16 yards, and it contains 960 square yards. Required the length and breadth.

Let x, and $x-16$ be the length and breadth, $x^2-16x=960$, and $x^2-16x+64=1024$, $x-8=32$, and $x=40=$ the length and the breadth 24 yards, Ans.

4. A person being asked his age, answered, If you add the square root of it to half of it, and subtract 12, there will remain nothing. Required his age. Ans. 16.

Let $x=$ his age ; then $\frac{1}{2}x+\sqrt{x}-12=0$, $x-2\sqrt{x}+1=25$, and $\sqrt{x}+1=5$, $\sqrt{x}=4$, and $x=16$, Ans.

5. Two casks of wine were bought for \$58, one of which contained 5 gallons more than the other, and the price per gallon was \$2 less than one-third of the number of gallons in the less. Required the number of gallons in each, and the price per gallon ?

Let $3x$, $3x+5$, and $x-2$ be the number of gallons in the less and the greater, and the price of a gallon. Then we have, $\therefore (6x+5)(x-2)=58$, and $6x^2-7x=68$. By case 2, $\therefore x^2-\frac{7}{6}x+\frac{49}{144}=\frac{68}{6}+\frac{49}{144}=\frac{1681}{144}$, and $x-\frac{7}{12}=\frac{41}{12}$, and $x=4$, and the numbers are 12 and 17, and the price \$2 per gallon.

6. From two places, at the distance of 320 miles, two persons, A and B, set out at the same time to meet each other. A travelled 8 miles a day more than B, and the number of days in which they met was equal to half the number of miles B went in a day. How many miles did each travel per day, and how far did each travel ?

Let $2x$, $2x+8$, and $x=$ the number B and A went per day, and number of days ; $\therefore 4x^2+8x=320$, and by case 1, $x=8$; $\therefore$ A went 24 and B 16 miles per day, and the distances travelled by them were 128 and 192 miles.

7. The difference between the hypothenuse and base of a right

angled triangle is $=6$, and the difference between the hypothenuse and the perpendicular is $=3$. What are the sides?

Let x, $x-6$, and $x-3$ be the hypothenuse, base and perpendicular, and $x^2=(x-6)^2+(x-3)^2=2x^2-18x+45$, and $\therefore x^2-18x+45=0$, and by case 3, $(x^2-18x=-45)$, or $x^2-18x+81=81-45=36$, and $x=15$, and 12, and 9, the sides, Ans.

8. In a parcel which contains 25 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many are there of each? Ans. 6 of one, and 18 of the other.

Let x. and $24-x$ be the numbers; $\therefore 2x(24-x)=18\times12$, or $x^2-24x=-108$, by case 3; $x=18$ or 6.

9. Bought a number of books, consisting of folios, quartos, and octavos, for \$1932. Fourteen folios (which was the whole number) cost three times as much as all the quartos; and one quarto cost as many dollars as there were quartos. The number of octavos was 32, and their value was such, that 4 of them cost as much as one quarto. Required the value of each, and the number of quartos. Ans. There were 21 quartos, each folio cost \$94½, each quarto \$21, and each octavo \$5 25 cts.

Let x be the number of quartos, $x^2=$ their value; and $3x^2$ and $8x=$ the value of the folios and octavos; then by the question, $4x^2+8x=1932$, and $x=21$.

10. Bought two flocks of sheep for £65, 13*s*., one containing 5 more than the other. Each sheep cost as many shillings as there were sheep in the flock. Required the numbers in each flock. Ans. 23, and 28.

Let x, and $x+5$ be the numbers; then $x^2+(x+5)^2=1313$, or $2x^2+10x+25=1313$, and $x=\frac{51}{2}-\frac{5}{2}=23$, and 28, Answers.

11. A regiment of soldiers, consisting of 1066 men, is formed into two squares, one of which has four men more in a side than the other. What number of men are in a side of each of the squares? Ans. 21, and 25.

Let x, and $x+4$ be the numbers; then $x^2+(x+4)^2=1066$, and $x=21$ and 25 are the numbers, Ans.

12. What number is that, to which if 23 be added, and the square root of the sum extracted, this root shall be less than the original quantity by 18? Ans. 25.

Let $x=$ the number; $\therefore \sqrt{(x+24)}=x-18$, and $x^2-37x=-300$. Hence $x=\pm\frac{13}{2}+\frac{37}{2}=25$, or 12, Ans.

13. After taking the kings, queens, and knaves out of a pack of cards, the rest were divided into three heaps. The number of pips contained in the second heap was found to be four times the

square of the number in the first heap; and had the third heap contained 5 more pips than it did, the number in it would have been exactly half of what the first and second heap contained. Required the number of pips in each heap.

Let x, and $4x^2$ be the first and second numbers, and the third number will be $=\frac{4x^2+x}{2}-5$, and $\frac{3}{2}(4x^2+x)-5=220$; $\therefore 4x^2+x=150$, and $4x^2+x+\frac{1}{16}=\frac{2401}{16}$; $\therefore 2x+\frac{1}{4}=\frac{49}{4}$, and $2x=12$, and $x=6$, 144, and 70 are the Ans.

14. A regiment of foot was ordered to send 216 men on garrison duty, each company being to furnish an equal number; but before the detachment marched, 3 of the companies were sent on another service, when it was found that each company that remained was obliged to furnish 12 additional men, in order to make up the complement 216. How many companies were there in the regiment, and what number of men was each company ordered to send at first?

Let $x=$ the number; then the number each was to send was $=\frac{216}{x}$, and $(x-3)(\frac{216}{x}+12)=216$, and $x^2-3x=54$, $x=\frac{15}{2}+\frac{3}{2}=9$, and each was to send 24 men.

15. A poulterer bought 15 ducks and 12 turkeys for five guineas. He had two ducks more for 18 shillings than he had of turkeys for 20 shillings. What was the price of each?

Let $x=$ the price of a duck, $(\frac{105-15x}{12}=)\frac{35-5x}{4}=$ the price of a turkey, and $x:1::18:$ the number of ducks for 18*s* equal $\frac{18}{x}$, and $\frac{4\times4}{7-x}=$ the number of turkeys for 20*s*; $\therefore \frac{18}{x}=\frac{16}{7-x}+2$, or $\frac{9}{x}=\frac{8}{7-x}+1$; $\therefore 63-9x=8x+7x-x^2$, or $x^2-24x=-63$, and $x=3$ or 21, and the prices were 3*s* and 5*s*.

16. A and B entered into a speculation, to which B subscribed \$15 more than A. After 4 months, C was admitted, who added \$50 to the stock; and at the end of twelve months from C's admission they found they had gained \$159; when A withdrawing, received for principal and gain \$88. What did he originally subscribe? Ans. \$40.

Let $x=$ the sum; $\therefore x+15=$B's subscription, and $16(2x+15)+12\times40:16x::159:88-x$, or $(4x+105)(88-x)=318x$, and $4x^2+71x+(\frac{71}{4})^2=9240+\frac{5041}{16}=\frac{152881}{16}$, whence $2x+\frac{71}{4}=\frac{391}{4}$, and $2x=80$, $\therefore x=40$.

17. A wall was built round a rectangular court to a certain

height, Now the length of one side of the court was two yards less than eight times the height of the wall, and the length of the adjacent side was 5 yards less than 6 times the height of the wall and the number of square yards in the court was greater than the number in the wall by 178. Required the dimensions of the court, and the height of the wall.

Let x=the height; $\therefore$ therefore the sides are $8x-2$, and $6x-5$, then $(8x-2)(6x-5)=2x(14x-7)+178$, and $10x^2-19x=84$, and $x^2-\frac{19}{10}x+(\frac{19}{20})^2=\frac{84}{10}+\frac{361}{400}=\frac{3721}{400}$; $\therefore x-\frac{19}{20}=\frac{61}{20}$, and $x=4$, or the side was 30 and 19, and the height 4 yards.

18. A ship containing 74 sailors, and a certain number of soldiers, beside officers, took a prize. The sailors received each one-third as many dollars as there were soldiers, and the soldiers recieved \$3 apiece less, and \$768 fell to the share of the officers. Had the officers however received nothing, the soldiers and sailors might have received half as many dollars per man, as there were soldiers. How many soldiers were there, and how much did each receive?

Let $3x$ = the number of soldiers; $\therefore$ x= what each sailor received, and $74x+3x^2-9x+768$ = the value of the prize:

$$\therefore \frac{3x^2+65x+768}{3x+74}=\frac{3x}{2}; \therefore 3x^2+92x=1536, \text{ and by case 1,}$$

$x^2+\frac{92}{3}x+(\frac{46}{3})^2=\frac{1536}{3}+\frac{2116}{9}=\frac{6724}{9}$; $\therefore x+\frac{46}{3}=\frac{82}{3}$, and $x=12$. There were 36 soldiers; each soldier received \$9, and each sailor \$12, Ans.

19. A poulterer going to market to buy turkeys, met with four flocks. In the second were 6 more than three times the square root of double the number in the first. The third contained three times as many as the first and second; and the fourth contained 6 more than the square of one-third of the number in the third; and the whole number was 1938. How many were there in each flock?

Let x = the number in the first, then $3\sqrt{2x}+6$, $3(x+3\sqrt{2x}+6)$, and $(x+3\sqrt{2x}+6)^2+6$ be the number in the 2d, 3d, and 4th; then $\therefore (x+3\sqrt{2x}+6)^2+4(x+3\sqrt{2x}+6)+4=1936$, and $x+3\sqrt{2x}+6+2=44$, $\therefore x+3\sqrt{3x}=36$, and $2x+6\sqrt{2x}+9=72+9=81$, $\therefore \sqrt{2x}+3=9$, $\sqrt{2x}=6$, and $x=18$, and the numbers were 18, 24, 126, and 1770.

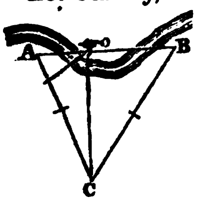
20. A body of men are just sufficient to form a hollow equilateral wedge, three deep; and if 597 be taken away, the remainder will form a hollow square, four deep, the front of which contains one man more than the square root of the number contained in a front of the wedge. What is the number of men? Ans. 693.

Let x = the number in a side of the first triangle; $\therefore 3(x-1)$

$3(x-3)-3$, $3(x-6)-3$, and $9x-36$, be the number in the first, in the 2d, in the third, and the number of men. Now $\sqrt{x}+1=$ the number of men in the *a* side of the hollow square, $\therefore (\sqrt{x}+1)^2-(\sqrt{x}-7)^2=$ the number of men in the hollow □, and $9x-36=16x^{\frac{1}{2}}-48+597$; $\therefore 9x-16x^{\frac{1}{2}}=585$, and $9x-16x^{\frac{1}{2}}+\frac{64}{9}=585+\frac{64}{9}=\frac{5329}{9}$, $\therefore 3x^{\frac{1}{2}}-\frac{8}{3}=\frac{73}{3}$, and $3x^{\frac{1}{2}}=27$; $\therefore x^{\frac{1}{2}}=9$, $x=81$, and the number is $9\times 77=693$, Ans.

21. From the middle of a town two streets branched off, and crossed a river that ran in a straight course, by two bridges, A and B. From their junction a sewer equally inclined to both streets, led to a point in the river at the distance of 6 chains from the bridge A, and a distance from B less by 11 chains than the length of the sewer: the expense of making it amounting to as many dollars per chain as there were chains in the street leading to A. The sewer, however, being insufficient to carry off the water, an additional drain was made from a point in this street, distant 4 chains from the bridge A, which entered the river at the same point with the sewer, and was equally inclined to the river and sewer. Now it was found that a drain down the middle of each street, at the rate of \$9 per chain would have cost only \$54 more than the expense of the sewer. Required the lengths of the streets and the sewer; and the distance of its mouth from the bridge B.

Let $CA=y$, $x=$ the length of the sewer, $DB=x-11$, and $xy=$ the expense. Also $AC : CB :: 6 : x-11$; $\therefore CB=\frac{y}{6}(x-11)$, and $AD : DC :: AE : EC$ or $6 : x :: 4 : y-4$; $\therefore 2x=3y-12$. Also, $9y+\frac{3}{2}(x-11)y=xy+54$, and $xy-15y=108$, whence $\frac{3}{2}y^2-21y=108$, and $y^2-14y=72$; $\therefore y=18$, and $x=21$, $=CD$, then $DB=10$, and $CB=30$, Ans.



22. Two plantations, one of an oblong and the other of a square form, contain the same number of trees, and they have one fence common to both, viz. that which bounds the end of the oblong one. Upon every pole in the square are planted as many trees as there are poles in the square, and upon every pole in the oblong four times as many trees as there are poles in the breadth, besides 144 in the hedges. Also the area of the oblong wants 6 poles to be to the area of the square as 3 to 2. Required the number of trees.

Ans. 1296.

Let $x^2=$ the number of poles in the square; $\therefore x^4=$ the number of trees, and $\frac{3x^2}{2}-6=$ the area of the oblong; hence

$(\frac{3}{2}x^2-6)4x+144=x^4$, or $6x^3-24x+144=x^4$; $\therefore 6x^3+144=x^4+24x$, or $6(x^3+24)=x(x^3+24)$; $\therefore x=6$, and the number $=1296$.

23. A and B undertake to perform a piece of work in four days, for which they are to receive a certain number of dollars; but after some time, finding that they shall not be able to finish it in the time proposed, they call in C to assist them, and upon an equitable division of the money, C receives a sum equal to the square root of the whole number of dollars; but had they been obliged to call in C to their assistance $1\frac{1}{9}$ day sooner, his share of the money would have been two fifths more. How long did C work, and what did he receive?

Let $x^2=$ the sum to be divided, $x=$ what C received, and $\frac{1}{2}(x^2-x)=$ what A received; $4:1::\frac{1}{2}(x^2-x):$ their daily pay $=\frac{1}{8}(x^2-x)$, and $\frac{1}{8}(x^2-x):x::1:$ the time C worked equal $\frac{8}{x-1}$. Also $\frac{7}{5}x=$ C's pay on the second supposition; therefore $\frac{1}{10}(5x^2-7x)=$ A's receipt, and $4:1::\frac{5x^2-7x}{10}:$ their daily pay $=\frac{5x^2-7x}{40}$, and $\frac{5x^2-7x}{40}:\frac{7x}{5}::1:$ the time C worked in the 2d case $=\frac{56}{5x-7}$; $\therefore \frac{56}{5x-7}=\frac{8}{x-1}+\frac{10}{9}$; or, $\frac{28}{5x-7}=\frac{4}{x-1}+\frac{5}{9}$, and then $\therefore 9\times 28\times(x-1)=4\times 9(5x-7)+5(5x-7)(x-1)$, or $25x^2-132x=-35$; hence $25x^2-132x+(\frac{66}{5})^2=\frac{4356}{25}-35=\frac{3481}{25}$, and $5x-\frac{66}{5}=\frac{59}{5}$; $\therefore 5x=25$, $\therefore$ he worked 2 days, and received \$5.

24. A cask, whose content is 20 gallons, is filled with brandy, a certain quantity of which is then drawn off into another cask of equal size; this last cask is then filled with water; after which the first cask is filled with the mixture, and it appears, that if $6\frac{2}{3}$ gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of brandy in each. Required the quantity of brandy first drawn off. Ans. 10 gallons.

Let $x=$ the quantity; $\therefore 20-x$ the quantity remaining, or the quantity of water in the second, and $20:x::x:$ quantity of spirits returned to the first $=\frac{x^2}{20}$, and $x-\frac{x^2}{20}=$ the quantity in the second, and $20:20-x+\frac{x^2}{20}::\frac{20}{3}:$ quantity of spirits in $6\frac{2}{3}$ gallons $=\frac{x^2-20x+400}{60}$; $\therefore \frac{20x-x^2}{20}+\frac{x^2-20x+400}{60}=\frac{2}{3}\times\frac{x^2-20x+400}{20}$, or $60x-3x^2=x^2-20x+400$, or $x^2-20x=-100$, and $x=10$, Ans.

25. There are three numbers, the difference of whose differences is 5; their sum is 20; and their continual product 130. Required the numbers. Ans. 2, 5, and 13.

Let $x-y$, x, and $x+y+5$ be the numbers; $\therefore 3x+5=20$, and $x=5$; hence $(5-y)\cdot 5\cdot(10+y)=130$, or $50-5y-y^2=26$, and $y+\frac{5}{2}=\pm\frac{11}{2}$; $\therefore y=3$, or -8, and then 2, 5, 13 is the Ans.

26. There are three numbers, the difference of whose differences is 3; their sum is 21; and the sum of the squares of the greatest and least is 137. Required the numbers.

Let $x-y$, x, and $x+y+3$ be the numbers; $\therefore 3x+3=21$, and $x=6$; hence $(6-y)^2+(9+y)^2=137$; $\therefore 2y^2+6y=20$, and $y=2$, or -5. Ans. the numbers are 4, 6, and 11.

27. There is a number consisting of 2 digits, which when divided by the sum of its digits gives a quotient greater by 2 than the first digit. But if the digits be inverted, and then divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Required the number.

Let $10x+y$ be the number; then $\frac{10x+y}{x+y}=x+2$, and $10x+y=x^2+2x+xy+2y$; $\therefore 8x-y=x^2+xy$; also $\frac{10y+x}{x+y+1}=x+4$, $10y+x=x^2+xy+5x+4y+4$; $\therefore 6y-4x-4=x^2+xy=8x-y$, and $7y=12x+4$, $y=\frac{12x+4}{7}$; and $8x-\frac{12x+4}{7}=x^2+\frac{1}{7}(12x^2+4x)$, $56x-12x-4=7x^2+12x^2+4x$, or $19x^2-40x=-4$, and $x^2-\frac{40}{19}x+(\frac{20}{19})^2=\frac{400}{361}-\frac{4}{19}=\frac{324}{361}$, and $x=\pm\frac{18}{19}+\frac{20}{19}=2$, and $y=4$. Ans. 24.

28. A certain sum was to be raised on three estates belonging to A, B and C, at the rate of one dollar per acre. Now the number of acres A and B had were as 3 to 7; and if the number of acres in the whole were divided by one third of the product of the numbers in the first and third, the quotient would be $\frac{3}{4}$. Also the sum paid by A and C was 36 dollars less than the sum of three times the money paid by C, and two sevenths of the money paid by B. Of how many acres did each estate consist, and what was the whole sum to be raised?

Let $3x$, $7x$, and y be the number of acres A, B, and C each had; $\therefore 3x+y+36=3y+2x$, and $x=2y-36$. Also, we have $\frac{10x+y}{xy}=\frac{3}{4}$; or $\frac{21y-360}{y(2y-36)}=\frac{3}{4}$; $\therefore \frac{7y-120}{y^2-18y}=\frac{1}{2}$, or $y^2-18y=14y-240$, and $y^2-32x=-240$, and $y=20$ or 12, and $x=4$, and the numbers are 12, 28, and 20, and the sum was \$60.

29. A butcher bought a certain number of calves and sheep, and for each of the former gave as many dollars as there were

sheep, and for each of the latter one fourth as much. Now had he given \$4 more for each of the former, and \$2 more for each of the latter, he would have paid \$140 more. But had a sheep cost as much as a calf, he would have expended \$1128. How many did he buy of each; and what were their prices?

Let x and y be the number of calves and sheep; then $xy+\frac{1}{4}y^2+70=x(y+4)+(\frac{1}{4}y+2)y=xy+4x+\frac{1}{4}y^2+2y$; $\therefore$ $70=2x+y$, and $x=\frac{1}{2}(70-y)$. Also $(x+y)y=1128$ or $y^2+\dfrac{70y-y^2}{2}=1128$; $\therefore$ $y^2+70y+35^2=2256+1225=3481$, $y=59-35=23$, and $x=23$.

30. A and B comparing their wages, observe that if A had received per day in addition to what he does receive, a sum equal to $\frac{1}{4}$th of what B received per week, and had worked as many days as B received dollars per day, he would have received \$48; and had B received \$2 a day more than A did, and worked for a number of days equal to half the number of dollars he received per week, he would have received \$98. What were their daily wages? Ans. A's \$5, and B's 4.

Let x and y be A's and B's daily wages; then by the question, $\therefore$ $(x+\dfrac{7y}{4})y=48$, and $(x+2)\dfrac{7y}{2}=98$, and $\dfrac{49}{4}y^2-14y=140$, $\therefore \dfrac{7y}{2}-2=12$, and $\dfrac{7y}{2}=14$; $\therefore$ $y=4$, and $x=5$, Ans.

31. There are four towns in the order of the letters A B C D. The difference between the distances from A to B and from B to C is greater by four miles than the distance from B to D. Also the number of miles between B and D is equal to two thirds of the number between A and C. And the number between A and B is to the number between C and D as seven times the number between B and C : 26. Required the respective distances.

Let $x=AB$, and $y=BC$; $\therefore$ $x-y=CD+y+4$, or $CD=x-2y-4$, and $x-y-4=\frac{2}{3}(x+y)$. Also $x : x-2y-4 :: 7y : 26$; hence $21y^2-74y=312$, or $y^2-\frac{74}{21}y+(\frac{37}{21})^2=\dfrac{1369}{21^2}+\dfrac{312}{21}=\dfrac{7291}{21^2}$; $\therefore$ $y-\frac{37}{21}=\frac{89}{21}$, and $y=6$, $\therefore$ $x=42$.miles.

32. A person bought a quantity of cloth of two sorts for £7 18s. For every yard of the better sort he gave as many shillings as he had yards in all; and for every yard of the worse as many shillings as there were yards of the better sort more than of the worse. And the whole price of the better sort was to the whole price of the worse as 72 to 7. How many yards had he of each?

Let x and y be the number of better and worse; then we have $x^2+xy : xy-y^2 :: 72 : 7$; $\therefore x^2+xy : (x^2+2xy-y^2=)158 :: 72 : 79$,

or, $x^2+xy=144$; $\therefore xy-y^2=14$, hence $(x+y=\frac{144}{x})-(x-y=\frac{14}{y})=2y=\frac{144}{x}-\frac{14}{y}$, and $y=\frac{72}{x}-\frac{7}{y}=\frac{72y}{y^2+14}-\frac{7}{y}$; $\therefore y^2=\frac{72y^2}{y^2+14}-7$, or $y^4-51y^2+(\frac{51}{2})^2=\frac{2601}{4}-98=\frac{2209}{4}$; $\therefore y^2=\frac{47}{2}+\frac{51}{2}=49$, and $y=7$, $x=9$.

33. A farmer sold a certain number of bushels of barley, and ten bushels of wheat for £7 19s. Now each bushel of wheat cost within 3 shillings as much as two bushels of barley. He afterwards sold as many bushels of barley and four more, and fifteen bushels of wheat, and received two shillings per bushel more for his wheat and barley than he did before; when he found that if he had received £1 4s more, he should just have received twice as much as he did before. How many bushels of barley did he sell the first time; and what were the prices per bushel of the wheat and barley?

Let $x=$ the number of bushels of barley, and $y=$ the price of a bushel; $\therefore 2y-3=$ the price of a bushel of wheat, and $xy+20y-30=159$, or $xy+20y=189$. Now $(x+4)(y+2)+15(2y-1)+24=318$. Hence $(xy-2x+34y=301)-(xy+20y=189)=2x+14y=112$, and $x=56-7y$, whence $56y-7y^2+20y=189$, and $y^2-\frac{76}{7}y+(\frac{38}{7})^2=\frac{1444}{49}-\frac{189}{7}=\frac{121}{49}$; $\therefore y=\frac{11}{7}+\frac{38}{7}=7$, and $x=7$.

34. In digging among some ruins the workmen found 9 urns, together containing 60 gold coins; the second and eighth containing 8 and 4 respectively. They secreted a certain number of these, greater than the number they left; which being afterwards recovered, it was found that the number of urns secreted was to the number left as the number of coins secreted was to the number remaining. Now if instead of taking the second urn they had carried off the eighth, then the number of coins taken away would have been to the number remaining as the square of the number of urns secreted to the difference between that square and 20 times the number of urns remaining. Required the number of urns and coins secreted.

Let $x=$ the number of coins secreted, $60-x$ the number remaining, $y=$ the the number of urns secreted; $\therefore 9-y=$ the number remaining, and $x:60-x::y:9-y$; $\therefore x:60::y:9$, and $3x=20y$. Also $x-4:64-x::y^2:20(9-y)-y^2$; $\therefore x-4:60::y^2:20(9-y)$; then $29y^2-192y=-108$, or $y^2-\frac{192}{29}y+(\frac{96}{29})^2=\frac{9216}{29^2}-\frac{108}{29}=\frac{6084}{29^2}$; $y=\frac{78}{29}+\frac{96}{29}=6$, and $x=40$.

35. Two men, A and B, set out from the same place to travel. A goes in 6 days twice as many miles as B goes in 5 days, but

does not arrive at the end of his journey till 5 days after B has arrived at the end of his, when he finds that he has travelled 259 miles more than B. But had B travelled 2 miles per day more than he did, and A stopped 6 days sooner, A would then have gone only 37 miles more than B. How many miles did each travel per day, and how many days did they travel?

Let x, and $\frac{5x}{3}$ be the number B and A went per day; then y, and $y+5$ will be the number of days travelled by B and A; then

$$\tfrac{5}{3}x(y+5)-xy=\tfrac{2}{3}xy+\tfrac{25}{3}x=259,$$

$$\text{and } \tfrac{5}{3}x(y-1)-(x+2)y=\tfrac{2}{3}xy-\tfrac{5}{3}x-2y=37$$

$$\therefore \text{ by subtraction, } 10x+2y=222$$

and $y=111-5x$, $\frac{2}{3}x(111-5x)+\frac{25}{3}x=259$, or $222x-10x^2+25x=777$, and $x^2-\frac{247}{10}x+(\frac{247}{20})^2=\frac{61009}{400}-\frac{777}{10}=\frac{29929}{400}$, and $y=\frac{173}{20}+\frac{247}{20}=21$, and $x=6$, Ans.

36. Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for two thirds of the time that Silenus would have taken to empty the whole cask. After that Silenus awoke, and drank what Bacchus had left. Had they drunk both together, it would have been emptied two hours sooner, and Bacchus would have drunk only half what he left Silenus. Required the time in which they would empty the cask separately. Ans. Bacchus in 6 hours.

Let $y=$ the time in which Bacchus would empty the cask, and $3x=$ the time in which Silenus would empty it; $\therefore 2x=$ the time Bacchus drank and $y:1::2x:$ the quantity Bacchus drank $=\frac{2x}{y}$; $\therefore 1-\frac{2x}{y}=$ the quantity Silenus drank, and $1:3x::1-\frac{2x}{y}$: the time Silenus was drinking $=$ $3x-\frac{6x^2}{y}$; $\therefore 5x-\frac{6x^2}{y}=$ the time of emptying. Also $\frac{1}{2}-\frac{x}{y}=$ the quantity Bacchus would have had, and $1:y::\frac{1}{2}-\frac{x}{y}$: the time of Bacchus drinking that quantity, $=\frac{y}{2}-x$; $\therefore 5x-\frac{6x^2}{y}=\frac{y}{2}-x+2$, Also, $1:3x::\frac{1}{2}+\frac{x}{y}$: time of Silenus drinking $=\frac{3x}{2}+\frac{3x^2}{y}$ $\therefore \frac{y}{2}-x=\frac{3x}{2}+\frac{3x^2}{y}$, and $6x^2=y^2-5xy$; $\therefore y^2-5xy+\frac{25x^2}{4}=\frac{49x^2}{4}$, and $y-\frac{5x}{2}=\frac{7x}{2}$, or $y=6x$, hence $5x-x=3x-x+2$, or $2x=2$, and $x=1$, and Silenus would empty the cask in 3 hours,

37. Two persons, A and B, comparing the distances they have travelled, found that the square of the number of miles which A usually walked per hour, exceeded the square of the number B usually walked by 5; and that if to the square of the product of those numbers there be added the square of the sum of their fourth powers, augmented by the product of the square of the difference of their squares into the square of the product of the numbers themselves, the aggregate amount would be 10345. How many miles did each walk per hour? Let x and y be the numbers; then $x^2 - y^2 = 5$, and $(x^4 + y^4)^2 + 2x^2y^2(x^2 - y^2)^2 + x^2y^2 = 10345$, or $625 + 100x^2y^2 + 4x^4y^4 + 26x^2y^2 = 10345$; $\therefore 4x^4y^4 + 126x^2y^2 + (\frac{63}{2})^2 = 9720 + \frac{3969}{4} = \frac{42849}{4}$; $\therefore 2x^2y^2 + \frac{63}{2} = \frac{207}{2}$, and $(x^2y^2 = 36)$; hence I have $(x^2 - y^2 = 5)^2 + 4(x^2y^2 = 36)$, or $(x^4 - 2x^2y^2 + y^4 = 25) + (4x^2y^2 = 144) = (x^2 + y^2)^2 = 169$, and $x^2 + y^2 = 13$. But $x^2 - y^2 = 5$; $\therefore x^2 = 9$, and $x = 3$; then $y^2 = 4$, and $y = 2$.

Ans. walked 3 and 2 miles per hour.

38. The roof of a storehouse is formed of 2 squares terminated by two equal and parallel isoceles triangles; the height of the walls being equal to the base of either of these triangles. The quantity of wood which the storehouse will hold, increased by six cubical piles each of the same length of the building, is to the quantity which the same storehouse would hold if its roof were flat, in the proportion of 11 : 2. The roof cost as many cents per square foot as there are feet in its ridge, and the flooring was laid at the same rate. Both together cost 5000 dollars Required the dimensions of the storehouse.

Let x = the length of the building, and $2y$ = the base of either triangle, = the height of the walls; $\therefore \sqrt{(x^2 - y^2)}$ = the perpendicular altitude of the triangle, and $4y^2$ = the area of the wall; $\therefore 4y^2x$ = the contents of the body and $4y^2x + xy\sqrt{(x^2 - y^2)}$ = the contents of the whole barn; $\therefore 4y^2x + xy\sqrt{(x^2 - y^2)} + 6x^3 : 4y^2x :: 11 : 2$, and $xy\sqrt{(x^2 - y^2)} + 6x^3 : 4y^2x :: 9 : 2$; then $y\sqrt{(x^2 - y^2)} + 6x^2 = 18y^2$, and $x^2 + \frac{y}{6}\sqrt{(x^2 - y^2)} = 3y^2$; $\therefore (x^2 - y^2) + \frac{y}{6}\sqrt{(x^2 - y^2)} + \frac{y^2}{144} = 2y^2 + \frac{y^2}{144} = \frac{289}{144}y^2$, and $\sqrt{(x^2 - y^2)} + \frac{1}{12}y = \frac{17}{12}y$; $\therefore \sqrt{(x^2 - y^2)} = \frac{4}{3}y$; whence $x^2 - y^2 = \frac{16y^2}{9}$, and $x^2 = 2\frac{5}{9}y^2$; $\therefore x = \frac{5}{3}y$, or $y = \frac{3}{5}x$. Now one square foot in the roof cost x cents, and there are $2x^2$ feet; therefore the whole expense is $2x^3$, and the area of the floor $= 2xy$; $\therefore$ its expense $= 2x^2y$, and $(2x^3 + 2x^2y =) 2x^3 + \frac{6}{5}x^3 = 50000$; $\therefore 8x^3 = 125000$, and $2x = 50$; $\because x = 25$, and $y = 15$. **Ans. the length is 25 and the height 15 feet.**

39. If 9 gentlemen or 15 ladies will eat up 16 apples in 4 hours, and 15 gentlemen and 15 ladies can eat up 47 apples of a similar size in 12 hours, the apples growing uniformly, how many boys will eat up 188 apples in 60 hours, admitting that 112 boys can eat the same number as 18 gentlemen and 26 ladies.

ladies gent apples

$$\left.\begin{array}{l}\text{If } 15 : 4 :: 16\\ \quad 40 \;\; 12\end{array}\right\} = \frac{40\times12\times16}{15\times4} = \frac{10\times4\times16}{5} = 2\times4\times16 = 128 \text{ apples Ans.}$$

which 40 ladies will eat in 12 hours, admitting the apples do not grow. But 40 ladies eat only 47 apples in 12 hours; ∴ $(12-4)=8$ hours, 47 apples by the growth of the apples, are equivalent to 128 apples. Hence $128-47=81$ apples, the increase upon 47 apples in $(12-4)=8$ hours. Now the real quantity which 15 ladies will eat in 4 hours without any increase was 17 apples. Hence the time of increase upon 188 apples is only $(60-4)=56$ hours. ap. ho. in.

$$\left.\begin{array}{l}\text{And if } 47 : 8 :: 81\\ \quad 188 : 56 ::\end{array}\right\} = \frac{188\times81\times56}{47\times8} = \frac{4\times81\times56}{8} = 4\times81\times7$$

$=2268$ apples, the increase on 188 apples in 56 hours. Hence $2268+188=2456$ apples, which is to support the required number of ladies for 60 hours, admitting they eat nothing but green apples.

$$\left.\begin{array}{l}\text{Again } 15 : 4 :: 16\\ \quad 60 \;\; 2456\end{array}\right\} = \frac{2456\times15\times4}{60\times16} = \frac{2456\times15}{15\times16} = \frac{2456}{16} =$$

$153\frac{1}{2}$ ladies to eat the required number of apples. Lastly,

If 9 gen. : 15 la. :: 15 gen : 25 lad. and $25+15=40$ lad.

9 : 15 :: 18 : 30 ladies, and $30+26 = 56$ ladies.

Then if 56 : 112 boys :: $153\frac{1}{2}$ ladies : 307 boys, Answer.

40. If 12 oxen will eat up $3\frac{1}{3}$ or $3\frac{1}{2}$ acres of grass in 4 weeks, and 21 oxen will eat up 10 acres in 9 weeks, how many oxen will eat up 24 acres in 18 weeks, the grass being allowed to grow uniformly? $a = 12$ oxen $c = 4$ weeks $b = 3\frac{1}{3}$ or $3\frac{1}{2}$ acres

$d = 21$ oxen $f = 9$ weeks $e = 10$ acres

$h = 18$ weeks $g = 18$ acres

$$\left.\begin{array}{l}a : c :: b\\ d : f\end{array}\right\} : \frac{d\times f\times b}{a\times c}$$, the result of this statement is $\frac{bfd}{ac}$ acres,

which d oxen will eat in f weeks, admitting the grass do not grow. But d oxen eat only e acres in f weeks; therefore in $f-c$ weeks, e acres, by the growth of the grass, are equivalent to $\frac{bfd}{ac}$ acres. Hence $\frac{bfd}{ac} - e = \frac{bfd-ace}{ac}$ acres, the increase upon e acres in $f-c$ weeks. Now the real quantity which a oxen will eat in c weeks, without any increase, was b acres. Hence the time of increase upon g acres is only $h-c$ weeks.

crease on g acres in $h—c$ weeks.

$$\left\{\begin{matrix} e\,.\,f—c :: \\ g : h—c \end{matrix}\right. \frac{bfd—ace}{ac} : \frac{(bfd—ace)\times\{(h—c)\times g\}}{ace\times f—c}$$ acres, the increase on g acres in $h—c$ weeks.

Hence $g+\dfrac{(bfd—ace)\times\{(h—c)\times g)}{ace\times(f—c)}$, the whole quantity;

$\dfrac{aceg\times(f—c(+\{(bfd—ace)\times(h—c)\times g\}}{ace\times(f—c)}$ of grass which is to support the required number of oxen for h weeks.

$$\begin{matrix} a : c :: b \\ h :: \end{matrix} \frac{aceg\times(f—c)+(bfd—ace)\times(h—c)\times g}{ace\times(f—c)};$$

the result of this stating is equal to the number of oxen sought $=\dfrac{aceg\times(f—c)+\{(bfd—ace)\times(h—c)\times g\}}{beh\times(f—c)}$ A general rule for all questions of a similar nature. This theorem, by $\times$tion becomes $\dfrac{bdfgh+ecfga—ecagh—bdcgf}{bfeh—bceh}=36$ or $37\frac{113}{175}$ oxen, Ans.

Again: oxen. weeks. acres.

$$\left.\begin{matrix} 12 : 4 :: 3\tfrac{1}{2} : \\ 21 : 9 :: \end{matrix}\right\} \frac{21\times9\times3\frac{1}{2}}{12\times4}=\frac{7\times9\times3\frac{1}{2}}{4\times4}=\frac{441}{32}$$

$13\frac{25}{32}$ acres; whence $13\frac{25}{32}—10=3\frac{25}{32}$ increase upon 10 acres in five weeks.

acres. weeks. acres inc.

$$\left.\begin{matrix} 10 : & 5 :: & 3\tfrac{25}{32} \\ 24 : & 14 :: & \end{matrix}\right\}=\frac{24\times14\times3\frac{25}{32}}{10\times5}=\frac{12\times14\times3\frac{25}{32}}{5\times5}=$$

$\dfrac{168\times3\frac{25}{32}}{25}=\dfrac{20328}{800}=25\frac{41}{100}$ acres increase, and (24 acres $+25\frac{41}{100}$) $=49\frac{41}{100}$.

Lastly: oxen. weeks. acres.

$$\left.\begin{matrix} 12 : 4 :: 3\tfrac{1}{2} \\ 18 :: 49\tfrac{41}{100} : \end{matrix}\right\}=\frac{12\times4\times49\frac{41}{100}}{18\times3\frac{1}{2}}=\frac{2\times4\times49\frac{41}{100}}{3\times3\frac{1}{2}}$$

$$=\frac{4941\times8}{100\times3\frac{1}{2}\times3}=\frac{39528}{350\times3}=\frac{39528}{1050}=37\frac{113}{175}$$ oxen, the Ans.

Otherwise

$$\left.\begin{matrix} 12 : 4 :: 3\tfrac{1}{3} \\ 21 : 9 \end{matrix}\right\}=\frac{21\times9\times3\frac{1}{3}}{12\times4}=\frac{7\times9\times3\frac{1}{3}}{4\times4}=\frac{63\times3\frac{1}{3}}{16}=\frac{630}{48}=13\tfrac{1}{8}\text{ acr.}$$

$$\left.\begin{matrix} 10 : 5 :: (13\tfrac{1}{8}—10) \\ 24 : 14 \end{matrix}\right\}=\frac{24\cdot14\cdot3\frac{1}{8}}{10\times5}=\frac{12\cdot14\cdot3\frac{1}{8}}{5\times5}=\frac{525}{25}=21\text{ acres.}$$

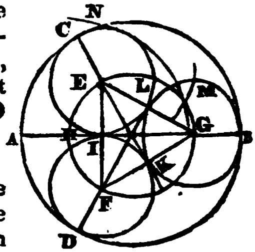
$$\left.\begin{matrix} 12 : 4 :: 3\tfrac{1}{3} \\ 18 :: 45 \end{matrix}\right\}=\frac{12\cdot4\times45}{18\times3\frac{1}{3}}=\frac{12\cdot4\times5}{2\times3\frac{1}{3}}=\frac{6\cdot4\cdot5}{3\frac{1}{3}}=\frac{120\times3}{3\frac{1}{3}\times3}=\frac{360}{10}$$

$=36$ oxen;

41. A certain widow owned a farm in the form of a circle. She had three daughters, to each of whom she gave as much land as could be contained in one of three equal circles, to be described within her farm, and each of the daughter's farms to touch on

another, and also that of the mother's, in their boundary lines respectively. Now the mother's farm contained 700 acres, and she wished to know how much each daughter would have, and how far their houses would be from hers, if each was built in the centre of her farm, as the mothers was?

Let ACBD be the given farm, of which O is the centre, of the circle and AB the diameter. From the centre A with radius AO describe an arc, cutting the given circle in the points C and D, join CO and DO; bisect AO in the point H, and from the centre H with radius AO describe the arc NM, and from the centre, N with the same radius describe an arc cutting NM in the point M; then from the centre O, with radius OM, describe the circle EFG, cutting BO, CO, and DO, in the points G, E, F, so will GB, EC, FD, be the radi required. Join EF, FG, GH, and produce EO & FO to K and L. Now, because OF=OE, and OD=OC, FD=EC, therefore the circle DIK is equal to the circle CIL. In like manner it may be shown that each of them is equal to the circle BKL, Again, because OE = OF, and OI common to the two triangles EOI and FOI, and they have their included angles equal, the base EI = the base FI, and the angles EIO and FIO are right angles; therefore the circles EIL, FIK touch the right line AO, and also touch each other in the point I.



In like manner it may be demonstrated, that they touch the circle BKL in the points K and L. It is evident also, that they touch the widow's circle, or the given circle in the points B, C, D. The number of square yards in an acre being 4840, a circle being equal to 3.14159, &c. drawn into the square of the radius; $\therefore \frac{4840 \times 700}{3{\cdot}1445926}$ = square of the radius, and $\therefore$ the radius itself will be $\sqrt{(\frac{4840 \times 700}{3.1416}} = \sqrt{(\frac{3388000}{3.1416})} = 1038{\cdot}475$ yards $= a =$ the radius of the widow's farm $=$ DO, FI $= x$, then will FO$=a-x$, and by similar triangles FE : EL :: FO : OI. But FE$=$2FL, $\therefore$FO$=$2OI, $\therefore \frac{a-x}{2}=$OI, then $x^2 = (\frac{a-x}{2})^2 - (a-x)^2$; that is, $x = 2a\sqrt{3} - 3a = a(2\sqrt{3} - 3), = 482.$ or $x = 481.956$ yards $=$FI, & 963.912 yards the distance of each daughter's house from one another, and 556.52 yards the distance of each daughter's house from her mother's house. Also, if each daughter had as many dollars as their farms contained square yards. they would each have \$ 729735,83 cents., nearly.

42. There are two sorts of metal, each being a mixture of gold and silver, but in different proportions. Two coins from these metals of the same weight are to each other in value as 11 to 17; but if to the same quantities of silver as before in each mixture double the former quantities of gold had been added, the values of two coins from them of equal weights would have been to each other as 7 to 11. Determime the proportion of gold to silver in each mixture, the values of equal weights of gold and silver being as 13 to 1. Let $1=$ the weight of gold in each mixture, $x=$ the weight of silver in the first, and $y=$ the weight in the second, and then let the value of weight 1 of the first and value of the weight of 1 of the second be

$\frac{x+13}{x+1}$, and $\frac{y+13}{y+1}$; $\therefore$ $\frac{x+13}{x+1} : \frac{y+13}{y+1} :: 11 : 17$. Also, $\frac{x+26}{x+2} : \frac{y+26}{y+2} :: 7 : 11$; from the first, and from the 2d we have $(x+13)(y+1) : (y+13)(x+1) :: 11 : 17$, and $(x+26)(y+2) : (y+26)(x+2) :: 7 : 11$, and $y=\frac{21x-13}{x+35}=\frac{40x-52}{x+68}=y$; or $21x^2+1415x-884=40x^2+1348x-1820$; or $19x^2-67x=936$; $x=9$, and $y=4$. Ans. The proportion of gold to silver is as, $1:9$ in the first mixture, and $1:4$ in the second mixture.

43. A mason has 2 cubical pieces of white marble of exactly the same size, and two cubical equal pieces of black, larger than the other. The number of solid yards in the four pieces is 9 more than 11 times the number of yards in a side of a white one, together with 12 times the number in a side of a black one. He afterwards finds another block, the length of which is 2 yards longer than a side of one of the white pieces, and the width 4 times the length of a side of the black one; and this when laid on its largest side occupies a space greater by 3 yards than the difference between 4 times the space occupied by a black, and 3 times the space occupied by a white one. Required the dimensions of the blocks Ans. The side of a white block is 1 yard, and of a black one 3 yards; the length of the other is 3 yards, an the width 12 yards. Let $x=$ the side of the white one, $y=$a side of the black one then

$(x+2)\cdot 4y=4y^2-3x^2+3$, and $3x^2+4xy=4y^2-8y+3$; $\therefore x^2+\frac{4}{3}xy+\frac{4}{9}y^2=\frac{16}{9}y^2-\frac{8}{3}y+1$, and $x+\frac{2}{3}y=\frac{4}{3}y-1$; $\therefore x=\frac{2}{3}y-1$. Also $2x^3+2y^3=\frac{22}{3}y-11+9+12y=\frac{58}{3}y-2$, or $\frac{16}{27}y^3-\frac{8}{3}y^2+4y-2+2y^3=\frac{58}{3}y-2$, or $70y^2-72y=414$, and $y^2-\frac{36}{35}y+(\frac{18}{35})^2=\frac{207}{35}+\frac{324}{1225}=\frac{7569}{1225}$, and $y=\frac{87}{35}+\frac{18}{35}=3$, and $x=1$.

44. A and B travelled on the same road and at the same rate from Huntingdon to London. At the 50th milestone from London, A overtook a drove of geese which were proceeding at the rate of

3 miles in 2 hours; and two hours afterwards met a stage wagon, which was moving at the rate of 9 miles in 4 hours. B overtook the same drove of geese at the 45th milestone, and met the same stage wagon exactly forty minutes before he came to the 31st milestone. Where was B when A reached London?

Let $x =$ the rate at which A or B travels, the geese travel at the rate of $\frac{3}{2}$ per hour, and the wagon at the rate of $\frac{9}{4}$. B approaches the wagon at the rate of $x + \frac{9}{4}$, and he overtakes the geese $\frac{10}{3}$ hours after A. Then $\frac{10}{3}x - 5 =$ B's distance from A, A meets the wagon $50 - 2x$ miles from London, and B meets it $31 + \frac{2}{3}x$ miles from London; $\therefore$ it had travelled in the interim $\frac{8}{3}x - 19$ miles, and $\therefore$ the interval $= (\frac{8}{3}x - 19)\cdot\frac{4}{9}$. Also, A's distance from B $= (\frac{8}{3}x - 19)\cdot\frac{4}{9}\cdot(x + \frac{9}{4})$; $\therefore \frac{4}{9}(\frac{8}{3}x - 19)\cdot(x + \frac{9}{4}) = \frac{10}{3}x - 5$, and $\frac{16}{9}x^2 - \frac{41}{3}x = 21$; $\therefore (\frac{4}{3}x)^2 - \frac{41}{4}(\frac{4}{3}x) + (\frac{41}{8})^2 = \frac{3025}{64}$, and $\frac{4}{3}x = \frac{55}{8} + \frac{41}{8} = \frac{96}{8} = 12$, and $x = 9$, $\therefore$ the distance required $= 25$.

45. The hold of a vessel partly full of water (which is uniformly increased by a leak) is furnished with two pumps worked by A and B, of whom A takes three strokes to two of B's; but four of B's throw out as much water as five of A's. Now B works for the time in which A alone would have emptied the hold; A then pumps out the remainder, and the hold is cleared in 13 hours and 20 minutes. Had they worked together, the hold would have been emptied in 3 hours and 45 minutes; and A would have pumped out 100 gallons more than he did. Required the quantity of water in the hold at first, and the horary influx at the leak.

Since A takes 6 strokes while B takes 4, but four of B's throw out as much as 5 of A's, the water thrown out in a given time by A is to that thrown out by B :: 6 : 5. Let $z =$ the number of gallons in the hole at first, $y =$ the influx of gallons at the leak per hour, $x =$ the time B worked; $\therefore x =$ the time in which A would have cleared the hold. In x hours the influx at the leak would be xy gallons; $\therefore 6 : 5 :: z + xy :$ quantity thrown out by B on the first supposition $= \frac{5}{6}(z + xy)$, and $z + \frac{40}{3}y =$ the whole quantity pumped out in 13h. 20′; $\therefore z + \frac{40}{3}y - \frac{5}{6}(z + xy) =$ the quantity pumped out by A on the first supposition; now $\frac{40}{3} - x =$ the time A worked; $\therefore x : \frac{40}{3} - x :: z + xy :$ the quantity pumped out by A on the first supposition $=$

$$\frac{(40 - 3x)(z + xy)}{3x};\ \therefore \frac{(40 - 3x)(z + xy)}{3x} = \frac{z + 80y - 5xy}{6}, \text{ or}$$

$$80z + 80xy - 6xz - 6x^2y = xz + 80xy - 5x^2y, \text{ and } 80z - 7xz = x^2y,$$

$\therefore z = \dfrac{x^2y}{80 - 7x}$. But had they worked together, the hold would have been cleared in $\frac{15}{4}$ hours; $\therefore z + \frac{15}{4}y$ would equal the whole

quantity pumped out on this supposition ; and $x : \frac{15}{4} :: \frac{5}{8}(z+xy)$: quantity pumped out by B in this case $= \frac{5\times5(z+xy)}{4\times2\times x}$;
and $z : \frac{15}{4} :: z+xy$: quantity pumped out by A in this case, $= \frac{15}{4x}(z+xy)$; $\therefore$ $\frac{25(z+xy)}{8x} + \frac{15}{4x}(z+xy) = z + \frac{15y}{4}$, or $55(z+xy)$ $= 8xz + 30xy$, $8xz - 55z = 25xy$, $z = \frac{25xy}{8x-55} = \frac{x^2y}{80-7x}$, and $25(80-7x) = 8x^2 - 55x$, and $8x^2 + 5\times 24x = 2000$, and $x^2+15x+(\frac{15}{2})^2 = \frac{225}{4}+\frac{1000}{4} = \frac{1225}{4}$, and $x = \frac{35}{2} - \frac{15}{2} = 10$. Again $\frac{15}{4x}(z+xy) - 100 = \frac{z+80y-5xy}{6}$; $\therefore$ $\frac{3}{8}(z+10y) - 100 = \frac{z+30y}{6}$
whence $9z+90y-2400 = 4z+120y$,

or, $z-6y=480$. Now $z = \frac{x^2y}{80-7x} = \frac{100y}{10} = 10y$, whence $(10y-6y) = 4y = 480$, and $y=120$, $z=1200$, Ans.

46. There are three numbers in arithmetical progression, whose sum is 21 ; and the sum of the first and second is to the sum of the second and third as 3 to 4. Required the numbers.

Let $x-y$, x, and $x+y$, be the numbers ; $\therefore$ $3x = 21$, $x = 7$. Also, $2x-y : 2x+y :: 3 : 4$, and $2x : y :: 7 : 1$, or $y = 2$.

47. There are 6 towns in the order of the letters A B C D E F, whose distances from each other are in an increasing arithmetical progression. The distance from A to C is 16 miles, and from C to E is 24 miles. Required their respective distances.

Let $x-2y$, $x-y$, x, $x+y$, $x+2y$, be the distance ; then $(2x-3y=16)-(2x+y=24) = 4y = 8$, and $y=2$, $x=11$.

48. A person makes a mixture of 51 gallons, consisting of brandy, rum, and water, the quantities of which are in arithmetical progression. The number of gallons of brandy and rum together is to the number of gallons of rum and water together as 8 to 9. Required the quantities of each. Ans. 15, 17, 19.

Let $x-y$, x, $x+y$ be the quantities ; $\therefore$ $3x = 51$, and $x = 17$. Also, $2x-y : 2x+y :: 8 : 9$, and $4x : 2y :: 17 : 1$, or $y = 2$.

49. A number consisting of three digits which are in arithmetical progression, being divided by the sum of its digits, gives a quotient 48; and if 198 be subtracted from it, the digits will be inverted. Required the number.

Let $x+y$, x, $x-y$ be the digits ; then, by the question I have $\frac{100(x+y)+10x+x-y}{3x} = 48$, or $111x+99y = 3\times 48x$; $\therefore$ $37x+33y = 48x$; and $33y = 11x$, and $x = 3y$. Also $100(x+y)+10x+x-y-198 = 100(x-y)+10x+x+y$, or $198y = 198$, and $y=1$; $\therefore$ $x=3$, and the number is 432.

50. During a scarcity, a person wished to make a mixture of 24 bushels, consisting of wheat, oats, and barley, the quantities of each forming an increasing arithmetical progression. Not being able, however, to procure any barley, he mixed additional quantities of wheat and oats in the proportion of 2 to 3, so as to complete his 24 bushels, when he found the whole quantities of wheat and oats to be in the proportion of 5 to 7. How many bushels of each did he originally intend to mix?

Let $x-y$, x, and $x+y$ be the quantities; $\therefore 3x=24$, and $x=8$. Also, $8-y+\frac{2}{5}(8+y) : 8+\frac{3}{5}(8+y) : : 5 : 7$, or $56-3y : 64+3y : : 5 : 7$, or $56-3y : 120 : : 5 : 12$, and $56-3y=50$ and, $y=\frac{6}{3}=2$, and the quantities were 6, 8, and 10.

51. The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4. What are the numbers?

Let xy^3, xy^2, xy, and x be the numbers; then, by the question, $(xy^3-xy^2=)xy^2(y-1)=36$, and $(xy-x=)x(y-1)=4$, whence $y^2=9$, & $y=3$ and $x=2$, and the numbers are 54, 18, 6 and 2.

52. A person employed three workmen, whose daily wages were in arithmetical progression. The number of days they worked was equal to the number of shillings that the second received per day. The whole amount of their wages was seven guineas, and the best workman received 28 shillings more than the worst. What were their daily wages?

Let $x-y$, x, and $x+y$ denote their wages; whence $3x^2=147$, and $x^2=49$; $\therefore x=7$, and $(x^2+xy)-(x^2-xy)=2xy=28$; $\therefore xy=14$, or $y=2$, and their wages were 5, 7, and 9 shillings.

53. There are three numbers in geometrical progression; the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

Let x, xy, and xy^2 be the numbers: $\therefore x+xy=9$, and $x+xy^2=15$; whence $\frac{9}{1+y}=\frac{15}{1+y^2}$, and $y^2-\frac{5}{3}y+\frac{25}{36}=\frac{2}{3}+\frac{25}{36}=\frac{49}{36}$; $\therefore y=\frac{7}{6}+\frac{5}{6}=2$, and $x=3$, and the numbers are 3, 6, 12.

54. There are three numbers in geometrical progression, whose sum is 14; and the sum of the first and second is to the sum of the second and third as 1 to 2. Required the numbers.

Let $\frac{x}{y}$, x, and xy denote the numbers; $\therefore \frac{x}{y}+x : x+xy : : 1 : 2$; or $\frac{1}{y}+1 : 1+y : : 1 : 2$; $\therefore \frac{1}{y} : 1 : : 1 : 2$, and $y=2$. Hence $(\frac{x}{2}+x+2x=)\frac{7x}{2}=14$, and $x=4$, and the numbers are 2, 4, 8.

55. There are three numbers in geometrical progression, whose

continued product is 64, and the sum of their cubes is 584. Required the numbers.

Let $\frac{x}{y}$, x, xy be the numbers ; $\therefore x^3=64$, and $x=4$. Also, $\frac{x^3}{y^3}+x^3+x^3y^3=584$, and $\frac{1}{y^3}+1+y^3=\frac{584}{64}=\frac{73}{8}$; $\therefore y^6-\frac{65}{8}y^3+(\frac{65}{16})^2=\frac{4225}{256}-1=\frac{3969}{256}$, and $y^3-\frac{65}{16}=\pm\frac{63}{16}$; $\therefore y^3=8$ or $\frac{1}{8}$, and $y=2$ or $\frac{1}{2}$, and the numbers are 2, 4, and 8.

56. There are four numbers in geometrical progression, the second of which is less than the fourth by 24 ; and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers. Ans. 1, 3, 9, 27.

Let x, xy, xy^2, xy^3 be the numbers ; $x+xy^3 : xy+xy^2 :: 7 : 3$, and $1-y+y^2 : y :: 7 : 3$; $\therefore 1+y^2 : y :: 10 : 3$, and $3y^2+3=10y$; $\therefore y^2-\frac{10}{3}y+\frac{25}{9}=\frac{25}{9}-1=\frac{16}{9}$; $\therefore y=\pm\frac{4}{3}+\frac{5}{3}=3$, or $\frac{1}{3}$, and $(xy^3-xy)=24x=24$, and $x=1$.

57. From two towns which were 168 miles distant, two persons, A and B, set out to meet each other ; A went 3 miles the first day, 5 the next, 7 the third, and so on ; B went 4 miles the first day, 6 the next, and so on. In how many days did they meet ?

Let $x=$ the number of days; $\therefore \{6+(x-1)\cdot 2\}\frac{x}{2}=x^2+2x=$ the number of miles A went, and $\{8+(x-1)2\}\frac{x}{2}=x^2+3x=$ the number B went ; $\therefore 2x^2+5x=168$, and $x=\frac{37}{4}-\frac{5}{4}=\frac{32}{4}=8$, A.

58. A traveller set out from a certain place, and went 1 mile the first day, 3 the second, 5 the next, and so on, going every day 2 miles more than he had gone the preceding day. After he had been gone three days, a second sets out, and travels 12 miles the first day, 13 the second, and so on. In how many days will the second overtake the first ? Ans. 2 and 9 days.

Let $x=$ the number of days the first travels ; then I have $\{2+(x-1)2\cdot\}\frac{x}{2}=x^2=$ the number of miles he travels, and $\{24+(x-4)\cdot 1\}\cdot\frac{x-3}{2}=\frac{(x+20)\cdot(x-3)}{2}=$ the number the second travels ; $\therefore 2x^2=x^2+17x-60$, and $x^2-17x+\frac{289}{4}=\frac{289}{4}-60=\frac{49}{4}$ $\therefore x=\pm\frac{7}{2}+\frac{17}{2}=5$ or 12, and the number is 2 or 9.

59. A person has two pieces of ground, one of which is in the form of an equilateral triangle, and the other of a rectangular parallelogram, one side of which is equal to a side of the triangle, and the other side is 8 yards less. These he plants with trees at the distance of two yards from each other, and finds that there are

5 more on the rectangle than on the triangle. What are the lengths of the sides?

Let $2x$ = a side of the triangle $x.\frac{1}{2}(x+1)=x(x-4)-5$, or $x^2+x=2x^2-8x-10$, and $x^2-9x+\frac{81}{4}=\frac{81}{4}+10=\frac{121}{4}$, and $x=\pm\frac{11}{2}+\frac{9}{2}=10$. The sides of the $\triangle$ are 20 and $\square$ 20 and 12 yards.

60. There are four numbers in arithmetical progression, whose sum is 28; and their continual product is 585. Required the numbers. Ans. the numbers are 1, 5, 9, 13.

Let $x-3y$, $x-y$, $x+y$, $x+3y$, be the numbers; $\therefore$ $4x=28$, $x=7$, and also $(x^2-9y^2)(x^2-y^2)=585$, and $9y^4-490y^2+2401=585$, or $9y^4-490y^2+\dfrac{60025}{9}=\dfrac{60025}{9}-1816=\dfrac{43681}{9}$, and $3y^2=\pm\frac{209}{3}+\frac{245}{3}=12$, $y^2=4$, and $y=2$.

61. There are four numbers in arithmetical progression; the sum of the squares of the first and second is 34, and the sum of the squares of the third and fourth is 130. Required the numbers.

Let $x-3y$, $x-y$, $x+y$, and $x+3y$ be the numbers; then $\therefore$
$$\left.\begin{array}{l}2x^2-8xy+10y^2=34\\2x^2+8xy+10y^2=130\end{array}\right\}=4x^2+20y^2=164,$$ and $x^2+5y^2=41$; also $16xy=96$; $\therefore$ $xy=6$. Hence $\dfrac{36}{y^2}+5y^2=41$, and $5y^4-41y^2=-36$, and $y^4-\frac{41}{5}y^2+\frac{1681}{100}=\frac{1681}{100}-\frac{36}{5}=\frac{961}{100}$, and $y^2=\pm\frac{31}{10}+4\frac{1}{10}=1$, and $y=1$, and $x=6$, and the numbers are 3, 5, 7, and 9.

62. The sum of \$700 was divided among four persons, whose shares were in geometrical progression; and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares?

Let x, xy, xy^2, xy^3, be the shares; then by the question I have $xy^3-x : xy^2-xy :: 37 : 12$, or $y^2+y+1 : y :: 37 : 12$; $\therefore$ $y^2+1 : y :: 25 : 12$, and $y^2-\dfrac{25}{12}y+\dfrac{625}{576}=\dfrac{625}{576}-1=\dfrac{49}{576}$, and $y=\pm\dfrac{7}{24}+\dfrac{25}{24}=\dfrac{4}{3}$, or $\dfrac{3}{4}$. Hence $x+\dfrac{4}{3}x+\dfrac{16}{9}x+\dfrac{64}{27}x=700$; $\therefore$ $175x=27\times700$, and $x=108$, and $\therefore$ their shares are 108, 144, 192, and \$250, Ans.

63. Five persons undertake to reap a field of 87 acres. The five terms of an arithmetical progression, whose sum is 20, will express the times in which they can severally reap an acre; and they altogether can finish the undertaking in 60 days. In how many days can each separately reap an acre?

Let $x-2y$, $x-y$; x, $x+y$, $x+2y$, be the number of days; $\therefore$ $5x=20$, and $x=4$. Now $\dfrac{1}{x-2y}+\dfrac{1}{x-y}+\dfrac{1}{x+y}+\dfrac{1}{x+2y}=\dfrac{87}{60}-\frac{1}{4}=$ [illegible]; whence

$\left.\begin{array}{l}(x^2-y^2)(x+2y)+(x^2-4y^2)(x+y)+\\(x^2-4y^2)(x-y)+(x^2-y^2)(x-2y)\end{array}\right\}=\frac{6}{5}(x^2-y^2)(x^2-4y^2)$, or $\{2x(x^2-y^2)+2x(x^2-4y^2)=\}2x(2x^2-5y^2)=\frac{6}{5}(x^2-y^2)(x^2-4y^2)$, and $x(2x^2-5y^2)=\frac{3}{5}(x^4-5x^2y^2+4y^4)$, or since $x=4$, $2x^2-5y^2=\frac{3}{5}(x^3-5xy^2+y^4)$. Hence $y^4-\frac{35}{3}y^2+\frac{1225}{36}=\frac{1225}{36}-\frac{32}{3}=\frac{841}{36}$, and $y^2=\pm\frac{29}{6}+\frac{35}{6}=\frac{64}{6}$, or 1, and $y=1$. Ans. 3, 4, 5, 6 days.

64. Out of a vessel containing 24 gallons of pure spirit, a vintner drew off at three successive times a certain number of gallons, which formed an increasing arithmetical progression, in which the difference between the squares of the extremes was equal to 16 times the mean, and filled up the vessel with water after each draught, till he found what he last drew off reduced to one sixth of its original strength. Required the number of gallons of pure spirit drawn off each time. Ans. 12, 8, $3\frac{1}{3}$.

Since the difference of the squares of the extremes is equal to 16 times the means, if $x=$ the mean, the extreme will be $x-4$, and $x+4$. Now $x-4$ being drawn off, there remains $28-x$; $\therefore$ $24:28-x::x:$ the spirits drawn off the second time, $=\frac{x}{24}(28-x)$; and there now remains $28-x-\frac{x}{24}(28-x)=\frac{(24-x)(28-x)}{24}$; and this by the supposition $=\frac{24}{6}=4$, $x^2-52x+672=96$, or $x^2-52x+676=676-576=100$, and $x-26=\pm 10$, and $x=16$ or 36, the latter of which cannot answer the conditions; $\therefore$ $x-4=12=$ the quantity first drawn, and $(28-x)\frac{x}{24}=8=$ the quantity drawn the second time, whence $4=$ the quantity now remaining, and $24:4::(x+4)=20:$ the spirit drawn the third time $=3\frac{1}{3}$.

65. A number of persons purchased a field for \$345. The youngest contributed a certain sum, the next \$5 more, the third \$5 more than the second, and so on to the oldest. For the greater accommodation of the seniors, the field was divided into two parts, the younger half taking a portion proportional to the sum they had subscribed; and in order that each might have an equal share in this portion, they agreed to equalize their contributions, and each to pay \$22. Required the number of persons, and the sums paid by each.

Let $x=$ the sum paid by the youngest, and $2y=$ the number of persons; $\therefore$ $\{2x+(2y-1)5\}y=$ the value of the field, $=345$, and $\{2x+(y-1)5\}\frac{y}{2}=22y$; $\therefore$ $2x+5(y-1)=44$, and $2x+5y=49$; $\therefore$ $(2xy+5y^2=49y)-(2xy+10y^2-5y=345)=5y^2-5y=$

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$345-49y$, and $y^2+\frac{44}{5}y+\frac{484}{25}=\frac{484}{25}+\frac{345}{5}=\frac{2209}{25}$, and $y=\frac{47}{5}-\frac{22}{5}=5$, and $x=12$, and the number of persons was 10.

66. The number of deaths in a besieged garrison amounted to 6 daily; amd allowing for this diminution their stock of provisions was sufficient to last for 8 days. But on the evening of the sixth day 100 men were killed in a sally, and afterwards the mortality increased to 10 daily. Supposing the stock of provisions unconsumed at the end of the sixth day to support 6 men for 61 days; it is required to find how long it would support the garrison; and the number of men alive when the provisions were exhausted.

Let a = each man's daily provision, x = the number of men at first; $\therefore a(2x-42)4=(8x-168)a$ = stock of provisions, $a(2x-30)3=(6x-90)a$ = the stock exhausted at the end of the 6th day; $\therefore (2x-78)a$ = remainder $=366a$; $\therefore x=222$, and $222-136=86$ = the number of men after the sally. Let n = the number of days the provisions lasted afterwards, $\{172-(n-1)10\}\frac{n}{2}.a=366a$, or $91n-5n^2=366$; $\therefore n^2-\frac{91}{5}n+\frac{8281}{100}=\frac{8281}{100}-\frac{366}{5}=\frac{961}{100}$, and $n=\pm\frac{31}{10}+\frac{91}{10}=6$, whence $86-60=26$ = the number of men remaining after the provisions were exhausted.

67. A ship with a crew of 175 men set sail with a store of water sufficient to last to the end of the voyage. But in 30 days the scurvy made its appearance and carried off three men every day, and at the same time a storm arose, which protracted the voyage three weeks. They were, however, just enabled to arrive in port, without any diminution in each man's daily allowance of water. Required the time of the passage, and the number of men alive when the vessel reached harbor.

Let = the number of days the voyage was expected to last; $\therefore 175x$ = the quantity of water laid in, supposing each man to drink daily a pint of water. On the 31st day the quantity drank was 172, on the 32d, 169; and thus the quantity of water consumed daily, after 30 days, forms a decreasing arithmetical progression, whose common difference is 3, and the number of terms is $x-9$; $\therefore$ the quantity drunk after 30 days =

$\{344-(x-10)3\}.\frac{x-9}{2}$, which must $=(x-30).175$; $\therefore 374x-3x^2-3366+27x=350x-10500$, or $51x-3x^2=-7134$, and $x^2-17x+\frac{289}{4}=2378+\frac{289}{4}=\frac{9801}{4}$, and $x=\frac{99}{2}+\frac{17}{2}=58$; that is, $58+21$

$=79$ days the voyage lasted; and $172-(x-10)3=28$, the number of men alive when the vessel entered the harbor.

68. The Fly starts 10 miles before the Telegraph; but the Fly coachman having made an appointment with the driver of the Telegraph, walks his horses so as to be overtaken at the end of the second mile. Now it is observed that the number of revolutions made in a given time by the hinder wheel of the Fly, its fore wheel, and the hinder wheel of the Telegraph increase in arithmetical progression, and that the circumference of these wheels, viz. of the fore wheel of the Fly, its hinder wheel, and the hinder wheel of the Telegraph, increase in a geometrical progression, whose common ratio is the same as the common difference of the Arith'l. Prog. find the ratio that the wheels bear to each other.

Let $1=$ the circumference of the fore wheel of the Fly, $x=$ the common ratio; $\therefore 1, x, x^2$, are the proportional lengths of the three wheels; the distances described by the two coaches in the same time are as $12:2::6:1$; $\therefore$ the number of revolutions of a wheel in a given time, $\propto \frac{\text{distance}}{\text{circumference}}$; $\therefore \frac{1}{x}, 1, \frac{6}{x^2}$ are the revolutions made by the three wheels in the same time; $x, x^2, 6$, are also revolutions made in the same time, since they are equimultiples of the former; but these revolutions increase in an arithmetical progression, whose common difference is x; $\therefore x^2=x+x=2x$, and $x=2$; $\therefore 1, 2, 4$ are the proportional lengths of the wheels, and 2, 4, and 6 the revolutions in a given time.

69. A company of Merchants fitted out a privateer, each subscribing \$100. The captain subscribed nothing, but was entitled to a \$100 share at the end of every certain number of months. In the course of 25 months he captured three prizes, which were in geometrical progression, the middle term being one fourth of the cost of the equipment, the common ratio the number of months which entitled the captain to his \$100 share, and their sum \$1375 more than the cost of the equipment. After deducting \$875 for the crew, the captain's share of the remainder amounted to one fifth of that of the company. Required the number of merchants, and the captain's pay.

Let $x=$ the number of merchants, and $y=$ the number of months at the end of which the captain was entitled to a 100 dollars share; $\therefore y:25::1:\frac{25}{y}=$ the number of the captain's shares, $25x=$ the middle term, and $y=$ the common ratio; $\therefore \frac{25x}{y}+25x+25xy=100x+1375$, or $\frac{x}{y}+xy+x=4x+55$, or $\frac{x}{y}-3x+xy=55$. Also, $100x+500=$ the sum to be divided after de-

ducting prize-money. Now the captain's share $=\frac{1}{5}$ company's share, and $\therefore=\frac{1}{6}$ whole number of shares.

Hence $\frac{100x+500}{6}:\frac{500x+2500}{6}::\frac{25}{y}:x$; $\therefore x=\frac{125}{y}$ and $\frac{125}{y^2}-\frac{375}{y}=-70$, or $\frac{25}{y^2}-15.\frac{5}{y}+\frac{225}{4}=\frac{225}{4}-14=\frac{164}{4}$ $\therefore \frac{5}{y}-\frac{15}{2}=\pm\frac{13}{2}$, and $\frac{5}{y}=1$ or 14; $\therefore y=5$, and $x=25$, Ans.

70. On the institution of Savings Banks, an industrious laborer, with his wife and children, saved each a certain number of pence in a degreasing arithmetical progression. The sum saved monthly was less by 3*s*. 3*d*. than would have purchased one sixth of as many bushels of wheat as the seventh child saved pence: the price of wheat being such that the sum saved by the eldest and fifth child augmented by 10*s*. would buy two bushels. But wheat rising 2*s*. per bushel, and work being scarce, the family find the sum saved would not buy as much wheat as their former savings by two bushels; when it appears that at this rate the sum annually saved would be less by five guineas than by the forner. Now the two youngest dying, it is found that if the remaining members of the family saved each one shilling less than the oldest child had done before the rise of wheat, their monthly account with the bank would not be affected by the deaths of the two youngest; but if they saved only 2*d*. less than the oldest had done, their monthly account would be 2*s*. 1*d*. less than it was at the first institution. Of how many did the family consist? What were the sums saved by each? and what was the price of wheat?

Let $x=$ the sum saved by the eldest child, or third in the famiiy, $y=$ the sum saved by the seventh or ninth in the family; $\therefore \frac{y}{6}=$ the number of bushels, and sum saved by the fifth child $=\frac{9-7}{9-3}.x+\frac{7-3}{9-3}.y=\frac{x+2y}{3}$; $\therefore$ a bushel cost $\frac{1}{2}(\frac{4x+2y}{3}+120)$ $=\frac{2x+y}{3}+60$, and the sum monthly saved $=\frac{y}{6}.\frac{2x+y+180}{3}-39=\frac{2xy+y^2}{18}+10y-39$. Now, after the rise, a bushel cost $\frac{2x+y}{3}+84$, and the number of bushels $=\frac{1}{6}y-2$; $\therefore$ the sum saved $=\frac{2xy+y^2}{18}+14y-\frac{4x+2y}{3}-168$, which $\therefore$ is $=\frac{2xy+y^2}{18}+10y-39-105$, whence $4y-\frac{4x+2y}{3}=24$; and $\therefore 2x=5y-36$. But,

after two died, the number in the family $=$

$$\frac{\text{sum saved}}{x-12}=\frac{\frac{1}{3}(y^2-6y)+14y-\frac{1}{3}(12y+72)-168}{\frac{5}{2}y-30}=\frac{\frac{1}{3}y^2+8y-144}{\frac{5}{2}y-30}$$

$\therefore \dfrac{\frac{1}{3}y^2+8y-144}{\frac{5}{2}y-30}\times(\tfrac{5}{2}y-20)=\dfrac{2xy+y^2}{18}+10x-39-25=\dfrac{y^2}{3}+8y-64$, & $(\frac{1}{3}y^2+8y)\frac{5}{2}y-360y-\frac{20}{3}y^2-160y+2880=(\frac{1}{3}y^2+8y)\frac{5}{2}y-160y-\frac{30}{3}y^2-240y+1920$; $\therefore \frac{10}{3}y^2-120y=-960$, and $y^2-36y+18^2=324-288=36$; $\therefore y-18=\pm 6$, and $y=24$, and $x=42$, and the family at first consisted of 10, and wheat cost $96d.=8s.$

71. A Grocer sold 80 pounds of tea, and 100 pounds of coffee for \$65; but he sold 60 pounds more of coffee for \$20, than he did of tea for \$10. What was the price of a pound of each?

Let $x=$ the price of a pound of tea (in dollars), and $y=$ the price of a lb. of coffee; then $x : 10 :: 1 :$ the number of lbs. of tea for \$10 $=\dfrac{10}{x}$. In the same way $\dfrac{20}{y}=$ the number of lbs. of coffee for \$20; $\therefore \dfrac{20}{y}=60+\dfrac{10}{x}$, or $\dfrac{2}{y}=6+\dfrac{1}{x}=\dfrac{6x+1}{x}$; and $\therefore y=\dfrac{2x}{6x+1}$. Again, $80x+100y=65$, or $16x+20y=13$; $\therefore 16x+\dfrac{40}{6x+1}=13$, and $96x^2+56x=78x+13$; by transposition, $96x^2-22x=13$, and $x^2-\dfrac{22}{96}\cdot x=\dfrac{13}{96}$, or $x-\dfrac{11}{96}=\pm\dfrac{37}{96}$; and $\therefore x=\frac{1}{2}$, or $-\frac{13}{48}$, which last does not answer the conditions; and $y=\frac{1}{4}$. $\therefore$ the price of a pound of tea is 50 cents, and of a pound of coffee, is 25 cents.

72. A and B engage to reap a field for \$90; and as A alone could reap it in 9 days, they promise to complete it in 5 days. They found however that they were obliged to call in C, an inferior workman, to assist them for the last two days, in consequence of which B received \$$3\frac{3}{4}$ less than he otherwise would have done. In what time could B or C, alone reap the field?

Let $x=$ the number of days in which B could reap the field, and $y=$ the number in which C could reap it; then $\frac{1}{9}+\dfrac{1}{x} : \dfrac{1}{x} :: 90 :$ the number of dollars B would have received $=\dfrac{810}{9+x}$, and $\dfrac{5}{x}\times 90=\dfrac{450}{x}=$ the number he did receive; $\therefore \dfrac{810}{9+x}-\dfrac{450}{x}=3\frac{3}{4}=\dfrac{15}{4}$, or $\dfrac{54}{9+x}-\dfrac{30}{x}=\dfrac{1}{4}$; $\therefore 216x-1080-120x=9x+x^2$; by transposition, $x^2-87x=-1080$; and $x=72$, or 15.

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73. Given $(x+1)(x^2+1)\times(x^3+1)=30x^3$, to find x by a quadratic. By performing the multiplication and ordering the terms of the equation, I have $x^6+x^5+x^4+2x^3+x^2+x+1=30x^3$, which, divided by x^3, and by putting $x+\frac{1}{x}=y$, becomes $y^3+y^2-2y=30$, or $y^3-9y^2+27y-27=(y-3)^3=-(10y^2-29y-3)=-(y-3)\times(10y+1)$; or I have $(y-3)^3+(y-3)\times(10y+1)=(y-3)\times(y-3)^2+(10y+1)=0$, which is satisfied by putting $y-3=0$, and $y=3$. But $x+\frac{1}{x}=y$; $\therefore x+\frac{1}{x}=3$, or $x^2-3x=-1$; $\therefore x^2-3x+\frac{9}{4}=\frac{5}{4}$ and $x=\frac{1}{2}(3\pm\sqrt{5})$, as required.

74. Find the values of x in the equation $x^5+15x^3+45x=18$. Put $x=y+z$; then, by substitution, reducing, and ordering the equation, I have $y^5+5(yz+3)\cdot(y^3+z^3)+5(yz+3)\cdot(2yz+3)\times(y+z)+z^5=18$. Assume $yz+3=0$, or $y=-\frac{3}{z}$; and it becomes $y^5+z^5=18$, and by substituting the value of z, I have $\frac{3^5}{y^5}=18$, or $y^{10}-18y^5=243$, and by completing the square, and $y^5=27$, and $y=\sqrt[5]{27}=3^{\frac{3}{5}}$; $\therefore z=-\frac{3}{y}=-3^{\frac{2}{5}}=-\sqrt[5]{9}$. Hence $x=y+z=\sqrt[5]{27}-\sqrt[5]{9}$, as required.

75. Find three numbers such, that if to the product of any two of them the remaining number be added, their sum shall be a $\square$.

Let x, y, and z denote the numbers; then $xy+z=m^2$, and $xz+y=n^2$, and $yz+x=\square=r^2$; then by taking

$$\left.\begin{aligned} xy+z&=m^2\\ xz+y&=n^2\end{aligned}\right\}\text{ sum}$$
$$\overline{xy+xz+y+z=m^2+n^2;}$$

or $(y+z)(x+1)=(m^2+n^2)$, from whence it appears that m^2+n^2 must consist of two factors at least, and from the known nature of such factors, each of them will be the sum of two squares, wherefore, put $y+z=a^2+b^2$, and $x+1=c^2+d^2$, or $x=c^2+d^2-1$, and then I may have $m=ac+bd$, and $n=ad-bc$, as is very well known. Again, taking $\left.\begin{aligned}(xy+y&=m^2,\\ xz+y&=n^2,\end{aligned}\right\}$ and

$$\overline{xy-xz-(y-z)=(y-z)(x-1)=m^2-n^2=(m+n)(m-n).}$$

Take $y-z=m-n=ac+bd+bc-ad=a(c-d)+b(c+d)$, and then $x-1=m+n=ac+bd+ad-bc=a(c+d)-b(c-d)$, or $x=a\times(c+d)-b(c-d)+1$; therefore I have $x=c^2+d^2-1=a(c+d)-b(c-d)+1$; whence $b=\frac{a(c+d)-(c^2+d^2)+2}{c-d}$ Now, in order to have b an integer, take $c-d=1$, or $c=d+1$; then $x=c^2+d^2-z=2d^2+2d=2d(d+1)$; $b=a(2d+1)-(2d^2+2d-1)$; $y+z=a^2+b^2=2a^2(2d^2+2d+1)-2a(2d+1)\times(2d^2+2d-1)+$

$(2d^2+2d-1)^2$, and $y-x=a(c-d)+b(c+d)=a+b(2d+1)=$ $2a(2d^2+2d+1)-(2d+1)\times(2d^2+2d-1)$; whence I have $y=a^2\times(2d^2+2d+1)-2a(2d^3+2d^2-d-1)+(d^2-1)(2d^2+2d-1)$; $z=a^2(2d^2+2d+1)-2a(2d^3+4d^2+d)+(d^2+2d)(2d^2+2d-1)$, and $x=2d(d-1)$, which are general expressions for the three required numbers, in which a, d, are arbitrary, or may be taken at pleasure. Let $a=0$, and $d=2$; then $x=12$, $y=33$, and $z=88$. Let $a=1$, $d=2$, $x=12$, $y=4$, and $z=33$. Let $a=-1$, $d=1$, then $x=4$, $y=9$, and $z=28$; the same numbers which *Fermat* gives by a very different solution.

76. Find 3 squares in geometrical progression such, that if each be diminished by its root, the remainders shall be a square.

Let x^2, xy, and y^2 be the sides of the three squares sought, then I have to make x^4-x^2, or $x^2-1=\square$, $x^2y^2-xy=\square$, and y^4-y^2, or $y^2-1=\square$. Put $p-x$, and $q-x$ for the sides of the first and 3d conditions or squares, then I get $x=\frac{p^2+1}{2p}$, $y=\frac{q^2+1}{2q}$. By substituting these values of x and y in the second equation or square, and reducing the whole to a common denominator, I have $(p^2+1)^2\times(q^2+1)^2-4pq\times(p^2+1)(q^2+1)=\square$, or $(q^2+1)^2p^4-4q\times(q^2+1)p^3+2(q^2+1)^2\times p^2-4q(q^2-1)+(q^2+1)^2=\square$. By putting $(q^2+1)p^2-2qp+(\frac{q^4+1}{q^2+1})$ for the side of this square, I shall find $p=\frac{q^4+q^2+1}{2q(q^2+1)}$. If I take $q=2$, I find $p=\frac{21}{20}$, $x=\frac{841}{840}$, $y=\frac{5}{4}$ and $(\frac{841}{840})^2$, $\frac{841}{840}$, and $(\frac{5}{4})^2$ are the sides of the three squares.

Let a^2, a^2x^2, and a^2x^4 be the numbers sought; then $a^2-a=\square$, $a^2x^2-ax=\square$, and $a^2x^4-ax^2=$, or $a^2x^2-a=\square$. Let $a^2x^2-a=(ax-p)^2=a^2x^2-2ap+p^2$; $x=\frac{p^2+a}{2ap}$, and $a^2x^2-ax=\square$; or, by substitution, $(\frac{p^2+a}{2p})^2-\frac{p^2+a}{2p}=\square=$. Multiply this by $4p^2$, and it becomes $(p^2+a)^2-2p(p^2+a)=p^4-2p^3+2ap^2-2ap+a^2=\square=(p^2-p+r)^2=p^4-2p^3+(r+1)p^2-2rp+r^2$. Here two of the first terms vanish on each side of the equation, and the third also. If I make $2r+1=2a$, or $r=a-\frac{1}{2}$, and I have $a^2-2ap=-(2a-1)p+(a-\frac{1}{2})^2$, whence $a-\frac{1}{4}=p$, it remains to make $a^2-a=\square=b^2a^2$, and $a=\frac{1}{1-p^2}$, from which a variety of sets of numbers may be found. If $b=\frac{2}{3}$, then $a=\frac{9}{5}$, $p=\frac{31}{20}$, $x=\frac{1681}{2232}$. Hence the numbers are $\frac{81}{25}$, $\frac{81}{25}\times(\frac{1681}{2232})^2$, $\frac{81}{25}\times(\frac{1681^4}{2232^4})$. If $b=\frac{5}{7}$, then $a=\frac{49}{24}$, and $p=\frac{43}{24}$, $x=\frac{3025}{4214}$. Again, let $p^4-2p^3+2ap^2-2ap+a^2=$

$(a - p + rp^2)^2 = r^2p^4 - 2rp^3 + (1 + 2ra)p^2 - 2ap + a^2$, $p = \frac{4a}{4a-1}$; also, if $a^2 - a = (a-q)^2 = a^2 - 2aq + q^2$, $a = \frac{q^2}{2q-1}$, again, take $b = \frac{3}{4}$, and the three roots will be $\frac{165649}{88920}$, $\frac{165649}{118560}$, and $\frac{165649}{158080}$; take $b = \frac{4}{5}$, and then the numbers will be $\frac{1343281}{857472}$, $\frac{1343281}{1071840}$, and $\frac{1343281}{1339800}$.

77. Find $(x+y+z)(x+y-z)(x-y+z) = \square$ in whole numbers.

Let $y = 2z + 1$, $x = 3z - 1$, and the given expression will be transformed to $\{6z \times 4z \times 2(z-1) \times 2\} = 16z^2(6z-6)$, whence $6(z-1) = 6z - 6$ must be a square. Let it $= m^2$, then $z = \frac{1}{6}m^2 + 1$, $y = \frac{1}{3}m^2 + 3$, and $x = \frac{1}{2}m^2 + 2$, and by multiplying each of them by 6, I have $z = m^2 + 6$, $y = 2m^2 + 18$, and $x = 3m^2 + 12$ the relation required. Thus, if $m = 1$, then $z = 7$, $y = 20$, $x = 15$, and if $m = 2$, then $z = 10$, $y = 26$, $x = 24$, it is evident that the given expression will be a $\square$ when each of its factors are. Let $x + y + z = 4p^2q^2$, $x - y + z = (q^2 + r^2 - p^2)^2$, and $-x + y + z = 4p^2r^2$, the sum is $x + y + z = (q^2 + r^2 + p^2)^2$. By subtracting each equation from this sum, I respectively obtain $x = \frac{(q^2+r^2+p^2)^2 - 4p^2r^2}{2}$,

$$z = \frac{(q^2+r^2+p^2)^2 - 4p^2q^2}{2}; \quad y = \frac{(q^2+r^2+p^2)^2 - (q^2+r^2-p^2)^2}{2}$$

Let $p = 2$, $r = 3$, $q = 1$, $x = \frac{196-16}{2} = 90$, $y = \frac{196-36}{2} = 80$ $z = \frac{1}{2}(196 - 144) = 26$. The above question is a formula that ex- the square of the area of a triangle whose sides are $2x$, $2y$, $2z$; therefore, if x, y and z denote any set of numbers constituting the sides of a right angle, the given formula will be a square, $x = 3$, $y = 4$, $z = 5$.

78. Find 3 square numbers, (to represent the sides of a right angle triangle,or) in arithmetical progression such that if each number be diminished by its root, the remainder shall be a square.

Denoting the three sides by ax, bx, cx, then per question, $a^2x^2 - ax = \square$, $b^2x^2 - bx = \square$, and $c^2x^2 - cx = \square$.

By division, $x^2 - \frac{1}{a}x = \square$, $x^2 - \frac{1}{b}x = \square$, $x^2 - \frac{1}{c}x = \square$. Put the respective coefficients of x, $= p$, q, r, then $x^2 - px = \square$, $x^2 - qx = \square$, $x^2 - rx = \square$. Make $x^2 - px = (m - x)^2 = m^2 - 2mx + x^2$, and $x = \frac{m^2}{2m-p}$. By subs. in $x^2 - qx = (\frac{m^2}{2m-p})^2 - q(\frac{m^2}{2m-p}) = \square$. Divide by the square $(\frac{m^2}{2m-p})^2$, and it becomes $m^2 - 2mq + pq = \square$. In like manner the third expression reduces to $m^2 - 2mr + rp = \square$, then the second to be made $m^2 - 2mq + pq = (s - m)^2 = s^2$

$-2sm+m^2$, whence $m=\dfrac{s^2-pq}{2s-2q}$. Also, make $m^2-2rm+rp=(t-m)^2=t^2-2tm+m^2$; $m=\dfrac{t^2-rp}{2t-2r}$. Equate these two values of m by making $s^2-pq=t^2-rp$, and $2s-2q=2t-2r$, from which I get $s^2-t^2=pq-rp$, and $s-t=q-r$, whence $\dfrac{s^2-t^2}{s-t}=\dfrac{pq-rp}{q-r}$, or $s+t=p$, from this $s=\frac{1}{2}(q+p-r)$, and $x=\dfrac{(s^2-pq)^2}{4s(s-q)\times(s-p)}$. Now, if a, b, and c be the numbers representing rational sides of a right angled triangle, the problem is solved. The most simple numbers are $a=3$, $b=4$, $c=5$, then $p=\frac{1}{3}$, $q=\frac{1}{4}$, $r=\frac{1}{5}$, $s=\frac{23}{120}$, and $x=\frac{450241}{1313760}$. Hence $3x=\frac{450241}{437920}$, $4x=\frac{450241}{328440}$, and $5x=\frac{450241}{262752}$, which are answers. Otherwise,

Let ax, bx and cx = the sides. Put $\dfrac{1}{a}=p$, $\dfrac{1}{b}=q$, $\dfrac{1}{c}=r$; then $x^2-px=\square=(d-x)^2$, and $x=\dfrac{d^2}{2d-p}$, $x^2-qx=\square=(\dfrac{d^2}{2d-p})^2-q(\dfrac{d^2}{2d-p})=\square$, or $x^2-rx=\square$, $(\dfrac{d^2}{2d-p})^2-r(\dfrac{d^2}{d^2-p})=\square$, and these become $d^2-2dp+pq=\square$, $d^2-2dr+rp=\square$.

Let $d^2-2dp+pq=\square=(t-d)^2$, and $d=\dfrac{t^2-pr}{2(t-r)}$. Again, let $d^2-2dr+rp=\square$; $\therefore x=\dfrac{d^2}{2d-p}$, $s=\frac{1}{2}(p+q-r)$, $x=\dfrac{(s^2-pq)^2}{4s(s-q)(s-p)}$.

Hence the sides are $\dfrac{a(s^2-pq)^2}{4s(s-q)\,(s-p)}$, $\dfrac{b(s^2-pq)^2}{4s(s-q)(s-p)}$, and $\dfrac{c(s^2-pq)^2}{4s(s-q)(s-p)}$, where a, b and c are the sides of any right angled triangle, and p, q, and r the reciprocals. Also, a, b, and c may be taken any numbers in arithmetical progression, as $a=31$, $b=41$, and $c=49$, and $p=\frac{1}{31}$, $q=\frac{1}{41}$, and $r=\frac{1}{49}$, and the squares are $(\dfrac{31\times7112635^2\times31}{46422774\times194560596})^2$; $(\dfrac{41\times7112635^2\times31}{46422774\times194560596})^2$; and $(\dfrac{49\times7112635^2\times31}{46422774\times194560596})^2$, and are, I think, the least.

Let the numbers be $9x^2-3x=\square$, $25x^2-5x=\square$, and $16x^2-4x=\square$. Let $9x^2-3x=(3x-r)^2=9x^2-6rx+r^2$, $x=\dfrac{r^2}{6r-3}$. And by substituting in the other two, I have $\dfrac{4r^2}{(6r-3)^2}\times\{4r^2-6r+3)=\square$, and $\dfrac{r^2}{(6r-3)^2}\times(25r^2-30r+15)=\square$, the first factor is already a square

Let $4r^2-6r+3=(2r-t)^2=4r^2-4rt+t^2$, when $r=\dfrac{t^2-3}{4t-6}$; and

by sub. in the last expression becomes $\frac{1}{(4t-6)^2}\times\{25t^4-120t^3+270t^2-360t+225\}=\square=[5t^2+12t-15]^2=25t^4+120t^3-6t^2-360t+225$, or $270t^2-120t^3=120t^3-6t^3$, or $t=\frac{23}{20}$, and $r=\frac{t^2-3}{4t-6}=\frac{671}{560}$, $x=\frac{r^3}{6r-3}=\frac{450241}{1313760}$, as before.

79. Find two square numbers such, that their product, when added to the square of each, shall make a square.

Let x^2, and y^2, denote the numbers or squares; then $x^2y^2+x^4=\square$, or $y^2+x^2=\square$, (1.) Also, $x^2y^2+y^4=\square$, or $x^2+y^2=\square$, (2.) (1) and (2) are satisfied by making $y=2mn$, and $x=m^2-n^2$; for $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, whence m and n are arbitrary, provided that $m>n$. Let $m=2$, $n=1$, then $2mn=4$, $m^2-n^2=3$; $\therefore\ y=4$, $x=3$.

80. Required a general method or theorem, by means of which any natural number may be divided any number of ways into 3 rational cubes.

Let $z^3=a-x^3-y^3$, take $x=p+q$, and $y=p-q$, then $z^3=a-2p^3-6pq^2$, or $q^2=\frac{a-z^3-2p^3}{6p}=\frac{(a-z^3-2p^3)6p}{36p^2}=\frac{6pa-6m^3p+36p^2(m-p)^2}{36p^2}$, (when $z=m-2p$)(and when $p=\frac{av^2}{6}$) $=\frac{a^2-am^3+a^2v^2(m-\frac{av^2}{6})^2}{a^2v^2}$. Assume $a-av(m-\frac{av^2}{6})$ for the root of the numerator; then $m^3=2av(m-\frac{av^2}{6})$. Let $m=dv$, then $v=\frac{2ad}{d^3+\frac{1}{3}a^2}=\frac{6ad}{3d^3+a^2}$, or $m=\frac{6ad^2}{3d^3+a^2}$, $p=\frac{av^2}{6}=\frac{6a^3d^2}{(3d^3+a^2)^2}$, $z=\frac{18ad^5-6a^3d^2}{(3d^3+a^2)^2}$, and $q=\frac{1-vm+\frac{av^3}{6}}{v}$
$\frac{(9d^6-30a^2d^3+a^4)\ (3d^3+a^2)+36a^4d^3}{6ad(3d^3+a^2)^2}$, or

$$x=\frac{(9d^6-30a^2d^3+a^4)(3d^3+a^2)+72a^4d^3}{6ad(3d^3+a^2)^2},\text{ and } y=\frac{30a^2d^3-9d^6-a^4}{6ad(3d^3+a^2)^2}.$$

Ex. 1. Let $a=2$, and $d=\frac{4}{3}$, then I have $(\frac{27475}{25^3\times36})^3+(\frac{3529}{25^3\times36})^3+(\frac{1209}{25^3\times36})^3=2$. Let $a=3$, and $d=3$, then I have $(\frac{10}{75})^3+(\frac{17}{75})^3+(\frac{108}{75})^3=3$.

Note. Of all the values of a, when $a=1$, positive values are

the most limited, as d must be greater than 1 and less than $1\frac{1}{3}$; therefore the fractions will be large in this case.

81. Find three numbers such, that their product, when added to the square of each particular number, shall be a square.

Let $x+y$. $x-y$, and y^2, denote the numbers. Then $(x^2-y^2)y^2$ $=$ their product, which will be a square when added to $(y^2)^2=y^4$, gives x^2y^2, which is a square, and one condition is satisfied. But $(x^2-y^2)y^2+(x+y)^2=\square=a^2(x+y)^2$, or

$x=\dfrac{y(a^2-1+y^2)}{y^2-(a^2-1)}$, and $x+y=\dfrac{2y^3}{y^2-(a^2-1)}$; $\therefore x-y=\dfrac{2(a^2-1)y}{y^2-(a^2-1)}$

and $x^2-y^2=\dfrac{4y^4(a^2-1)}{\{y^2-(a^2-1)\}^2}$. But $(x^2-y^2)y^2+(x-y)^2=\square$, or

$\dfrac{4y^6(a^2-1)}{\{y^2-(a^2-1)\}^2}+\dfrac{4y^2(a^2-1)^2}{\{y^2-(a^2-1)\}^2}=\square$, or, by neglecting the square factor, I have $y^4(a^2-1)+(a^2-1)^2=\square$, which becomes a square by making $y=1$ and $a=\dfrac{m^2+n^2}{2mn}$. Let $m=2$, $n=1$, then

$a^2-1=\dfrac{7}{16}$. Hence $x+y=\dfrac{32}{7}$, $x-y=\dfrac{18}{7}$, and the numbers required are 1, $\frac{32}{7}$, and $\frac{18}{7}$.

82. Find three numbers such, that their product, when increased by each particular number, may be a square. Let x, $4x+4$, and 1, denote the numbers; then their product $=4x^2+4x$, and $4x^2+4x+1=(2x+1)^2$. Also, I have $4x^2+4x+4x+4=4(x+1)^2$; $\therefore$ two of the conditions are answered by the assumption. It remains to make $4x^2+4x+x=\square$, or $4x^2+5x=\square=a^2x^2$; therefore $x=\dfrac{5}{a^2-4}$. Let $a=4$, then $x=\dfrac{5}{12}$, and the numbers are $\dfrac{5}{12}$ $\dfrac{17}{3}$, and 1. If $a=7$, the numbers are $\dfrac{1}{9}$, $\dfrac{40}{9}$, and 1.

83. Find x, y, z, so that $xy+z=\square$, and $xz+y$, $yz+x$, may all be squares. Put $z=x+y+n$; then $xy+z=xy+x+y+n=\square=a^2$, (1) by assumption, and $xz+y=$(by (1))$=x^2+(n-1)x+a^2-n=\square$ (2); similarly $yz+x=y(n-1)y+a^2-n=\square$ (3), and (2) and (3) become squares by making $a=4$, and $n=7$; then (1) becomes $xy+x=9-y$, and $x=\dfrac{9-y}{y+1}$. Put $y=1$, then $x=4$ and $z=12$, which numbers answer the question.

84. Find x, y, and z, so that $x^2+x+y+z=\square$, $y^2+x+y+z=\square$, $z^2+x+y+z=\square$. Put $x+y+z=r$, and assume $x^2+r=(a-x)^2$, or $x=\dfrac{a^2-r}{2a}$; similarly I put $y=\dfrac{b^2-r}{2b}$, $z=\dfrac{c^2-r}{2c}$. By adding these values of x, y, and z, and putting r for $x+y+z$, I

have $r=\frac{a^2-r}{2a}+\frac{b^2-r}{2b}+\frac{c^2-r}{2c}$; $r=\frac{abc(a+b+c)}{2abc+ab+ac+bc}$.

Let $a=2, b=3, c=4$; then $r=\frac{108}{37}$, and $y=\frac{15}{37}$, $z=\frac{121}{37}$, $x=\frac{10}{37}$.

85. Find 3 whole numbers in arithmetical progression such, that the sum of any two shall be a square.

Let x, and z be the extremes, and y the means; then by the nature of arithmetical proportion, $x+z=2y$. Put $x+y=a^2$; then $2y=a^2$, or $\frac{1}{2}a^2$. But $x+y=x+\frac{a^2}{2}=b^2$; $\therefore x=b^2-\frac{a^2}{2}$; also put $z+y=z+\frac{a^2}{2}=c^2$, then $z=c^2-\frac{a^2}{2}$. By adding the values of x and z, I have $x+z=a^2=b^2+c^2-a^2$, or $b^2+c^2=2a^2$, or $b^2=2a^2-c^2=\square$. Put $c=a-m$, then $2a^2-c^2=a^2+2am-m^2=(a-n)^2=a^2-2an+n^2$. Hence I have $a=\frac{m^2+n^2}{2(m+n)}$, & $b=\sqrt{(2a^2-c^2)}=a-n=\frac{m^2-2mn-n^2}{2(m+n)}$, and $c=\frac{n^2-2mn-m^2}{2(m+n)}$. Let $m=10$, $n=-8$, then $b=49$, $c=31$, $a=41$, whence $x=b^2-\frac{a^2}{2}=1560\frac{1}{2}$, $y=\frac{a^2}{2}=840\frac{1}{2}$, and $z=c^2-\frac{1}{2}a^2=120\frac{1}{2}$, as required.

86. Find 3 square numbers in arithmetical progression.

Let y^2 be the first term, and $2xy+x^2$ the common difference; then will y^2, $y^2+2xy+x^2$, and $y^2+4xy+2x^2$, be the three terms of the progression, the two first of which are already squares; we have, therefore, only to make $y^2+4xy+2x^2=$ a square. Let $(y-rx)^2=y^2-2rxy+r^2x^2$, and, by reduction, $x=\frac{(2r+4)y}{r^2-2}$ where y and r may be any numbers taken at pleasure, provided that r^2 be greater than 2. If $y=1$, and $r=2$, then $x=4$, and the numbers are 1, 25, and 49.

Otherwise, let x^2, y^2, and z^2, be the numbers required; then by the nature of arithmetical progression, $x^2+z^2=2y^2$; or $\left(\frac{x+z}{2}\right)^2+\left(\frac{z-x}{2}\right)^2=y^2$. We have, therefore, to find two squares, $\left(\frac{y+z}{2}\right)^2$ and $\left(\frac{z-x}{2}\right)^2$, such, that their sum may be a square. This will be accomplished by taking $\frac{1}{2}(z+x)=2rs$, and $\frac{1}{2}(z-x)=r^2-s^2$, for then y will be $=r^2+s^2$, as it plainly appears. Hence $x=\frac{z+x}{2}-\frac{z-x}{2}=2rs-r^2+s^2$; and $z=\frac{z+x}{2}+\frac{z-x}{2}=2rs+r^2-s^2$; therefore $2rs-r^2+s^2$, r^2+s^2, and $2rs+r^2-s^2$, are the the roots of three squares in arithmetical progression, where r and s may

be any two unequal numbers taken at pleasure. If $r = 2$, $s = 1$.

87. Find 3 square numbers in harmonical progression.

Let x^2, y^2, and z^2, be the three numbers required; then, by the nature of harmonical proportion, $x^2 - y^2 : y^2 - z^2 :: x^2 : z^2$;

$$\therefore y^2 = \frac{2x^2z^2}{x^2+z^2} = \frac{x^2z^2}{\frac{1}{2}(x^2+z^2)} = \frac{x^2z^2}{\left(\frac{x+z}{2}\right)^2 + \left(\frac{x-z}{2}\right)^2} = \text{a} \ \square;$$

therefore, (since the numerator is already a square,) $\left(\frac{x+z}{2}\right)^2 + \left(\frac{x-z}{2}\right)^2$ must be a square; and it appears from the last example, that this will be the case when $x = 2rs - r^2 + s^2$, and $z = 2rs + r^2 - s^2$; then y, or the root of $\frac{2x^2z^2}{x^2+z^2}$, will be equal $\frac{6r^2s^2 - r^4 - s^4}{r^2+s^2}$. Therefore $2rs - r^2 + s^2$, $\frac{6r^2s^2 - r^4 - s^4}{r^2+s^2}$, and $2rs + r^2 - s^2$, will be the roots of three squares in harmonical proportion. Or, if each number be multiplied by r^2+s^2, we shall have $2r^3s + 2rs^3 - r^4 + s^4$, $6r^2s^2 - r^4 - s^4$, and $2r^3s + 2rs^3 + r^4 - s^4$, for the three roots, where r and s may be taken at pleasure. If $r = 1$, and $s = 2$, then $x = 35$, $y = 7$, and $z = 5$; and the three numbers are 35^2, 7^2, and 5^2, or 1225, 49, and 25.

88. Find 2 numbers in the proportion as 8 is to 15 and such that the sum of their squares shall be a square number.

Let $8x$ and $15x$ be the numbers required; then $64x^2 + 225x^2 = 289x^2 =$ a square, which it evidently is, the root being $17x$; therefore x may be any number taken at pleasure. If $x = 72$, then $8x = 576$, and $15x = 1080$, the Answer.

89. Find 2 numbers, such that their difference may be equal to the diference of their squares and that the sum of their squares shall be a square number.

Let x and y denote the numbers sought; then, by the question, $x^2 - y^2 = x - y$, and $x^2 + y^2 = \square$. The first equation, divided by $x - y$, gives $x + y = 1$; hence $x = 1 - y$, and $x^2 + y^2 = 1 - 2y + y^2 =$ a square.

Assume $1 - ry$ for its side; then $1 - 2y + y^2 = 1 - 2ry + r^2y^2$; whence $y = \frac{2r-2}{r^2-2}$, and $x = 1 - y = \frac{r^2-2r}{r^2-2}$, where r may be taken at pleasure, provided it be greater than 2. If $r = 3$, then $y = \frac{4}{7}$, and $x = \frac{3}{7}$, the Answer.

90. Find 2 numbers such that if their product be added to the sum of their squares, it shall make a square number.

Let x and y be the two numbers; then $x^2 + xy + y^2$ must be a square. Assume its side $= x + r$; then $x^2 + xy + y^2 = x^2 + 2xr$

$+r^2$, and $x=\frac{y^2-r^2}{2r-y}$, where y and r may be taken at pleasure, provided that y be greater than r, but less than $2r$. If $y=3$, and $r=2$, then $x=5$. If $y=5$, and $r=3$, then $x=16$. Otherwise;

Let $x^2+xy+y^2=\left(\frac{r}{s}y-x\right)^2=\frac{r^2}{s^2}y^2-2\frac{r}{s}yx+x^2$; then $xy+y^2=\frac{r^2}{s^2}y^2-2\frac{r}{s}yx$, or $x+y=\frac{r^2}{s^2}y-2\frac{r}{s}x$; and, by reduction, $(s^2+2rs)x=(r^2-s^2)y$. Hence, if x be taken $=r^2-s^2$, then y will be taken $=s^2+2rs$, and $x^2+xy+y^2=\left(\frac{r}{s}y-x\right)^2=(r^2+rs+s^2)^2$.

Cor. It is evident from the above, that if x be taken $=r^2-s^2$, $y=s^2+2rs$, and $z=r^2+rs+s^2$; then $z^2=x^2+xy+y^2$, r and s being any two unequal numbers whatever.

91. If a^2 and b^2 be two squares whose sum is to be divided into two other squares m^2, n^2, then $m=\frac{ar^2-as^2-2rsb}{r^2+s^2}$, and n equal $\frac{2rsa+br^2-bs^2}{r^2+s^2}$. If $a=4$, and $b=1$; and r be taken $=4$, and $s=1$; then $m=\frac{52}{17}$, and $n=\frac{47}{17}$.

92. Find three numbers such, that whether their sum be added to, or subtracted from, the square of each particular number, the numbers thence arising shall be all squares.

It is evident from the *Cor.* to the 90th Example, that if a be taken $=r^2-s^2$, $b=2rs+s^2$, and $c=r^2+rs+s^2$, then $c^2=a^2+ab+b^2$; and from this it is easy to show, that

$$(c^2+b^2)^2\pm 4abc(a+b),$$
$$(c^2+a^2)^2\pm 4abc(a+b),$$
$$\{c^2+(a+b)^2\}^2\pm 4abc(a+b),$$

are all squares. For, since $c^2=a^2+ab+b^2$, we have $c^2-b^2=a(a+b)$, $c^2-a^2=b(a+b)$, $c^2-(a+b)^2=-ab$; and, by substitution, the three formula become

$$(c^2+b^2)^2\pm 4bc(c^2-b^2)$$
$$(c^2+a^2)^2\pm 4ac(c^2-a^2)$$
$$\{c^2+(a+b)^2\}^2\pm 4(a+b)(c-ab);$$

the roots of which are evidently $2bc+c^2-b^2$, or $2bc-c^2+b^2$; $2ac+c^2-a^2$, or $2ac-c^2+a^2$; $2c(a+b)+c^2-(a+b)^2$, or $2c(a+b)-c^2+(a+b)^2$. Put $m=c^2+b^2$, $n=c^2+a^2$, $p=c^2+(a+b)^2$, and $q=4abc(a+b)$; then $m^2\pm q=\square$, $n^2\pm q=\square$, and $p^2\pm q=\square$. Or, if each be multiplied by x^2, then $m^2x^2\pm qx^2=\square$, $n^2x^2\pm qx^2=\square$, and $p^2x^2\pm qx^2=\square$.

Now it is evident that all the conditions of the question will be satisfied, if mx, nx, and px, be the three numbers, and $qx^2=$ to their sum; that is, if $qx^2=mx+nx+px$, or $x=\frac{m+n+p}{q}$; whence we have $mx=\frac{m}{q}(m+n+p)$, $nx=$

$\frac{n}{q}(m+n+p)$, and $px = \frac{p}{q}(m+n+p)$, for the three numbers required. If r be taken $= 2$, and $s = 1$, then $a = 3$, $b = 5$, $c = 7$ $m = c^2 + b^2 = 74$, $n = c^2 + a^2 = 58$, $p = c^2 + (a+b)^2 = 113$, $q = 4abc(a+b) = 3360$, $mx = \frac{245 \times 74}{3360} = \frac{518}{96}$, $nx = \frac{245 \times 58}{3360} = \frac{406}{96}$, & $px = \frac{245 \times 113}{3360} = \frac{791}{96}$, the Ans.

If the values of m, n, p, be expressed in terms of r and s, we have

$$m = r^4 + 2r^3s + 7r^2s^2 + 6rs^3 + 2s^4$$
$$n = 2r^4 + 2r^3s + r^2s^2 + 2rs^3 + 2s^4$$
$$p = 2r^4 + 6r^3s + 7r^2s^2 + 2rs^3 + s^4$$
$$q = 4s(r^4 + 2r^3s - r^2s^2 - 2rs^3)(2r^3 + 3r^2s + 3rs^2 + s^3);$$

where r and s may be any numbers taken at pleasure, so that from hence we may obtain an indefinite number of answers.

93. Find 3 square numbers such, that the difference of every two of them shall be a square number.

Let x^2, y^2, and z^2, be the numbers sought. Then $x^2 - z^2 = \square$, $y^2 - z^2 = \square$, and $x^2 - y^2 = \square$;

Or, $\frac{x^2}{z^2} - 1 = \square$, $\frac{y^2}{z^2} - 1 = \square$, and $\frac{x^2}{z^2} - \frac{y^2}{z^2} = \square$; And, by putting $\frac{x}{z} = \frac{r^2+1}{r^2-1}$, and $\frac{y}{z} = \frac{s^2+1}{s^2-1}$, we shall have $\frac{x^2}{z^2} - 1 = \frac{4r^2}{(r^2-1)^2}$, and $\frac{y^2}{z^2} - 1 = \frac{4s^2}{(s^2-1)^2}$, which are both evidently squares; and, therefore it remains only to make $\frac{x^2}{z^2} - \frac{y^2}{z^2} =$ a square.

But $\frac{x^2}{z^2} - \frac{y^2}{z^2} = \left(\frac{r^2+1}{r^2-1}\right)^2 - \left(\frac{s^2+1}{s^2-1}\right)^2 = \frac{4(r^2s^2-1)(s^2-r^2)}{(r^2-1)^2(s^2-1)^2} = \square$; $\therefore$ $(r^2s^2-1)(s^2-r^2) = \square$, or $(r^2s^2-1)\left(\frac{s^2}{r^2} - 1\right) = \square$.

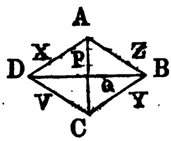
Now there are two obvious cases in which this expression will be a square, viz. when r or $s = 1$, and when $r = \frac{4}{5}$, and $s = \frac{13}{4}$. In the first case, the expression becomes $(s^2-1)^2$, or $(r^2-1)^2$; but z being then $= 0$, this value of r or s is inadmissible. In the second case, both the factors $(r^2s^2 - 1)$ and $\left(\frac{r^2}{s^2} - 1\right)$ are squares the first being equal to $(\frac{12}{5})^2$, and the second equal to $(\frac{63}{16})^2$; and therefore $(r^2s^2-1)\left(\frac{r^2}{s^2} - 1\right) = \frac{12^2 \times 63^2}{5^2 \times 16^2} =$ a square. Now $\frac{x}{z} = \frac{r^2+1}{r^2-1} = -\frac{41}{9}$, and $\frac{y}{z} = \frac{s^2+1}{s^2-1} = \frac{185}{153}$; or $x = -\frac{41z}{9}$, and $y = \frac{185z}{153}$ hence, if z be taken $= 153$, in order that x and y may be whole

numbers, we shall have $x = -697$, and $y = 183$; and therefore $x^2=485809$, $y^2=34225$, and $z^2=23409$, the Answer.

To obtain other Answers, put $n=\frac{r}{s}$, and $r=1+v$; then $(r^2s^2-1)\left(\frac{r^2}{s^2}-1\right)=(n^2r^4-1)(n^2-1)=(1+4n^2v+6n^2v^2+4n^2v^3+n^2v^4)(n^2-1)$; or, dividing by $(n^2-1)^2$, and putting $w=\frac{n^2}{n^2-1}$, we shall have $1+4wv+6wv^2+4wv^3+wv^4=\square$. Assume its root $=1+2wv+(3w-2w^2)v^2$, the square of which is equal to $1+4wv+6wv^2+4w(3w-2w^2)v^3+(3w-2w^2)^2v^4$; and, by comparing this with the preceding equation, we have $4wv^3+wv^4=4w(3w-2w^2)v^3+(3w-2w^2)^2v^4$, or $2+v=4(3w-2w^2)+w\{3-2w\}^2v$; whence

$$v=\frac{8w^2-12w+4}{4w^3-12w^2+9w-1}=\frac{4(2w-1)}{4w^2-8w+1};$$ and from this value we may obtain an infinite number of answers, because the number n in the value of w may be taken at pleasure.

94. Find a quadrilateral inscribed in a circle such, that its four sides, and each of its diagonals shall be expressed in whole numbers. Let ABCD denote the trapezium, having x, y, z, and v for its sides, p and q for its diagonals. Now $xz+yv=pq$, (1), $\frac{xy+vz}{xv+yz}=\frac{p}{q}$, (2). Multiply (1) by (2), and there results $(xz+yv)\times\left(\frac{xy+vz}{xv+yz}\right)=p^2$, (3). Divide (1) by (2), and there results $(xz+yv)\times\left(\frac{xv+yz}{xy+vz}\right)=q^2$, (4). Hence p and q will be found by (3) and (4), by making $xz+yv=\square$, and $\frac{xy+vz}{xv+yz}=\square$, rational square.



Let $y=mx$, $z=nx$, $v=p'x$, then $xz+yv=(n+p'm)x^2=\square$; $\therefore n+p'm=\square$. Also,
$\frac{xy+vz}{xv+yz}=\frac{m+p'n}{p'+mn}=\square=a^2$. (By assumption,) $p'=\frac{(a^2n-1)m}{n-a^2}$.

Hence $n+p'm=n+\frac{(a^2n-1)m^2}{n-a^2}=\frac{n(n-a^2)^2+(a^2n-1)\times(n-a^2)m^2}{(n-a^2)^2}=\square$, or $n(n-a^2)^2+(a^2n-1)\times(n-a^2)m^2=\square$. Put $n=r^2$; then $r^2(r^2-a^2)^2+(a^2r^2-1)\times(r^2-a^2)m^2=\square=\{r(r^2-a^2)+mb\}^2$,

and $m=\frac{2br(r^2-a^2)}{(a^2r^2-1)(r^2-a^2)-b^2}=\frac{2br(r^2-a^2)}{(a^2r^2-1)(r^2-a^2)-b^2}$, (5).

Otherwise, put $\frac{a^2n-1}{n-a^2}=b^2$, then $n=\frac{a^2b^2-1}{b^2-a^2}$. Hence

$$n+p'm=n+\frac{a^2n-1}{n-a^2}m^2=\frac{a^2b^2-1}{b^2-a^2}+b^2m^2=\square=(bm+c)^2=b^2m^2+$$

$2bcm+c^2$, or $m=(\frac{a^2b^2-1}{b^2-a^2}-c^2)\div 2bc$, or I have $m=\frac{a^2b^2-1+c^2(a^2-b^2)}{2bc(b^2-a^2)}$; (6). Again, put $\frac{xz+yv}{xv+yz}=a^2$; then $x=\frac{a^2yz-yv}{z-a^2v}$, and $xy+vz=y^2(\frac{a^2z-v}{z-a^2v})+vz=\square$. Put $\frac{a^2z-v}{z-a^2v}=b^2$; then $z=(\frac{a^2b^2-1}{b^2-a^2})v$; $\therefore y^2(\frac{a^2z-v}{z-a^2v})+vz=y^2b^2+(\frac{a^2b^2-1}{b^2-a^2})v^2=\square$ $=(yb+c)^2$, or $y=\{(\frac{a^2b^2-1}{b^2-a^2})v^2-c^2\}\div 2bc$, (7), and $x=b^2y$. Let $a=\frac{3}{2}$, $b=2$, $c=\frac{3}{2}$, $v=\frac{7}{4}$, then $x=\frac{47}{6}$, $y=\frac{47}{24}$, $z=8$, $p=\frac{65}{8}$, $q=\frac{423}{52}$, and $x=2444$, $y=611$, $z=2492$, $v=546$, $p=2535$, $q=2538$.

95. Find 4 whole numbers such, that their sum shall be a square, and the sum of their squares a biquadrate.

Let x, qz, yz, and vz, denote the numbers. Put $x^2+q^2z^2+y^2z^2+v^2z^2=a^4$, then $x^2=a^4-(q^2+y^2+v^2)z^2=\square=(a^2-pz)^2$, or $z=\frac{2a^2p}{p^2+v^2+y^2+q^2}$, and $x=a^2-pz=\frac{a^2(q^2+y^2+v^2-p^2)}{p^2+q^2+y^2+v^2}$; $\therefore\ x+qz+yz+vz=\frac{a^2(q^2+y^2+v^2-p^2)+2a^2p(v+y+q)}{p^2+q^2+y^2+v^2}=$ $\frac{a^2(q^2+y^2+v^2-p^2)+2a^2p(v+y+q)}{(p^2+q^2+y^2+v^2)^2}\times(p^2+q^2+y^2+v^2)=\square$, (per question), or $(v^2+y^2+q^2)^2-p^4+2p(v+y+q)(p^2+q^2+y^2+v^2)=$ $v^4+2pv^3+2(y^2+q^2+py+pq)v^2+2p(p^2+q^2+y^2)v+(y^2+q^2)^2+$ $2p(y+q)\times(p^2+q^2+y^2)-p^4=(v^2+pv+c)^2=v^4+2pv^3+(p^2+2c)v^2$ $+2pcv+c^2$. Hence I have

$$c=(\frac{2(y^2+q^2+py+pq)-p^2}{2}),\ v=\{\frac{\frac{5}{4}p^2+2yq-3p(y+q)}{3p-2(q+y)}\}.$$

Let $y=-3$, $q=-1$, $p=-2$, then $v=-\frac{13}{2}$. Hence the numbers are $x=193\times(\frac{a}{15})^2$, $vz=104\times(\frac{a}{15})^2$, $yz=48\times(\frac{a}{15})^2$, and and $qz=16\times(\frac{1}{15}a)^2$, or, by rejecting the square factor, $(\frac{1}{15}a)^2$, the numbers are 193, 104, 48, and 16. Or for three numbers, let $q=0$, $p=-2$, $y=-4$, then $v=-\frac{19}{2}$, $z=-\frac{16a^2}{441}$, $x=\frac{409a^2}{441}$, $yz=\frac{64a^2}{441}$, $vz=\frac{152a^2}{441}$, or by rejecting the common $\square$ factor, the numbers are 409, 64, 152.

96. Divide unity into three such positive parts, that each, diminished by unity, shall have the remainders cubes.

Let $x^3+y^3+z^3=4$, or $a^3-1+b^3-1+c^3-1=1$. Put $x=p+q$, $y=p-q$, then $2p^3+6pq^2=4-z^3$, or

$q^2 = \frac{4 - z^3 - 2p^4}{6p} = \frac{24p - 6pz^3 - 12p^4}{36p^2} = \square$. $\therefore 24p - 6pz^3 - 12p^4 = \square$. Let a and b denote such numbers, that when they are assumed for p and z, they may make $24p - 6pz^3 - 12p^4 = \square = c^2$; then generally $p = a - mx$, and $z = b + nx$; then $p^4 = a^4 - 4a^3mx + 6a^2m^2x^2 - 4am^3x^3 + m^4x^4$, and $z^3 = b^3 + 3b^2nx + 3bn^2x^2 + n^3x^3$, and $pz^3 = ab^3 + 3b^2anx + 3abn^2x^2 + an^3x^3 - mb^3x - 3b^2mnx^2 - 3bmn^2x^3 - mn^3x^4 = ab^3 + (3b^2an - mb^3)x + (3abn^2 - 3b^2mn)x^2 + (an^3 - 3bmn^2)x^3 - mn^3x^3$. Hence $pz^3 + 2p^4 = ab^3 + 2a^4 + (3b^2an - mb^3 - 8a^3m)x + (3abn^2 - 3b^2mn + 12a^2m^2)x^2 + (an^3 - 3bmn^2 - 8am^3)x^3 + (2m^4 - mn^3)x^4$; and $24p - 6pz^3 - 12p^4 = c^2 + (48a^3m + 6mb^3 - 18b^2an - 24m)x + 18(b^2mn - abn^2 - 4a^2m^2)x^2 + 6(3bmn^2 + 8am^3 - an^3)x^3 + 6(mn^3 - 2m^4)x^4 = c^2 + 6(8a^3m + mb^3 - 3b^2an - 4m)x + 18(b^2mn - abn^2 - 4a^2m^2)x^2 + 6(3bmn^2 + 8am^3 - an^3)x^3 + 6m(n^3 - 2m^3)x^4 = \square = \{c + 3(\frac{8a^3m + mb^3 - 3b^2an - 4m}{c})x + vx^2\}^2$. Hence

$$18(b^2mn - abn^2 - 4a^2m^2) + 6(3bmn^2 + 8am^3 - an^3)x + 6m(n^3 - 2m^3)x^2$$
$$= 2cv + \frac{9}{c^2}(8a^3m + mb^3 - 3b^2an - 4m)^2 +$$
$$6(\frac{8a^3m + mb^3 - 3b^2av - 4z}{c})vx \times vx + v^2x^2.$$ Assume $v = \frac{18(b^2mn - abn^2 - 4a^2m^2)c^2 - 9(8a^3n + mb^3 - 3b^2an - 4m)^2}{2c^3}$;

then $x = \frac{6(3bmn^2 + 8am^3 - an^3)c - 6(8a^3m + mb^3 - 3b^2an - 4m)v}{c(v^2 + 12m^4 - 6mn^3)}$.

Hence a and b being known, and m and n arbitrary, x is easily found, and thence the roots of the cube become known. Such is the general solution; but the following particular solution is more simple. Let $p = \frac{6}{5} - yx$, and $z = \frac{4}{5} + vx$, then $p^4 = \frac{1296}{625} - \frac{864}{125}yx + \frac{216}{25}y^2x^2 - \frac{24}{5}y^3x^3 + y^4x^4$, $z^3 = \frac{64}{125} + \frac{48}{25}vx + \frac{12}{5}v^2x^2 + v^3x^3$, and $pz^3 = \frac{384}{625} + \frac{4}{125}(72v - 16y)x + \frac{12}{25}(6v^2 - 4yv)x^2 + \frac{6}{5}(v^3 - 2yv^2)x^3 - yv^3x^4 = \frac{384}{625} + \frac{32}{125}(9v - 2y)x + \frac{24}{25}v(3v - 2y) + \frac{6}{5}v^2(v - 2y) - yv^3x^4$. Hence $pz^3 + 2p^4 = \frac{2976}{625} + \frac{32}{125}(9v + 56y)x + \frac{24}{25}(3v^2 - 2vy + 18y^2)x^2 + \frac{6}{5}(v^3 - 2v^2y - 8y^3)x^3 + (y^4 - yv^3)x^4$. Hence I have $4p - pz^3 - 2p^4 = \frac{24}{625} + \frac{4}{125}(129z^2y - 288v)x + \frac{24}{25}(2vy - 3v^2 - 18y^2)x^2 + \frac{6}{5}(8y^3 + 2v^2y - v^3)x^3 + (yv^3 - 2y^4)x^4$; $\therefore \frac{144}{625} + \frac{24}{125}(323y - 72v)x + \frac{144}{25}(2vy - 3v^2 - 18y^2)x^2 + \frac{36}{5}(8y^3 + 2v^2y - v^3)x^3 + 6(yv^3 - 2y^4)x^4 = \square =$
$\left(\frac{12}{25} + (\frac{323y - 72v}{5})x + ax^2\right)^2 = \frac{144}{625} + \frac{24}{125}(323y - 72v)x + \left(\left(\frac{323y - 72v}{5}\right)^2 x^2 + \frac{24a}{25}\right)x^2 + \frac{1}{5}(646y - 144v)ax^3 + a^2x^4$. Assume $a = \frac{1}{24}(46800vy - 5616v^2 - 106921y^2)$, (1), then I have $x =$

$\frac{288y^3+72yv^2-36v^3-646ay+144av}{5(a^2+12y^4-6yv^3)}$; (2.) If $y=1.02$; $v=3.5$; then $a=\frac{1080.0507}{2}$ and $x=\frac{1234807335200}{5(436710708571)}$ $p=\frac{49282337539044}{43167107085700}$; $z=\frac{21587268507484}{21583553542855}$. Let $v=3.4$, and $y=1$; $a=\frac{318049\cdot}{600}$, and $288+72v-36v^3=-294\cdot642$; $646-144v=157\cdot4=156.4$; $x=\frac{247830\cdot446}{15(a^2+12-6v^3)}$; $a^2=\frac{101155166401.}{36}$; $\therefore x=\frac{12\times247830\cdot446}{5(101074589761)}$, or $x=\frac{29739653520}{5(101074589761)}$; which answers. $x=\frac{2973965352\cdot}{5(101074589761)}$. and $\frac{323-72v}{5}x+ax^2=\frac{78604121846606\cdot}{5(101074589761)}x$, and $p=\frac{576707885046}{5(101074589761)}$, and $z=\frac{505413181012}{5(101074589761)}$; therefore $p+q=\frac{5662709430938 50697}{47004080983 5196035}$; $p-q=\frac{5065060702637 07307}{47004080983 5196035}$, and $z=\frac{505413181012}{505372948805}$, consequently $(p+q)^3$, $(p-q)^3$ and z^3, these three cube numbers answer the question. These 2 solutions I sent to France and to London in 1832.

100. Find any number of squares whose sum and product are equal.

In order to make the solution quite easy, I shall begin with finding two squares whose sum and product shall be equal to each other; and afterwards proceed gradually to finding three, four, and five squares, whose sum and continual product shall be equal to each other, till the way of extending the solution to any number of squares, is sufficiently clear and evident.

1. To find two square numbers whose sum and product shall be equal to other.

Let x^2 and y^2 denote the two squares; then, by the question, $x^2y^2=x^2+y^2$, whence $x^2=\frac{y^2}{y^2-1}=$ a square; therefore y^2-1 must be a square. Now y^2-1 will evidently be a square when $y=(m^2+n^2)\div2mn$ where m and n may be taken at pleasure. For then $x^2=\frac{y^2}{y^2-1}=\frac{(m^2+n^2)^2}{(m^2-n^2)^2}$, $y^2=\frac{(m^2+n^2)^2}{4m^2n^2}$.

Take $m = 2$ and $n = 1$, then $x^2 = \frac{25}{9}$ and $y^2 = \frac{25}{16}$.

98. Find three rational squares whose sum shall be equal to their continual product.

Let the squares be denoted by x^2, y^2 and z^2: Then, by the question, $x^2y^2z^2 = x^2+y^2+z^2$, hence $x^2 = \dfrac{y^2+z^2}{y^2z^2-1}$; whence it appears that y^2+z^2 and y^2z^2-1 must be both squares. Put $y^2+z^2 = (y+z-n)^2$, whence $z = \dfrac{2ny-n^2}{2(y-n)}$, and hence

$$y^2z^2-1 = \frac{y^2(2ny-n^2)^2}{4(y-n)^2}-1 = \frac{y^2(2ny-n^2)^2-4(y-n)^2}{4(y-n)^2} = \text{a square};$$

$y^2(2ny-n^2)^2-4(y-n)^2 = 4n^2y^4-4n^3y^3+n^4y^2-4y^2+8ny-4n^2 =$ a square. Assume $2ny^2-n^2y-\dfrac{1}{n}$ for its root; that is, put $4n^2y^4-4n^3y^3+n^4y^2-4y^2+8ny-4n^2 = \left(2ny^2-n^2y-\dfrac{1}{n}\right)^2 = 4n^2y^4-4n^3y^3+n^4y^2-4y^2+2ny+\dfrac{1}{n^2}$: whence $6ny = 4n^2+\dfrac{1}{n^2} = \dfrac{4n^4+1}{n^2}$ and $y = \dfrac{4n^4+1}{6n^3}$ where n may be taken at pleasure.

Take $n=1$, then $y = \frac{5}{6}$, $z = \dfrac{2ny-n^2}{2(y-n)} = -2$, $x^2 = \dfrac{y^2+z^2}{y^2z^2-1} = \frac{169}{64}$: Therefore the three squares are $\frac{169}{64}$, $\frac{25}{36}$, and 4.

Take $n=2$, then $y = \frac{65}{48}$, $z = \dfrac{2ny-n^2}{2(y-n)} = -\frac{34}{31}$, $x^2 = \dfrac{(2593)^2}{(1634)^2}$, and therefore the three squares in this case are

$$\frac{(2593)^2}{(1634)^2}, \frac{(65)^2}{(48)^2} \text{ and } \frac{(34)^2}{(31)^2}.$$

99. Find four squares, such, that their sum and continual product may be equal to each other.

Let v^2, x^2, y^2 and z^2 denote the four squares: Then, by the question, $v^2x^2y^2z^2 = v^2+x^2+y^2+z^2$; hence $v^2 = (x^2+y^2+z^2) \div x^2y^2z^2-1$. This last expression will manifestly be a square when $x^2+y^2+z^2$ and $x^2y^2z^2-1$ are both squares. Put $x^2+y^2+z^2 = (x+y-z)^2 = x^2+2xy+y^2-2z(x+y)+z^2$; whence $z = xy \div (x+y)$, and $x^2y^2z^2-1 = x^4y^4 \div (x+y)^2-1 = \{x^4y^4-(x+y)^2\} \div (x+y)^2 =$ a square; therefore $x^4y^4-(x+y)^2 = x^4y^4-x^2-2xy-y^2 =$ a square: Assume $x^2y^2-\dfrac{1}{2y^2}$ for the root, that is, put $x^4y^4-x^2-2xy-y^2 = (x^2y^2-\dfrac{1}{2y^2})^2 = x^4y^4-x^2+\dfrac{1}{4y^4}$; whence $x = -\dfrac{4y^6+1}{8y^5}$, where y may be taken at pleasure.

Take $y = 1$, then $x = -\frac{5}{8}$, $z = -\frac{5}{3}$ and $v^2 = 49$; and therefore 49, $\frac{25}{64}$, 1, and $\frac{25}{9}$ are four squares that will answer.

Since $x^2+y^2+z^2$ and $x^2y^2z^2-1$ are both to be squares: put $x^2+y^2+z^2=(x+y-m)^2=x^2+2xy+y^2-2m(x+y)+m^2$: then $x=\frac{2my+z^2-m^2}{2(y-m)}$, and $x^2y^2z^2-1=\frac{y^2z^2(2my+z^2-m^2)^2}{4(y-m)^2}-1=$ $\frac{y^2z^2(2my+z^2-m^2)^2-4(y-m)^2}{4(y-m)^2}=$ a square: therefore $y^2z^2(2my+z^2-m^2)^2-4(y-m)^2=4m^2z^2y^4+4mz^2y^3(z^2-m^2)+y^2z^2(z^2-m^2)^2-4y^2+8my-4m^2=$ a square. Assume $2mzy^2+zy(z^2-m^2)-g$ for its root; that is, put $4m^2z^2y^4+4mz^2y^3(z^2-m^2)+y^2z^2(z^2-m^2)^2-4y^2+8my-4m^2=\{2mzy^2+zy(z^2-m^2)-g\}^2=4m^2z^2y^4+4mz^2y^3(z^2-m^2)+z^2y^2(z^2-m^2)^2-4gmzy^2-2gzy(z^2-m^2)+g^2$. Put $4gmz=4$, or $g=\frac{1}{mz}$, in order to take away all the terms in which any power of y above the first is concerned; and from the equality of the remaining terms, viz. $8my-4m^2=-2gzy(z^2-m^2)+g^2$, $y=\frac{4m^2+g^2}{8m+2gz(z^2-m^2)}=\frac{4m^2+1\div m^2z^2}{8m+2(z^2-m^2)\div m}$ $=\frac{4m^4z^2+1}{6m^3z^2+2mz^4}=\frac{4m^4z^2+1}{2mz^2(3m^2+z^2)}$, where m and z may be taken at pleasure. Take z and m each $=1$, then $y=\frac{5}{8}$, $x=-\frac{5}{3}$, and $v^2=49$; therefore 49, 1, $\frac{25}{64}$ and $\frac{25}{9}$ are four squares that will answer. Take $z=1$ and $m=\frac{1}{2}$, then $y=\frac{5}{7}$, $x=\frac{41}{12}$, and $v^2=\frac{(305)^2}{(187)^2}$; therefore 1, $\frac{5^2}{7^2}$, $\frac{(41)^2}{(12)^2}$ and $\frac{(305)^2}{(187)^2}$, are four squares that will answer.

100. Find 5 squares whose sum and continual product shall be equal to each other. Let the squares be denoted by u^2, v^2, x^2, y^2, and z^2. By the question $u^2v^2x^2y^2z^2=u^2+v^2+x^2+y^2+z^2$: whence $u^2=\frac{v^2+x^2+y^2+z^2}{v^2x^2y^2z^2-1}$: put $v^2+x^2+y^2+z^2=(x+y-m)^2=x^2+2xy+y^2-2m(x+y)+m^2$: whence $x=\frac{2my+v^2+z^2-m^2}{2(y-m)}$, and hence

$$v^2x^2y^2z^2-1=\frac{v^2y^2z^2(2my+v^2+z^2-m^2)^2}{4(y-m)^2}-1=$$

$\frac{v^2y^2z^2(2my+v^2+z^2-m^2)^2-4(y-m)^2}{4(y-m)^2}=$ a square, and therefore $v^2y^2z^2(2my+v^2+z^2-m^2)^2-4(y-m)^2=4m^2v^2z^2y^4+4mv^2z^2y^3(v^2+z^2-m^2)+v^2z^2y^2(v^2+z^2-m^2)^2-4y^2+8my-4m^2=$ a square $=\{2mvzy^2+vzy(v^2+z^2-m^2)-g\}^2=4m^2v^2z^2y^4+4mv^2z^2y^3(v^2+z^2-m^2)+v^2z^2y^2(v^2+z^2-m^2)^2-4gmvzy^2-2gvzy(v^2+z^2-m^2)+g^2$. Put $4gmvz=4$, or $g=\frac{1}{mvz}$, in order to take away all the powers of

y above the first; and the remaining terms of the expression will be $8my - 4m^2 = -2gvzy(v^2 + z^2 - m^2) + g^2$; whence $y =$

$$\frac{4m^2+g^2}{8m + 2gvz(v^2 + z^2 - m^2)} = \frac{4m^2+1 \div m^2v^2z^2}{8m+(2\div m)(v^2+z^2-m^2)} =$$

$$\frac{4m^4v^2z^2+1}{8m^3v^2z^2 + 2mv^2z^2(v^2 + z^2 - m^2)} = \frac{4m^4v^2z^2+1}{6m^3v^2z^2+2mv^2z^2(v^2+z^2)} =$$

$\dfrac{4m^4v^2z^2+1}{2mv^2z^2(3m^2 + v^2 + z^2)}$, where m, v, and z may be taken at pleasure.

Take $m=1$, $v=2$, and $z=\frac{1}{2}$, then $y=\frac{10}{29}$, and $x=-\frac{457}{152}$; and hence $u^2=\dfrac{(16141)^2}{(1206)^2}$. Therefore, 4, $\dfrac{1}{4}$, $\dfrac{(10)^2}{(29)^2}$, $\dfrac{(457)^2}{(152)^2}$ and $\dfrac{(16141)^2}{(1206)^2}$ are five squares that will answer. If the process and results of the two preceding cases are attentively considered, there will be no difficulty in extending the solution to as many squares as we please, without any further calculation. For let w^2, u^2, v^2, x^2, y^2 and z^2 denote six squares whose continual product is equal to their sum: then we shall have $y=\dfrac{4m^4u^2v^2z^2+1}{2mu^2v^2z^2(3m^2 + u^2+v^2+z^2)}$;

$x = \dfrac{2my+u^2+v^2+z^2-m^2}{2(y-m)}$ and $w^2=\dfrac{u^2+v^2+x^2+y^2+z^2}{u^2v^2x^2y^2z^2-1}$.

It also appears from what has been done, that all the squares excepting three, may be assumed at pleasure: and the roots of the three other squares are to be found by means of the preceding formulæ.

101. Find two fractions such, that the sum and sum of their squares shall both be rational squares; and either of them being added to the square of the other shall make the same square.

Let the two fractions be denoted by $\dfrac{x}{x+y}$ and $\dfrac{y}{x+y}$, for the sum of these is obviously 1, and is therefore a square number. The sum of their squares is $\dfrac{x^2+y^2}{(x+y)^2}$, which will evidently be a square when $x^2 + y^2$ is a square. Again, either of the fractions being added to the square of the other is $\dfrac{x^2+xy+y^2}{(x+y)^2}$ which will also evidently be a square when $x^2 + xy + y^2$ is a square. The question is therefore reduced to finding such rational values of x and y as will make $x^2 + y^2$ and $x^2 + xy + y^2$ both squares. Put $x = m^2 - n^2$ and $y = 2mn$, then $x^2 + y^2 = (m^2 - n^2)^2 + 4m^2n^2=m^4+2m^2n^2+n^4=(m^2+n^2)^2=$ a square. Also, $x^2+xy+y^2=$

$m^4+2m^3n+2m^2n^2-2mn^3+n^4$ is to be a square. Assume m^2+mn+n^2 for the root of the preceding square; then $m^4+2m^3n+2m^2n^2-2mn^3+n^4=(m^2+mn+n^2)^2=m^4+2m^3n+3m^2n^2+2mn^3+n^4$; whence $\frac{m}{n}=\frac{4}{-1}$. Wherefore we may take $m=5$ and $n=-1$; but these values of m and n would manifestly give the value of y negative. In order therefore to obtain a positive value of y, take $m=4$ and $n=s-1$; then $x=m^2-n^2=15+2s-s^2$, and $y=2mn=8(s-1)$, and from hence $x^2+xy+y^2=169+36s+62s^2-12s^3+s^4$ which is to be a square. Assume $13+6s-s^2$ for its root, that is, put $169+36s+62s^2-12s^3+s^4=(13+6s-s^2)^2=169+156s+10s^2-12s^3+s^4$; whence $s=\frac{30}{13}$, $n=s-1=\frac{17}{13}$, $x=m^2-n^2=\frac{2415}{169}$, $y=2mn=\frac{1768}{169}$, and rejecting the common denominator, we may take $x=2415$, and $y=1768$, and hence the required fractions will be $\frac{x}{x+y}=\frac{2415}{4183}$ and $\frac{y}{x+y}=\frac{1768}{4183}$. See page 412.

102. Divide a given square number n^2, into two such parts that the sum of their squares and the sum of their cubes may both be rational squares.

Let x be one part, then n^2-x will be the other; hence $(n^2-x)^2+x^2$ and $(n^2-x)^3+x^3$, or $n^4-2n^2x+2x^2$ and $n^2(n^4-3n^2x+3x^2)$ must be squares. But it is evident that the latter would be a square if $n^4-3n^2x+3x^2$ was a square. Let $n^4-3n^2x+3x^2=n^4-2n^2px+p^2x^2$ and $n^4-2n^2x+2x^2=n^4-2n^2qx+q^2x^2$, then $-3n^2+3x=-2n^2p+p^2x$ and $-2n^2+2x=-2n^2q+q^2x$; from these equations I obtain $x=\frac{3n^2-2n^2p}{3-p^2}$ and $x=\frac{2n^2-2n^2q}{2-q^2}$, therefore putting these values of x equal to each other, and dividing by n^2, we have $\frac{3-2p}{3-p^2}=\frac{2-2q}{2-q^2}$. In order to determine p and q in rational numbers, let us assume $3-2p=2-2q$, then $3-p^2=2-q^2$, from the former equation we find $p=\frac{1+2q}{2}$, which substituted in the latter equation gives $2-q^2=3-\frac{1+4q+4q^2}{4}$; whence we have $q=\frac{3}{4}$ and this gives p, or $\frac{1+2q}{2}=\frac{5}{4}$. By substituting these values of p and q in the equation $x=\frac{n^2(3-2p)}{3-p^2}$ or $x=\frac{n^2(2-2q)}{2-p^2}$ I get $x=\frac{8n^2}{23}$, hence $n^2-x=\frac{15}{23}n^2$. Thus it appears that the numbers sought are $\frac{8n^2}{23}$ and $\frac{15n^2}{23}$. Let $n=10$, then $x=\frac{800}{23}$ and $n^2-x=\frac{1500}{23}$, hence $(n^2-x)^2+x^2=\frac{2890000}{529}=\left(\frac{1700}{23}\right)^2$ and

$(n^2-x)^3+x^3=\frac{3887000000}{12167}=\frac{169000000}{529}=\left(\frac{13000}{23}\right)^2$. Let $n=2$, then $x=\frac{32}{23}$ and $n^2-x=\frac{60}{23}$, hence $(n^2-x)^2+x^2=\left(\frac{68}{23}\right)^2$ and $(n^2-x)^3+x^3=(\frac{104}{23})^2$. Let $n=\frac{1}{3}$, then $x=\frac{8}{207}$ and $n^2-x=\frac{15}{207}$, hence $(n^2-x)^2+x^2=(\frac{13}{207})^2$ and $(n^2-x)^3+x^3=(\frac{17}{207})^2$. Otherwise,

Let $\frac{n^2x}{x+y}$ and $\frac{n^2y}{x+y}$ denote the two parts, then by the question $\frac{n^4x^2+n^4y^2}{(x+y)^2}=\frac{n^4}{(x+y)^2}\times(x^2+y^2)=$ a square ; therefore $x^2+y^2=$ a square. Again, by the question $\frac{n^6x^3+n^6y^3}{(x+y)^3}=\frac{n^6(x+y)(x^2-xy+y^2)}{(x+y)^3}=\frac{n^6}{(x+y)^2}\times(x^2-xy+y^2)=$ a square ; therefore $x^2-xy+y^2=$ a square. Wherefore the question is reduced to the finding of such rational values of x and y as shall make x^2+y^2 and x^2-xy+y^2 both rational squares. Put $x=r^2-s^2$ and $y=2rs$, then $x^2+y^2=(r^2-s^2)^2+4r^2s^2=(r^2+s^2)^2$, which being a square there only remains to make $x^2-xy+y^2=(r^2+s^2)^2-2rs(r^2-s^2)=r^4-2r^3s+2r^2s^2+2rs^3+s^4$ a square ; assume r^2-rs+s^2 for its root, that is, put $r^4-2r^3s+2r^2s^2+2rs^3+s^4=(r^2-rs+s^2)^2=r^4-2r^3s+3r^2s^2-2rs^3+s^4$; whence $r=4s$, $x=r^2-s^2=15s^2$, $y=2rs=8s^2$, $\frac{n^2x}{x+y}=\frac{15n^2}{23}$ and $\frac{n^2y}{x+y}=\frac{8n^2}{23}$ two parts that will answer. Other numbers that will answer may be found as follows ; put $r=4s-v$, then $r^2+s^2=v^2-8sv+17s^2$, $r^2-s^2=v^2-8sv+15s^2$, $2rs=2s(4s-v)$; whence the general expression $(r^2+s^2)^2-2rs(r^2-s^2)$, to be made a square, becomes $v^4-14sv^3+74s^2v^2-178s^3v+169s^4$. Take $v^2+\frac{89}{13}sv-13s^2$ for its root, that is, put $v^4-14sv^3+74s^2v^2-178s^3v+169s^4=(v^2+\frac{89}{13}sv-13s^2)^2=v^4+\frac{178}{13}sv^3+\frac{89^2}{169}s^2v^2-26s^2v^2-178s^3v+169s^4$; whence $v=\frac{2993s}{1560}$, $r=4s-v=\frac{3243s}{1560}$; $\frac{n^2x}{x+y}=\frac{8083449n^2}{18201609}$ and $\frac{n^2y}{x+y}=\frac{10118160n^2}{18201609}$, other two parts that will answer. Otherwise,

Let x and y represent the parts required, then x^2+y^2 and x^3+y^3 must be squares, or since $x^3+y^3=(x+y)(x^2+y^2-xy)=n^2(x^2+y^2-xy)$, we have only to make x^2+y^2 and x^2+y^2-xy into squares. Now $x=n^2+y$, and this value being substituted for x we have $x^2+y^2=n^4-2n^2y+2y^2$, and $x^2+y^2-xy=n^4-3n^2y+3y^2$.

To make the first a square, assume the root $=n^2-\frac{r}{s}y$, then

$n^4 - 2n^2y + 2y^2 = n^4 - \frac{2r}{s}n^2y + \frac{r^2}{s^2}y^2$, and by reduction $(r^2 - 2s^2)y = 2(rs - s^2)n^2$; therefore, to make $n^4 - 2n^2y + y^2$ a square, we may take $n^2 = r^2 - 2s^2$, and $y = 2rs - 2s^2$.

Again, to make the second expression a square, assume the root $= n^2 - \frac{e}{tw}y$, then by a similar process $\left(\frac{e^2 - 3t^2w^2}{w}\right)y = (2et - 3t^2w)n^2$, therefore $n^4 \times 3n^2y + 3y^2$ will be a square if $n^2 = \frac{e^2 - 3t^2w^2}{w}$ and $y = 2et - 3t^2w$. We have, therefore, by equating the values of n^2 and y, $w(r^2 - 2s^2) = e^2 - 3t^2w^2$ and $2rs - 2s^2 = 2et - 3t^2w$, to find the relation of r and s. To do this, suppose $s = t$, then the equations become $w(r^2 - 2s^2) = e^2 - 3s^2w^2$ and $2r - 2s = 2e - 3sw$.

From the first, $e^2 = w(r^2 - 2s^2) + 3s^2w^2$, and from the second, $4e^2 = 4r^2 + 4(3w - 2)rs + (3w - 2)^2s^2$; therefore $4w(r^2 - 2s^2) + 12s^2w^2 = 4r^2 + 4(3w - 2)rs + (3w - 2)^2s^2$; and making $4wr^2 = 4r^2$, in order that the equation may be divisible by s, we have $w = 1$; whence $-8s + 12s = 4r + s$ and $4r = 3s$; therefore we may take $r = 3$ and $s = 4$; then $y = \frac{2(rs - s^2)n^2}{r^2 - 2s^2} = \frac{8}{23}n^2$ and $x = n^2 - y = \frac{15}{23}n^2$ and these parts will fulfil all the conditions of the question; for their sum is $= n^2$, the sum of their squares $= (\frac{17}{23}n^2)^2$ and the sum of their cubes $= \left(\frac{13n^3}{23}\right)^2 \times n^2$.

103. To find three right-angled triangles, such, that their perimeters may be equal to each other, and their areas in arithmetical progression.

Let the sides of the first triangle be denoted by $2amn$, $a \times (m^2 - n^2)$ and $a \times (m^2 + n^2)$; the sides of the second by $2bpq$, $b \times (p^2 - q^2)$ and $b \times (p^2 + q^2)$; and the sides of the third by $2crs$, $c \times (r^2 - s^2)$ and $c \times (r^2 + s^2)$: then, by the question, the perimeters of the triangles being all equal, there will be had, $a \times (m^2 + mn) = b \times (p^2 + pq) = c \times (r^2 + rs)$. It is well known, that the area of any plane triangle is equal to that of a rectangle under half the perimeter, and the radius of its inscribed circle; and the perimeters of the three triangles being equal, their areas will be as the radii of their inscribed circles; and when these radii are in arithmetical progression, their areas will be so likewise.

Now the radius of the inscribed circle of any right-angled triangle is equal to half the difference between the hypothenuse and the sum of the two legs; wherefore $an \times (m - n)$, $bq \times (p - q)$, and $cs \times (r - s)$ are the respective radii of the inscribed circles of the three preceding triangles; and when these radii are in arithmeti-

cal progression, supposing the second the mean, we shall have $an\times(m-n)+cs\times(r-s)=2bq\times(p-q)$; and from this expression, exterminating b and c, by means of the equation $a\times(m^2+mn)=b\times(p^2+pq)=c\times(r^2+rs)$, it will become $an\times(m-n)+a\frac{(m^2+mn)}{r^2+rs}\times s(r-s)=2a\times\frac{(m^2+mn)}{p^2+pq}\times q\times(p-q)$; $\therefore\frac{m^2+mn}{n\times(m-n)}=\frac{(r^2+rs)\times(p^2+pq)}{(r^2+rs)\times 2q(p-q)-(rs-s^2)\times(p^2+pq)}=e$. Putting $e=\frac{m^2+mn}{n(m-n)}$; and the equation $\frac{(r^2+rs)\times(p^2+pq)}{(r^2+rs)\times 2q\times(p-q)-(rs-s^2)\times(p^2+pq)}=e$, by reduction, becomes $p^2+pq\times\frac{(1-2e)r^2+rs\times(1-e)-es^2}{r^2+rs\times(1+e)-es^2}=-\frac{2eq^2(r^2+rs)}{r^2+rs(1+e)-es^2}$.

Completing the square on the first side of the equation,

$$p^2+pq\times\frac{(1-2e)r^2+rs\times(1-e)-es^2}{r^2+rs\times(1+e)-es^2}+q^2\times\frac{\{(1-2e)r^2+rs\times(1-e)-es^2\}^2}{4\times\{r^2+rs\times(1+e)-es^2\}^2}=$$

$$q^2\times\frac{\left\{\begin{array}{l}\{(1-2e)r^2+rs\times(1-e)-es^2\}^2\\-8e\times(r^2+rs)\times\{r^2+rs\times(1+e)-es^2\}\end{array}\right\}}{4\{r^2+rs(1+e)-es^2\}^2}=\square;\therefore$$

$\left\{\begin{array}{l}\{(1-2e)r^2+rs(1-e)-es^2\}^2\\-8e\times(r^2+rs)\times\{r^2+rs\times(1+e)-es^2\}\end{array}\right\}$ $=r^4\times(1-12e+4e^2)+2r^3s\times(1-11e-2e^2)+r^2s^2\times(1-12e+5e^2)-2ers^3\times(1-5e)+e^2s^4=$ a square. Assume $r^2\times(1+10e)+re\times(1-5e)-es^2$ for the root; that is, put $r^4\times(1-12e+4e^2)+2r^3s\times(1-11e-2e^2)+r^2s^2\times(1-12e+5e^2)-2ers^3\times(1-5e)+e^2s^4=\{r^2(1+10e)+rs\times(1-5e)-es^2\}^2=r^4\times(1+20e+100e^2)+2r^3s\times(1+5e-50e^2)+r^2s^2\times(1-12e+5e^2)-2ers^3\times(1-5e)+e^2s^4$, whence $r^4\times(1-12e+4e^2)+2r^3s\times(1-11e-2e^2)=r^4\times(1+20e+100e^2)+2r^3s(1+5e-50e^2)$; from whence, after proper reduction, $\frac{r}{s}=\frac{3e-1}{3e+1}$; wherefore we may take $r=3e-1$, and $s=3e+1$; but r being here less than s, will obviously make one side of the triangle to which it belongs negative, it will therefore be necessary to obtain a value for s that shall be less than r.

We have hitherto proceeded generally with the solution, and the problem might, from what has been deduced, be done generally for any value of $e=\frac{m^2+mn}{n\times(m-n)}$; but some of the result-

ing expressions would be rather complex, yet still quite manageable. However, for the sake of abridging the labor of calculation, we shall, for the remaining part of the solution, proceed in a manner not quite so general; for which purpose, take $m = 2$, and $n=1$; then $e=\frac{m^2+mn}{n \times (m-n)} = 6$; $r=3e-1=17$, and $s=3e+1=19$; and in order to obtain a value for s that shall be less than r, put $r=17$, and $s = 19 - v$; then writing these values in the before found expression, viz. $p^2+pq \times \frac{(1-2e)r^2+rs(1-e)-es^2}{r^2+rs\times(1+e)-es} = -\frac{2eq^2\times(r^2+rs)}{r^2+rs(1+e)-es^2}$, it becomes $p^2+pq\times\frac{(6v^2-313v+6960)}{6v^2-109v-384} = -\frac{12q^2\times(17v-612)}{6v^2-103v-384}$. Completing the square, $p^2+\frac{pq\times(6v^2-313v+6960}{(6v^2-109v-384)}$

$$+q^2 \times \frac{(6v^2-313v+6960)^2}{4\times(6v^2-109v-384)^2} = \frac{q^2\times(6v^2-313v+6960)^2}{4\times(6v^2-109v-384)^2}$$

$$-\frac{48q^2\times(17v-612)\times(6v^2-109v-384)}{4\times(6v^2-109v-384)^2} =$$

$$q^2\times\frac{(36v^4-8652v^3+446689v^2-7245600v+37161216)}{4\times(6v^2-109v-384)^2} =$$

a square; and therefore $36v^4-8652v^3+446689v^2-7245600v+37161216$ must be a square. Assume $6v^2-721v+6096$ for its root, that is, put $36v^4-8652v^3+446689v^2-7245600v+37161216 = (6v^2-721v+6096)^2 = 36v^4-8652v^3+592993v^2-8790432v+37161216$; whence $v=\frac{1544832}{146304}=\frac{1341}{127}$; and therefore $s=19-v=19-\frac{1341}{127}=\frac{1072}{127}$, and $r=17=\frac{2159}{127}$; wherefore we may take $r=2159$, and $s=1072$. Recollecting now that I took $6v^2-721v+6096$ for the root of $36v^4-8652v^3+446689v^2-7245600v+37161216$, and taking the square root of each side of the expression $p^2+pq\times\frac{(6v^2-313v+6960)}{6v^2-109v-384}+q^2\times\frac{(6v^2-313v+6960)^2}{4\cdot(6v^2-109v-384)^2} = q^2\times\frac{(36v^4-8652v^3+446689v^2-7245600v+37161216)}{4\times(6v^2-109v-384)^2}$ we shall have $p-q\times\frac{(6v^2-313v+6960)}{2\times(6v^2-109v-384} = q\times\frac{(6v^2-721v+6096)}{2\times(6v^2-109v-384)}$;

and hence $\frac{p}{q}=\frac{204v+432}{384+109v-6v^2}=\frac{41710356}{13967314}=\frac{4572}{1531}$; by writing for v its value found above, and reducing the fraction to its lowest terms. Therefore we may take $p=4572$ and $q=1531$; and we found $r=2159$, and $s=1072$; also we took $m=2$ and $n=1$; whence, and from the equality of the perimeters, viz. $a\times(m^2+mn)$

$=b\times(p^2+pq)=c\times(r^2+rs)$, that is, $6a=27902916b=6975729c$; whence $a=4650486b$, and $c=4b$. Hence $2mna=4a=18601944b$, $(m^2-n^2)\times a=3a=13951458b$, and $(m^2+n^2)\times a=5a=23252430b$, which are the two legs and hypothenuse of the first triangle; and $2pqb=13999464b$, $(p^2-q^2)\times b=18559223b$, and $(p^2+q^2)\times b=23247145b$, which are the two legs and hypothenuse of the second or mean triangle. Also, $2rsc=18515584b$, $(r^2-s^2)\times c=14048388b$, and $(r^2+s^2)\times c=23241860b$, are the two legs and hypothenuse of the third triangle. Or, omitting the common factor b, 18601944, 13951458, and 23252430, are the sides of the first triangle; 13999464, 18559223, and 23247145, are the sides of the mean triangle; and 18515584, 14048388, and 23241860, are the sides of the other triangle. For it will appear, that all these triangles have the same perimeter, viz. 55805832; and the radii of their inscribed circles are 4650486, 4655771, and 4661056, which are in arithmetical progression. I shall now give a more general solution to the question; and for this purpose put $r=3e-1$ and $s=3e+1-v$; and these being written in the expression

$$p^2+pq\times\frac{(1-2e)r^2+rs(1-e)-es^2}{r^2+rs(1+e)-es^2}=-\frac{2eq^2(r^2+rs)}{r^2+rs(1+e)-es^2},$$ before deduced, it becomes

$$p^2+pq\times\frac{ev^2-v(9e^2-2e+1)+4e(9e^2-6e+2)}{ev^2-v(3e^2+1)-4e(3e-2)}$$

$$=-\frac{2eq^2\{v(3e-1)-6e(3e-1)\}}{ev^2-v(3e^2+1)-4e(3e-2)};$$ completing the square,

(a); $$p^2+pq\times\frac{ev^2-v(9e^2-2e+1)+4e(9e^2-6e+2)}{ev^2-v(3e^2+1)-4e(3e-2)}$$

$$+\frac{q^2\{ev^2-v(9e^2-2e+1)+4e(9e^2-6e+2)\}^2}{4\{ev^2-v(3e^2+1)-4e(3e-2)\}^2}$$

$$=\frac{q^2\{ev^2-v(9e^2-2e+1)+4e(9e^2-6e-2)\}^2}{4\{ev^2-v(3e^2+1)-4e(3e-2)\}^2}$$

$$-\frac{2eq^2\{v(3e-1)-6e(3e-1)\}}{ev^2-v(3e^2+1)-4e(3e-2)}=\frac{q^2}{4\{ev^2-v(3e^2+1)-4e(3e-2)\}^2}$$

$$\times\left\{\begin{matrix}e^2v^4-2ev^3(21e^2-6e+1)+v^2(369e^4-156e^3+62e^2-12e+1)\\-8ev(135e^4-126e^3+93e^2-24e+2)+16e^2(9e^2-12e+2)^2\end{matrix}\right\}$$

$=$ a square, and therefore

$e^2v^4-2ev^3(21e^2-6e+1)+v^2(369e^4-156e^3+62e^2-12e+1)-8ev(135e^4-126e^3+93e^2-24e+2)+16e^2(9e^2-12e+2)^2$, must be a $\square$. Let $ev^2-v(21e^2-6e+1)+4e(9e^2-12e+2)$ for the root of the last expression, that is, put $e^2v^4-2ev^3(21e^3-6e+1)+v^2(269e^4-156e^3+62e^2-12e+1)-8ev^3(135e^4-126e^3+93e^2-24e+2)+16e^2(9e^2-12e+2)^2=\{ev^2-v(21e^2-6e+1)+4e(9e^2-12e+2)\}^2=$

$e^2v^4-2ev^3(21e^2-6e+1)+v^2(513e^4-348e^3+94e^2-12e+1)-8ev(189e^4-306e^3-123e^2-24e+2)+16e^2(9e^2-12e+2)^2$, whence, by transposition, $v^2(144e^4-192e^3+32e^2)=8ev(54e^4-180e^3+30e^2)$, which gives $v=\frac{3e(9e^2-30e+5)}{9e^3-12e+2}$, and hence $r=3e-1=(3e-1)\times\frac{9e^2-12e+2}{9e^2-12e+2}=\frac{27e^3-45e^2+18e-2}{9e^2-12e+2}$, and $s=3e+1-v=3e+1-\frac{3e(9e^2-30e+5)}{6e^2-12e+2}=\frac{63e^2-21e+2}{9e^2-12e+2}$, and therefore we may take $r=27e^3-45e^2+18e-2$, and $s=63e^2-21e+2$; or r and s may be expounded by any integer numbers, in the ratio of $27e^3-45e^2+18e-2$, to $63e^2-21e+2$, that is, we may put $\frac{r}{s}=\frac{27e^3-45e^2+18e-2}{63e^2-21e+2}$. Again, taking the square root of each side of the expres (a), and recollecting what was taken for the root fo the irrational part, $p+q\times\frac{ev^2-v(9e^2-2e+1)+4e(9e^2-6e+2)}{2\{ev^2-v(3e^2+1)-4e(3e-2)\}}=q\times\frac{ev^2-v(21e^2-6e+1)+4e(9e^2-12e+2)}{2\{ev^2-v(3e^2+1)-4e(3e-2)\}}$ and by transposition, &c. $\frac{p}{q}=\frac{v(6e^2-2e)+12e^2}{4e(3e-2)+v(3e^2+1)-ev^2}$; and in this expression for v, writing its value found above $\frac{p}{q}=\frac{v(6e^2-2e)+12e^2}{4e(3e-2)+v(3e^2+1)-ev^2}$

$$=\left\{\frac{\frac{3e(9e^2-30e+5)(6e^2-2e)}{9e^2-12e^2+2}+12e^2}{4e(3e-2)+\frac{3e(3e^2+1)(9e^2-30e+5)}{9e^2-12e+2}-\frac{9e^3(9e^2-30e+5)^2}{(9e^2-12e+2)^2}}\right\}$$

$$=\frac{6e(9e^2-12e+2)(27e^3-81e^2+21e-1)}{\left\{\begin{array}{c}4(3e-2)(9e^2-12e+2)^2+3(3e^2+1)(9e^2-12e+2)9e^2-30e+7)\\-9e^2(9e^2-30e^2+5)\end{array}\right\}}.$$

and the denominator of this fraction being properly expanded by multiplication, &c. will be found to be divisible by $27e^3-81e^2+21e-1$, which is a factor of the numerator, by which means it finally reduces to

$\frac{3e(9e^2-12e+2)}{45e^2-15e+1}$; that is, $\frac{p}{q}=\frac{3e(9e^2-12e+2)}{45e^2-15e+1}$, or p and q may be expounded by any two numbers in the ratio of $3e(9e^2-12e+2)$ to $45e^2-15e+1$; and we have put $e=\frac{m(m+n)}{n(m-n)}$, where m and n may be taken at pleasure.

Example. Take $m=2$ and $n=1$; then $e=\frac{m(m+n)}{n(m-n)}=6$; and from what has been deduced, $\frac{r}{s}=\frac{27e^3-45e^2+18e-2}{63e^2-21e+2}=\frac{4318}{2144}=\frac{2159}{1072}$; therefore we may take $r=2159$, and $s=1072$. Also $\frac{p}{q}=\frac{3e(9e^2-12e+2)}{45e^2-15e+1}=\frac{4572}{1531}$; therefore we may take $p=4572$ and $q=1531$. The preceding values of the quantities r, s; p, q, obtained from the conclusions deduced in the general solution, being precisely the same as those obtained in the particular solution, prove the accuracy of the various calculations.

104. Find a right angled triangle, such that the area increased by a given number, (5) may be a square.

Let $(x^2+y^2)n$, $(x^2-y^2)n$, $2nxy$ be the sides of the triangle, then the area will be $(x^2-y^2).n^2xy$, and $(x^2-y^2).n^2xy=$ a square; to effect this take $x=5$, $y=4$, then $x^2-y^2=9$, and $(x^2-y^2)n^2xy+5=20\times 9n^2+5=$ a square, which is evidently the case if $9n^2=1$, or $x=\frac{1}{3}$, hence the sides of the triangle are $\frac{40}{3}$, $\frac{41}{3}$, $\frac{9}{3}$.

105. To find a right-angled triangle such, that its perimeter shall be a cube, and the perimeter together with the area a square.

Put $a=$ the perimeter, and $x=$ the area of the triangle; then, by the question, a must be a cube, and $a+x$ a square number. Let the sides about the right angle be denoted by y and z; then, since the rectangle of those sides is equal to double the area, yz is $=$ to $2x$, or $y=\frac{2x}{z}$, and the hypothenuse of the triangle is expressed by $a-\frac{2x}{z}-z$. But the square of the hypothenuse is equal to the sum of the squares of the two sides; $\therefore$ $\left(a-\frac{2x}{z}-z\right)^2=\frac{4x^2}{z^2}-z^2$, or $a^2-\frac{4ax}{z}-2az=-4ax$. Multiply by z, and divide by $2a$; then $z^2-\frac{a^2+4x}{2a}z=-2x$; and if $\left(\frac{a^2+4x}{4a}\right)^2$ be added to both sides of the equation, in order that the first side may be a complete square, we shall have $\left(z-\frac{a^2+4x}{4a}\right)^2=\frac{a^4-24a^2x+16x^2}{16a^2}$. Now it is evident from this equation, that z will be a rational number when $a^4-24a^2x+16x^2$ is a square; therefore all the conditions of the question will be satisfied if $a^4-24a^2x+16x^2$, and $a+x$ be squares, a being a cube number taken at pleasure. Assume $a=64$, which is both a square and a cube number; then a^3 will be a square; and since $a+x$ is

to be a □, the product, $a^3(a+x)=a^4+a^3x$, must also be a □.

Assume $a^4-24a^2x+16x^2=m^2$, and $a^4+a^3x=n^2$, and subtract the second equation from the first; then we have $16x^2-(24a^2+a^3)x=m^2-n^2=(m+n)(m-n)$, or $16x^2-88a^2x=(m+n)(m-n)$.

Resolve the expression $16x^2-88a^2x$ into two factors, $44x$, and $\frac{4}{11x}-2a^2$, so that $2a^2$ or twice the root of a^4, may be a term of one of the factors; then we have $44x\left(\frac{4}{11x}-2a^2\right)=(m+n)(m-n)$. Equate the respective factors by making $44x=m+n$, and $\frac{4}{11x}-2a^2=m-n$; then taking half the sum and half the difference, I have $m=\frac{244}{11}x-a^2$, and $n=\frac{240}{11}x+a^2$. These values being substituted for m and n, we have two equations, from either of which the value of x may be deduced. Thus, if $a^4+a^3x=n^2=\left(\frac{240}{11}x+a^2\right)$, then, by reduction,

$$x=\frac{11(11a-480)a^2}{240^2}=\frac{39424}{225}, \text{and } m=\frac{244x}{11}-a^2=-\frac{8096\times64}{225\times11}.$$

Also, since $\left(z-\frac{a^2+4x}{4a}\right)^2=\frac{m^2}{16a^2}$; z is $=\frac{a^2+4x}{4a}+\frac{m}{4a}=16+\frac{39424}{225\times64}-\frac{8096}{225\times11\times4}=\frac{4032}{225}$, $y=\frac{2x}{2}=\frac{4400}{225}$ and the hypothenuse $=a-y-z=64-\frac{4032}{225}-\frac{4400}{225}=\frac{5968}{225}$. If the numerators and denominators of these fractions be multiplied by 11, we have $\frac{44352}{2475}$, $\frac{48400}{2475}$, and $\frac{65648}{2475}$, for the three sides of the triangle.

106. Find two different isosceles triangles such, that their areas and perimeters shall be both equal.

Let x be one of the equal sides, y the base, and z the perpendicular of the first triangle; and x' one of the equal sides, y' the base, and z' the perpendicular of the second triangle; then $2x+y$ is the perimeter, and $\frac{yz}{2}$ the area of the first; also $2x'+y'$ the perimeter, and $\frac{y'z'}{2}$ the area of the second; and, by the question, $2x+y=2x'+y'$, and $\frac{yz}{2}=\frac{y'z'}{2}$. Now, in order that the sides of the triangle may be expressed by rational numbers, put $x=r^2+s^2$, and $\frac{1}{2}y=r^2-s^2$; then will $z=2rs$.

Also, put $x'=r^2+p^2$, and $\frac{1}{2}y'=r^2-p^2$; then will $z'=2rp$, and the perimeter of each triangle will be $4r^2$.

Moreover, $\frac{1}{2}yz=2rs(r^2-p^2)$, and $\frac{1}{2}y'z'=2rp(r^2-p^2)$; whence

$2rs(r^2-s^2)=2rp(r^2-p^2)$, or $s(r^2-s^2)=p(r^2-p^2)$, and $r^2=\frac{s^3-p^3}{s-p}=s^2+sp+p^2$; therefore s^2+sp+p^2 must be a square number, which it will be when s is $=m^2-n^2$, and $p=n^2+2mn$, for then r is $=m^2+mn+n^2$; hence we have

$$x=r^2+s^2=(m^2+mn+n^2)^2+(m^2-n^2)^2$$
$$y=r^2-s^2=2(m^2+mn+n^2)^2-2(m^2-n^2)^2$$
$$x'=r^2+p^2=(m^2+mn+n^2)^2+(n^2+2mn)^2$$
$$y'=r^2-p^2=2(m^2+mn+n^2)^2-2(n^2+2mn)^2,$$

where m and n may be taken at pleasure.

If $m=2$, and $n=1$, then x is $=58$, $y=80$, $x'=74$, and $y'=48$; or, as these numbers are all divisible by 2, we may take $x=29$, $y=40$, $x'=37$, and $y'=24$.

107. Find any number of numbers, *ad libitum*, such, that if their sum be either added to, or subtracted from the square of each particular number, the several results shall all be rational squares.

Let a, b, h; a', b', h'; a'', b'', h'', &c. denote the two sides and hypothenuse (all rational) of any number of right-angled triangles, having the same area; that is, $ab=a'b'=a''b''=a'''b'''$ &c. then we shall obviously have $a^2+b^2\pm 2ab=h^2\pm 2ab=$ square,
$a'^2+b'^2\pm 2a'b'=h'^2\pm 2a'b'=h'^2\pm 2ab=$ a square,
$a''^2+b''^2\pm 2a''b''=h''^2\pm 2a''b''=h''^2\pm 2ab=$ a square,
$a'''^2+b'''^2\pm 2a'''b'''=h'''^2\pm 2a'''b'''=h'''^2\pm 2ab=$ a square, &c.
and the same properties must necessarily still prevail, if each of the quantities, a, b, h; a', b', h', &c. is multiplied by any common factor, as g, that is, $g^2h^2\pm 2g^2ab$; $g^2h'^2\pm 2g^2ab$; $g^2h''^2\pm 2g^2ab$; $g^2h'''^2\pm 2g^2ab$; &c. must all be rational squares.

A method of finding the sides of any number of right-angled triangles, having the same area, in rational numbers, may be seen at page 476, question 117; and the hypothenuses of such right-angled triangles may be taken for the quantities, h, h', h'', h''', &c. and four times the areas of each of the triangles will manifestly be equal to $2ab$.

Now in order to apply what has been premised to the solution of the question, let gh, gh', gh'', gh''', &c. denote the required numbers, and take $2abg^2$, for their sum, that is, put $gh+gh'+gh''+gh'''$, &c. $=2abg^2$, whence $g=\frac{h+h'+h''+h'''\text{ \&c.}}{2ab}$.

and hence, the required numbers will be

$$gh=\frac{h(h+h'+h''+h'''\text{ \&c.})}{2ab};\ gh'=\frac{h'(h+h'+h''+h'''\text{ \&c.})}{2ab}$$
$$gh''=\frac{h''(h+h'+h''+h'''\text{ \&c.})}{2ab};\ gh'''=\frac{h'''(h+h'+h''+h'''\text{ \&c.})}{2ab}.$$

At page 476, I have 58, 74 and 113 for the hypothenuses of

three right-angled triangles, four times the area of which, or $2ab = 3360$.

Wherefore we may put $h=58$, $h'=74$, and $h''=113$; and hence $\frac{h(h+h'+h'')}{2ab}=\frac{203}{48}=\frac{406}{96}$; $\frac{h'(+h+h'+h'')}{2ab}=\frac{259}{48}=\frac{518}{96}$ and $\frac{h''(h+h'+h'')}{2ab}=\frac{791}{96}$, which are three numbers that will answer. In the same page we have also 218, 233 and 394 for the hypothenuses of three right-angled triangles, four times the area of each of which or $2ab=43680$; from whence, in the same manner, we shall find $\frac{2834}{672}$, $\frac{3029}{672}$, and $\frac{5122}{672}$ for three other numbers that will answer; that is, if their sums be either added to or subtracted from, the square of each particular number, the several results will all be rational squares.

108. To find four rational numbers, such, that if their sum be either added to, or, subtracted from the square of each particular number, the several results shall all be rational squares.

At page 476, question 115, I have 58, 74, 113 and $\frac{1412881}{1189}$ for the hypothenuses of four right-angled triangles, four times the area of each of which is $2ab = 3360$, wherefore we may put $h = 58$, $h'=74$, $h''=113$, and $h'''=\frac{1412881}{1189}$; whence

$$h+h'+h''+h'''=\frac{1704186}{1189}; \text{ and hence}$$

$$\frac{h(h+h'+h''+h''')}{2ab}=\frac{58}{3360}\times\frac{1704186}{1189}=\frac{98842788}{3995040}$$

$$\frac{h'(h+h'+h''+h''')}{2ab}=\frac{74}{3360}\times\frac{1704186}{1189}=\frac{126109764}{3995040}$$

$$\frac{h''(h+h'+h''+h''')}{2ab}=\frac{113}{3360}\times\frac{1704186}{1189}=\frac{192573018}{3995040}$$

$$\frac{h'''(h+h'+h''+h''')}{2ab}=\frac{1412881}{3360\times1189}\times\frac{1704186}{1189}=\frac{2407812019 66}{4750102560}$$

which four numbers will answer the question.

We may next take $h^{iv}=\frac{2579761}{39997}$, and proceeding as before, I shall obtain five numbers that will answer, but it is manifest from what has been done that they will be very large fractional numbers.

109. Find any number of fractions, ad libitum, such, that if the sum of their cubes, be either added to, or subtracted from, the

square of each particular fraction, the several results shall all be rational squares.

Let gh, gh', gh'', gh''', &c. denote the required fractions ; the quantities h, h', h'', h''', &c. being the same as in the preceding problem. Put $2abg^2$ for the sum of their cubes ; that is, put $g^3h^3+g^3h'^3+g^3h''^3+g^3h'''^3+\&c.=2abg^2$; whence

$$g=\frac{2ab}{h^3+h'^3+h''^3+h'''^3+\&c.};\ \therefore gh=\frac{2abh}{h^3+h'^3+h''^3+h'''^3\&c.}$$

$$gh'=\frac{2abh'}{h^3+h'^3+h''^3+h'''^3\&c.};\ gh''=\frac{2abh''}{h^3+h'^3+h''^3+h'''^3\&c.};$$

and $gh'''=\dfrac{2abh'''}{h^3+h'^3+h''^3+h'''^3+\&c.}$.

Let it be required to find three rational fractions, such, that if the sum of their cubes be either added to, or subtracted from the square of each particular fraction, the several results shall all be rational squares. Take $h=58$, $h'=74$, $h''=113$ and $2ab=3360$, and we shall have $g=\dfrac{2ab}{h^3+h'^3+h''^3}=\dfrac{3360}{2043233}$, $gh=\dfrac{58\times3360}{2043233}$ $=\dfrac{194880}{2043233}$, $gh'=\dfrac{74\times3360}{2043233}=\dfrac{248640}{2043233}$, and $gh''=\dfrac{113\times3360}{2043233}$ $=\dfrac{379680}{2043233}$, which are three fractions that will answer the proposed question ; that is, if the sum of their cubes be either added to, or subtracted from the square of each particular fraction, the several results shall be rational squares. If it were required to find four fractions of the kind, we shall have $h'''=\dfrac{1412881}{1189}$; it is therefore evident that the required fractions would be very large, or consist of a great number of figures. The manner of extending the question to any assigned number of fractions that will answer, will be obvious enough from what has been done in the preceding examples.

110. Find three numbers such that the sum of the first and second, and also the difference of the first and third, shall be squares, the sum of whose roots shall be a square equal to the sum of the three required numbers.

Let $8x^2$, x^2, $7x^2$, denote the numbers, then the sum of the first and second $=9x^2=(3x)^2$, and the difference of the first and third, $=8x^2-7x^2=x^2$, and the roots of these $=4x=16x^2$, which is a square; whence $x=\frac{1}{4}$, hence $\frac{8}{16}$, $\frac{1}{16}$, $\frac{7}{16}$, are the numbers required.

111. Given $\sqrt{(\frac{x+a}{x})}+2\sqrt{(\frac{a}{x+a})}=b^2.\sqrt{(\frac{x}{x+a})}$, to find the values of x. Multiplying the equation by

$\sqrt{(\frac{x+a}{x})}.\frac{x+a}{x}+2\sqrt{(\frac{a}{x})}=b^2$, or $1+\frac{a}{x}+2\sqrt{(\frac{a}{x})}=b^2$, or the square root, is $1+\sqrt{(\frac{a}{x})}=\pm b$; by trans. $\sqrt{(\frac{a}{x})}=\pm b-1$, and squaring both sides, $\frac{a}{x}=(b\mp1)^2$; $\therefore\ x=-\frac{a}{(b\mp 1)^2}$.

112. Given $\frac{\sqrt{(a+x)}}{a}+\frac{\sqrt{(a+x)}}{x}=\frac{\sqrt{x}}{c}$, to find the values of x. The equation by reduction becomes $\frac{a+x}{ax}\cdot\sqrt{(a+x)}=\frac{\sqrt{x}}{c}$, and $\therefore\ (a+x)^{\frac{3}{2}}=\frac{a}{c}\cdot x^{\frac{3}{2}}$; extracting the $(\frac{3}{2})^{\text{th}}$ root, $a+x=\overline{\frac{a}{c}}\Big|^{\frac{3}{2}}\cdot x$; $\therefore$ by transposition, $a=\{\overline{\frac{a}{c}}\Big|^{\frac{3}{2}}-1\}\cdot x$; and $\frac{a}{\left(\frac{a}{c}\right)^{\frac{3}{2}}-1}=x$.

113. Given $\frac{(a+x)^{\frac{1}{n}}}{a}+\frac{(a+x)^{\frac{1}{n}}}{x}=\frac{x^{\frac{1}{n}}}{c}$, to find the value of x.

The equation is $(a+x)^{\frac{1}{n}}\cdot\left(\frac{1}{a}+\frac{1}{x}\right)=\frac{x^{\frac{1}{n}}}{c}$, or $(a+x)^{\frac{1}{n}}\cdot\frac{a+x}{ax}=\frac{x^{\frac{1}{n}}}{c}$, or $(a+x)^{\frac{1}{n}+1}=\frac{a}{c}\cdot x^{\frac{1}{n}+1}$; $\therefore\ \left(\frac{a+x}{x}\right)^{\frac{n+1}{n}}=\frac{a}{c}$; the root, $\frac{a+x}{x}=\overline{\frac{a}{c}}\Big|^{\frac{n}{n+1}}$; or $\frac{a}{x}+1=\overline{\frac{a}{c}}\Big|^{\frac{n}{n+1}}$; by trans. $\frac{a}{x}=\overline{\frac{a}{c}}\Big|^{\frac{n}{n+1}}-1$;

$$\therefore x=\frac{a}{\overline{\frac{a}{c}}\Big|^{\frac{n}{n+1}}-1}.$$

114. Given $\frac{m}{n}\cdot x^{\frac{m}{n}-1}=\frac{r}{s}\cdot x^{\frac{r}{s}-1}$, to find the value of x.

(18. Cor 2.) $\frac{m}{n}\cdot x^{\frac{m}{n}}=\frac{r}{s}\cdot x^{\frac{r}{s}}$; $\therefore\ \frac{x^{\frac{m}{n}}}{x^{\frac{r}{s}}}=\frac{nr}{ms}$, or $x^{\frac{ms-nr}{ns}}=\frac{nr}{ms}$; $\therefore$ extracting the root, $x=\left(\frac{nr}{mn}\right)^{\frac{ns}{ms-nr}}$.

Find a quadrilateral inscribed in a circle, such that its four sides and its area shall be all expressed in whole numbers.

Ans. 2; 11; 18, and 23, area = 180.

115. To find 3 square numbers, and also a number which being added to, or subtracted from these squares, shall make the sums and remainders all square numbers. Let a denote the number, and x^2, x'^2, and x''^2, the three squares; then $x^2 \pm a$, $x'^2 \pm a$, $x''^2 \pm a$, must be squares. Assume
$x^2 = m^2 + n^2$, $a = 2mn$, $x'^2 = m'^2 + n'^2$, $a = 2m'n'$, $x''^2 = m''^2 + n''^2$, $a = 2m''n''$. Again, assume
$m = r^2 - s^2$, $n = 2rs$, $\therefore x = r^2 + s^2$
$m' = r'^2 - s'^2$, $n' = 2r's'$, $\therefore x' = r'^2 + s'^2$
$m'' = r''^2 - s''^2$, $n'' = 2r''s''$, $\therefore x'' = r''^2 + s''^2$. Hence we must $a = 4rs(r^2 - s^2) = 4r's'(r'^2 - s'^2) = 4r''s''(r''^2 - s''^2)$, or $rs(r^2 - s^2) = r's'(r'^2 - s'^2) = r''s''(r''^2 - s'')$: or if $r = r' = r''$, as we may evidently assume, then $rs(r^2 - s^2) = rs'(r^2 - s'^2) = rs''(r^2 - s''^2)$, hence $sr^2 - s^3 = s'r^2 - s'^3$, equating first and second; or $(s - s')r^2 = s^3 - s'^3$, or $r^2 = s^2 + ss' + s'^2$; from this we get $s = -\frac{1}{2}s' \pm \frac{1}{2}\sqrt{(4r^2 - 3s'^2)}$. To make $4r^2 - 3s'^2$ a square, assume $r = f^2 + 3g^2$, $s' = 4fg$, whence $s = -2fg \pm (f^2 - 3g^2)$, if $f = 2$, $g = 1$, then $s = -5$, or -3: $r = 7$, $s' = 8$, and since s has 2 values, we may assign one value to s'' in the third expression $rs''(r^2 - s''^2)$; so we have

$r = 7$, $s = -5$; $r = 7$, $s' = 8$; $r = 7$, $s'' = -3$, and therefore $a = 3360$, $x = 74$, $x' = 113$, $x'' = 58$. Or I have 218; 233; 394, and 43680.

116. Find 3 square numbers such, that the difference between every two of them and a 3d shall be a square number.

Let x, y, and z, denote the three numbers; then, by the question, $x^2 + y^2 - z^2$, $x^2 + z^2 - y^2$, and $y^2 + z^2 - x^2$, are to be squares. Assume $x = r^2 + s^2$, $y = r^2 + rs - s^2$, and $z = r^2 - rs - s^2$; then $x^2 + y^2 - z^2 = (r^2 - s^2 + 2rs)^2$, and $x^2 + z^2 - y^2 = (r^2 - s^2 - 2rs)^2$; so that two of the conditions are satisfied, and it only remains to make the last $y^2 + z^2 - x^2$, or its equal $r^4 - 4r^2s^2 + s^4$, a square. Put $r = (2 + t)s$; then $r^4 - 4r^2s^2 + s^4$ is $= s^4(1 + 16t + 20t^2 + 8t^3 + t^4)$; therefore $1 + 16t + 20t^2 + 8t^3 + t^4$ must be a square. Assume the root $= 1 + 8t + t^2$, so that the first, second, and last terms may go out; then $1 + 16t + 20t^2 + 8t^3 + t^4 = (1 + 8t + t^2)^2$, and, squaring and reducing we find $t = -\frac{22}{4}$. But r is $= (2 + t)s$, or $r = -\frac{14}{4}s$; therefore r may be taken $= 15$, and $s = -4$; then $x = r^2 + s^2 = 241$, $y = r^2 + rs - s^2 = 149$, and $z = r^2 - rs - s^2 = 269$; so that 241^2, 149^2, and 269^2, are the numbers.

117. Find the sides of three right-angled triangles in whole numbers, such, that their areas shall all be equal to each other.

Let $r^2 - s^2$, $2rs$, and $r^2 + s^2$, denote the sides of one of the triangles, and $r^2 - p^2$, $2rp$, and $r^2 + p^2$, the sides of another; then when the areas are equal, $rs(r^2 - s^2) = rp(r^2 - p^2)$, or $s(r^2 - s^2) = p(r^2 - p^2)$; whence $r^2 = \frac{s^3 - p^3}{s - p} = s^2 + sp + p^2$, or $s^2 + sp + \frac{1}{4}p^2 = r^2 - \frac{3}{4}p^2$, that is,

$(s+\frac{1}{2}p)^2=r^2-\frac{3}{4}p^2$; therefore $r^2-\frac{3}{4}p^2$ must be a square. Assume $r^2-\frac{3}{4}p=\left(r-\frac{m}{n}p\right)^2$; then $r^2-\frac{3}{4}p=r^2-\frac{2mpr}{n}+\frac{m^2}{n^2}p^2$, and by reduction, $r=\frac{(4m^2+3n^2)p}{8mn}$; therefore, if we take $p=8mn$, r will be $=4m^2+3n^2$; consequently $r-\frac{m}{n}p=3n^2-4m^2$, and $(s+\frac{1}{2}p)^2=(3n^2-4m^2)^2$. Whence $s+\frac{1}{2}p=\pm(3n^2-4m^2)$, or $s=\pm(3n^2-4m^2)-\frac{1}{2}p=3n^2-4m^2-4mn$, or $4m^2-3n^2-4mn$. Let the latter of these values be denoted by s'; then the sides of the 3d triangle will evidently be denoted by $2rs'$, $r^2-s'^2$ and $r^2+s'^2$), where m and n may be taken at pleasure. If m be taken $=1$, and $n=1$; then $s=3-8=-5$, $s'=4-3-4=-3$, $p=8mn=8$, $r=4m^2+3n^2=7$, and the sides of the first triangle $=r^2-s^2=40$, $2rs=42$, $r^2+s^2=58$, of the second, $r^2-p^2=15$, $2rp=112$, $r^2+p^2=113$; and of the third, $r^2-s'^2=24$, $2rs'=70$, $r^2+s'^2=74$.

118. Find 2 whole numbers such, that if unity be added to each of them, and also to their halves, the sums in both cases shall be squares.

First, to find a number x, such that, unity being added to it and to its half, the sums shall be squares; let $x+1=a^2$, and $\frac{1}{2}x+1=b^2$; then from the first equation $x=a^2-1$, and from the second $x=2b^2-2$; whence $2b^2-1=a^2$. Therefore $2b^2-1$ must be a square, which will evidently be the case when $b=1$. But this value gives $x=-1$, which will not answer. Let $b=r+1$; then, by substitution, $2b^2-1$ becomes $2r^2+4r+1$, which is to be a □. Assume its root $=1-\frac{n}{s}r$; then $2r^2+4r+1=1-\frac{2n}{s}r+\frac{n^2}{s^2}r^2$; and, by reduction, $r=\frac{4s^2+2ns}{n^2-2s^2}$, where n and s may be taken at pleasure. If n be taken $=2$, and $s=1$, then $r=4$, $b=4+1=5$, and $x=2b^2-2=48$, one of the Answers. Again, if n be taken $=3$, and $s=2$, then $r=28$, $b=28+1=29$, and $x=2b^2-2=1680$, the other.

119. Find 3 numbers such, that the sum or difference of any 2 of them shall be square numbers.

Let x, y, and z, denote the three numbers, of which y is the mean; then, by the question, $z-y=a^2$, $z-x=b^2$, and $y-x=c^2$: the third equation is evidently $=$ to the sum of the other two; therefore a^2+c^2 must be $=b^2$. Again, from the first equation, $z-a^2=y$, and from the third $x+c^2=y$: therefore, by multiplication, $zx+c^2z-a^2x-a^2c^2=y^2$. But y^2 is $=xz$ by the question; therefore $c^2z-a^2x=a^2c^2$; and if from this we subtract the second equation $z-x=a^2+c^2$, first multiplied by c^2, and then by a^2, we

get $x=\frac{c^4}{c^2-a^2}$, and $z=\frac{a^4}{c^2-a^2}$; consequently $y=\frac{a^2c^2}{c^2-a^2}$. But, since $a^2+c^2=b^2$, we may take $a=r^2-s^2$, $c=2rs$, and $b=r^2+s^2$; then $x=\frac{(r^2-s^2)^4}{6r^2s^2-r^4-s^4}$, $y=\frac{4r^2s^2(r^2-s^2)^2}{6r^2s^2-r^4-s^4}$, and $z=\frac{16r^4s^4}{6r^2s^2-r^4-s^4}$; or, to have whole numbers, x may be taken $=(r^2-s^2)^4\times(6r^2s^2-r^4-s^4)$, $y=4r^2s^2(r^2-s^2)^2(6r^2s^2-r^4-s^4)$, and $z=16r^4s^4(6r^2s^2-r^4-s^4)$, where r and s may be any numbers taken at pleasure. If $r=2$, and $s=1$, then $x=567$, $y=1008$, and $z=1792$.

120. Find 2 whole numbers such, that their sum shall be a square, and the sum of their squares a biquadrate.

Let x and y denote the numbers; then $x+y$ must be a square, and x^2+y^2 a biquadrate. Put $x=p^2-q^2$, and $y=2pq$; then x^2+y^2 will be $=(p^2+q^2)^2$, and, therefore, to make x^2+y^2 a biquadrate, p^2+q^2, must be a square. Put $p=r^2-s^2$, and $q=2rs$; then p^2+q^2 will be $=(r^2+s^2)^2$, and $x^2+y^2=(r^2+s^2)^4$, a biquadrate. From these suppositions we have $x=r^4-6r^2s^2+s^4$), and $y=4r^3s-4rs^3$; whence $x+y=r^4+4r^3s-6r^2s^2-4rs^3+s^4$, which is to be a square. Assume its root $=r^2-2rs+s^2$, then $r^4+4r^3s-6r^2s^2-4rs^3+s^4=r^4-4r^3s+6r^2s^2-4rs^3+s^4$, $4r^3s-6r^2s^2=-4r^3s+6r^2s^2$; dividing by $2r^2s$, we have $2r-3s=-2r+6s$, or $r=\frac{3s}{2}$; so that, if r be taken $=3$, s must be taken $=2$; x will then be $=-119$, a negative value. To find a value that will answer, put $r=\frac{3}{2}s+t$; substituting this value in the given formula, we have $\frac{1}{16}s^4+\frac{37}{2}s^3t+\frac{51}{2}s^2t^2+10st^3+t^4$, to make into a square; and therefore $s^4+296s^3t+408s^2t^2+160st^3+16t^4$ must be a square. Assume its root $=s^2+148st-4t^2$; then, by squaring and proceeding as before, we get $\frac{s}{t}=\frac{1344}{21488}=\frac{84}{1343}$; then taking $s=84$, and $t=1343$, we have $r=\frac{3}{2}s+t=1469$, and consequently $x=r^4-6r^2s^2+s^4=4565486027761$, and $y=4r^3s-4rs^3=1061652293520$, as required.

121. Find the sum (s) of n terms of the series, 1,2,3,4,5,6, &c.

Here $1+2+3+4+5+6+$&c. . . . . $+n=s$.
And $n+(n-1)+(n-2)+(n-3)+$ . . . $+1=s$.

Therefore, by addition,

$$(n+1)+(n+1)+(n+1)+(n+1)+(n+1)+\text{ \&c.}=2s.$$

Whence, $n(n+1)=2s$, or dividing by 2, $s=\frac{n(n+1)}{2}$

Or if 0 be made the first term of the series, instead of 1, its sum,

by putting $n-1$ for n, will, in that case, become $s=\frac{n(n+1)}{2}$

And the same substitution may be made in any other case, where 0 is taken for the first term of the series, and n for the number of terms.

122. To find the sum (s) of n terms of the series, 1, 3, 5, 7, 9, 11, &c. Here $1+3+5+7+9+$ &c. . . . $+(2n-1)=s$.

And $(2n-1)+(2n-3)+(2n-5)+$ &c. . . . $+1=s$.

Whence, by addition, $2n+2n+2n+2n+$ &c. taken to n terms $=2s$.

Consequently $2n\times n=2s$; or $s=n^2=$ sum required.

123. Required the sum (s) of n terms of the series, $a+(a+d)+(a+2d)+(a+3d)+(a+4d)+$ &c.

Here $a+(a+d)+(a+2d)+(a+3d)+(a+4d)$ . . . . . $+\{a+(n-1)d\}=s$.

And $\{a+(nd-d)\}+\{a+(nd-2d)\}+\{a+(nd-3d)\}+\{a+(nd-4d)\}+$ &c. . . . . $+a=s$.

Therefore, by addition, $\{2a+(nd-d)\}+\{2a+(nd-d)\}+\{2a+(nd-d)\}+$ &c. . . . . $+\{2a+(nd-d)\}=2s$; and consequently $\{(2a+nd-d)\}\times n=2s$, or $s=\{2a+(n-1)d\}\times\frac{n}{2}=$ sum required.

Or the sum may be obtained in a different manner, as follows:

$$a+(a+d)+(a+2d)+(a+3d)+(a+4d)+\&c.$$
$$=\left\{\begin{matrix}(+1+1+1+1+1+\&c.)\times a\\(+0+1+2+3+4+\&c.)\times d\end{matrix}\right\}=s$$

But n terms of $1+1+1+1+1+$&c. $=n$. And n terms of $0+1+2+3+4+$ &c. $+\frac{n(n-1)}{2}$. Whence $s=na+\frac{n(n-1)d}{2}=\{2a+(n-1)d\}\times\frac{n}{2}$, which is the same answer as before.

124. To find the sum (s) of n terms of the series $1, x, x^2, x^3, x^4$, &c. Here $1+x+x^2+x^3+x^4+$ &c. . . . $+x^{n-1}=s$.

And $x+x^2+x^3+x^4+x^5+$ &c. . . . $+x^n=sx$.

Whence, by subtraction, $x^n-1=sx-s$. Or $s=\frac{x^n-1}{x-1}=$ sum.

And if x be a proper fraction, the sum of the series, continued ad infinitum, may be found in the same manner. Thus, putting $1+x+x^2+x^3+x^4+x^5+$ &c. $=s$,

We shall have $x+x^2+x^3+x^4+x^5+$ &c. $=sx$; and, consequently, $-1=sx-s$; or $s-sx=1$. Whence $s=\frac{1}{1-x}=$ sum of an infinite number of terms, as was to be found.

125. Required the sum (s) of the circulating decimal ·999999... continued ad infinitum.

Here $\cdot 999999 \ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \&c.$

$$= 9\left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \&c.\right) = s.$$

Or, $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \&c. = \frac{s}{9}$. $\therefore 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \&c. = \frac{10s}{9}$, and $\therefore 1 = \frac{10s}{9} - \frac{s}{9} = \frac{9s}{9} = s$; whence $s = 1 =$ sum of the series.

126. Required the sum (s) of the series $a^2 + (a+d)^2 + (a+2d)^2 + (a+3d)^2 + (a+4d)^2 + \&c.$ continued to n terms.

Here $a^2 = a^2$; $(a+d)^2 = a^2 + 2\times 1ad + 1d^2$
$(a+2d)^2 = a^2 + 2\times 2ad + 4d^2$; $(a+3d)^2 = a^2 + 2\times 3ad + 9d^2$; and $(a+4d)^2 = a^2 + 2\times 4ad + 16d^2$; whence

$$s = \begin{cases} \text{Sum of } n \text{ terms of } (1+1+1+1+ \&c.)a^2 \\ + \ \ldots \text{ ditto of } (0+1+2+3+ \&c.)2ad \\ + \ \ldots \text{ ditto of } (0+1+4+9+ \&c.)d^2 \end{cases}$$

But n terms of $1+1+1+1+ \&c. = n$. And of $0+1+2+3+4 + \&c. = \frac{n(n-1)}{1.2}$. Also of $0+1+4+9+16+25+\&c = \frac{n(n-1)(2n-1)}{1.2.3}$; $\therefore s = na^2 + n(n-1)ad + \frac{n(n-1)(2n-1)}{1.2.3}d^2 =$ the whole sum of the series to n terms.

127. Required the sum (s) of the series $a^3 + (a+d)^3 + (a+2d)^3 + (a+3d)^3 + (a+4d)^3 + \&c.$ continued to n terms. Here $a^3 = a^3$

$$\begin{aligned} (a+d)^3 &= a^3 + 3\times 1a^2d + 3\times 1ad^2 + 1d^3 \\ (a+2d)^3 &= a^3 + 3\times 2a^2d + 3\times 4ad^2 + 8d^3 \\ (a+3d)^3 &= a^3 + 3\times 3a^2d + 3\times 9ad^2 + 27d^3 \\ (a+4d)^3 &= a^3 + 3\times 4a^2d + 3\times 16ad^2 + 64d^3 \\ (a+5d)^3 &= a^3 + 3\times 5a^2d + 3\times 25ad^2 + 125d^3 \end{aligned}$$

$$\text{Whence } s = \begin{cases} \text{Sum of } n \text{ terms of } (1+1+1+1+\&c.)\ a^3 \\ + \ldots\ldots\ldots \text{ditto of } (0+1+2+3+\&c.)3a^2d \\ + \ldots\ldots\ldots \text{ditto of } (0+1+4+9+\&c.)3a^2d \\ + \ldots\ldots\ldots \text{ditto of } (0+1+8+27+\&c.)\ d^3 \end{cases}$$

But n terms of $1+1+1+1- \&c. = n$.
Ditto ..... of $0+1+2+3+ \&c. = \frac{1}{2}n(n-1)$
Ditto ..... of $0+1+4+9+ \&c. = \frac{n(n-1)(2n-1)}{2.3}$
Ditto .... of $0+1+8+27+ \&c. = \frac{(n-1)^4 + 2(n-1)^3 + (n-1)^2}{4}$

Therefore $s = na^3 + \frac{n(n-1)3a^2d}{2} + \frac{n(n-1)(2n-1)3ad^2}{6} + \frac{(n-1)^4 + 2(n-1)^3 + (n-1)^2}{4}d^3 =$ sum required.

128. Required the sum (s) of n terms of the series $1+3+7+15+31+$ &c. Here the terms of the series are, evidently, equal to 1, $(1+2)$, $(1+2+4)$, $(1+2+4+8)$, &c. or to the successive sums of the geometrical series, 1, 2, 4, 8, 16, &c.

Let, therefore, $a=1$ and $r=2$, and we shall have $a+ar+ar^2+ar^3+ar^4+$ &c.$=1+2+4+8+16+$ &c. But the successive sums of 1, 2, 3, 4, &c. terms of this series are

1. $\frac{ar-a}{r-1}=(r-1)\times\frac{a}{r-1}$. 2. $\frac{ar^2-a}{r-1}=(r^2-1)\times\frac{a}{r-1}$.

3. $\frac{ar^3-a}{r-1}=(r^3-1)\times\frac{a}{r-1}$. 4. $\frac{ar^4-a}{r-1}=(r^4-1)\times\frac{a}{r-1}$.

&c. &c.

Therefore $s=\frac{a}{r-1}\times\left\{\begin{array}{l} n \text{ terms of } r+r^2+r^3+r^4+ \text{ \&c.} \\ -n \text{ terms of } 1+1+1+1+ \text{ \&c.}\end{array}\right.$

But $a+r^2+r^3+r^4+$&c.$=(r^n-1)\times\frac{r}{r-1}$, and $1+1+1+1+1+1+1+$ &c.$=n$. Whence $s=\frac{r(r^n-1)}{r-1}\times\frac{a}{r-1}-n\times\frac{a}{r-1}=2(2^n-1)-n=$ whole sum required.

129. Sum n terms of the series $\frac{1}{1}+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+\frac{31}{16}$ &c. Here, the terms of the series are the successive sums of the geometrical series $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$ &c. Let, therefore $a=1$ and $r=2$; then will $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$ &c. $=a+\frac{a}{r}+\frac{a}{r^2}+\frac{a}{r^3}+\frac{a}{r^4}$

The successive sums of 1, 2, 3, 4, &c. terms of this series are,

1. $\frac{(r-1)\times a}{(r-1)\times 1}=(r-1)\times\frac{a}{r-1}$. 3. $\frac{(r^3-1)\times a}{(r-1)\times r^2}=(r-\frac{1}{r^2})\times\frac{a}{r-1}$.

2. $\frac{(r^2-1)\times a}{(r-1)\times r}=(r-\frac{1}{r})\times\frac{a}{r-1}$. 4. $\frac{(r^4-1)\times a}{(r-1)\times r^3}=(r-\frac{1}{r^3})\times\frac{a}{r-1}$.

&c. &c. Therefore

$s=\frac{a}{r-1}\times\left|\begin{array}{l} n \text{ terms of } r+r+r+r+ \text{ \&c.} \\ -n \text{ terms of } \frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+ \text{ \&c.}\end{array}\right.$

But $r+r+r+r+r+r+$ &c. $= nr$.

And $\frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+$ &c. $=\frac{r^n-1}{(r-1)r^{n-1}}$. Whence $s=\frac{a}{r-1}\times\{nr-\frac{r^n-1}{(r-1)r^{n-1}}\}=\frac{(n-1)2^n+1}{2^{n-1}}=$ sum required.

130. Required the sum of the infinite series of the reciprocals of the triangular numbers $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+$&c. Let $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+$&c. ad infinitum $=$ s.

Or $\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{2.3}+\frac{1}{2.5}+\frac{1}{3.5}+$ &c. .... $=$ s.

Then $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+$ &c..... $=\frac{s}{2}$. That is $\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+$&c. $=\frac{s}{2}$.

Or, $\left.\begin{array}{l}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\text{ \&c.}\\ \quad-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\text{ \&c.}\end{array}\right|=\frac{s}{2}$. Whence $\frac{s}{2}$ $=\frac{1}{1}$; or s $=$ 2 sum required.

131. Find the sum of n terms of the same series, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+$ &c. Let $s=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$&c. .. to $\frac{1}{n}$.

Then $s-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$&c. to . . to $\frac{1}{n}$. And $s-\frac{1}{n}+\frac{1}{n+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$&c. to $\frac{1}{n-1}$. Therefore, by subtracting this from the first, we have $\frac{1}{1}-\frac{1}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+$ &c. to $\frac{1}{n}-\frac{1}{n+1}$. Or $\frac{n}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+$ &c. to $\frac{1}{n(n+)}$. Whence $\frac{2n}{n+1}=\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$&c. to $\frac{2}{n(n+1)}$. Or $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+$&c. to $\frac{2}{n(n+1)}=\frac{2n}{n+1}=$ sum of n terms of the series, as was required.

132. Of the infinite series $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+$ &c. required the sum.

Let s $=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+$&c. ad infinitum.

Then $s-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$&c. by transposition.

And $1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+$ &c. by subtraction.

Or $1-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+$ &c. by transposition.

Whence $\frac{1}{2}=\frac{4}{1.4.3}+\frac{6}{2.9.4}+\frac{8}{3.16.5}+$ &c. by subtraction.

Or $\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}+$ &c.

And $\frac{1}{2}\div2=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}$, by dividing each side by 2.

But $\frac{1}{2}\div2=\frac{1}{4}$; $\therefore\ \frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\frac{1}{5.6.7}$ &c. ad infinitum $=\frac{1}{4}$, which is the sum required.

133. And if it were required to find the sum of n terms of the same series $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$ &c.

Let $s=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$&c. to $\frac{1}{n(n+1)}$.

Then $s-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+$ &c. to $\frac{1}{n(n+1)}$.

And $s-\frac{1}{2}+\frac{1}{(n+1)(n+2)}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+\frac{1}{6.7}+\frac{1}{7.8}$ &c. continued to $\frac{1}{(n+1)(n+2)}$ terms. Therefore $\frac{1}{2}-\frac{1}{(n+1)(n+2)}=\frac{2}{1.2.3}=\frac{2}{2.3.4}+\frac{2}{3.4.5}+$ &c. to n terms, by subtraction. Whence $\frac{1}{4}-\frac{1}{2(n+1)(n+2)}=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+$ &c. to n terms, by division. And, consequently, $\frac{1}{1.2.3}$ $\frac{1}{2.3.4}+\frac{1}{3.4.5}+$ &c. continued to n terms $=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$ $=$ sum required.

134. Required the sum (s) of the series $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+$ &c. continued ad infinitum. Let $x=\frac{1}{2}$; in which case $s=x-x^2+x^3-x^4+$ &c. Then, if this be multiplied by $1+x$, we shall have $s(1+x)=(1+x)\times(x-x^2+x^3-x^4+x^5-$&c.), where, by performing the operation on the right hand member, there will arise $s(1+x)=x$. Therefore, $s=x-x^2+x^3-x^4+x^5-$&c. $=\frac{x}{1+x}$. Or $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-$&c. $=\frac{\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{3}$ $=$ sum required.

135. Required the sum (s) of the series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+$ &c. continued ad infinitum. Let $x=\frac{1}{2}$; in which case $s=x+2x^2+3x^3+4x^4+$ &c.

Then, if this be multiplied by $(1-x)^2$, we shall have $s(1-x)^2=(1-x)^2\times(x+2x^2+3x^3+4x^4+\text{ \&c.})$ Where, by performing the operation at length, on the right hand member, as before, there will arise $s(1-x)^2=x$. Therefore, also, $x+2x^2+3x^3+4x^4+5x^5+$ &c.

$$\frac{x}{(1-x)^2}.\quad \text{Or } \frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\text{\&c.}=\frac{\frac{1}{2}}{(1-\frac{1}{2})^2}=2=\text{sum.}$$

136. Required the sum (s) of the series $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}+\frac{25}{243}+$ &c. continued ad infinitum. Let $x=\frac{1}{3}$; in which case $s=x+4x^2+9x^3+16x^4+$ &c.

Then, if this be multiplied by $(1-x)^3$, we shall have $s(1-x)^3=(1-x)^3\times(x+4x^2+9x^3+16x^4+\text{\&c.})=x+x^2$, as will be found by performing the operation.

$$\text{Therefore } s(1-x)^3=x+x^2,\text{ or } s=\frac{x+x^2}{(1-x)^3}=\frac{x(1+x)}{(1-x)^3}.$$

$$\text{Or } \frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}+\text{\&c.}=\frac{\frac{1}{3}(1+\frac{1}{3})}{(1-\frac{1}{3})^3}=\frac{3}{2}=1\tfrac{1}{2}=\text{sum required.}$$

137. Required the sum (s) of the infinite series

$$\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\frac{a+3d}{mr^3}+\frac{a+4d}{mr^4}+\text{\&c.}$$

Let $x=\frac{1}{r}$; in which case $s=\frac{a}{m}+\frac{a+d}{m}x+\frac{a+2d}{m}x^2+$ &c.

Then, if this be multiplied by $(1-x)^2$, we shall have $s(1-x)^2=(1-x)^2=\left(\frac{a}{m}+\frac{a+d}{m}x=\frac{a+2d}{m}x^2+\frac{1+3d}{m}x^3+\text{ \&c.}\right)$. Or, by actually performing the operation, $s(1-x)^2=\frac{a}{m}-\frac{a-d}{m}x$; and, by by division, $s=\frac{a-(a-d)x}{m(1-x)^2}$.

Whence, restoring the value of x, we shall have, by substitution and reduction, $s=\dfrac{a-(c-d)\frac{1}{r}}{m(1-\frac{1}{r})^2}=\dfrac{ar^2-(a-d)r}{m(r-1)^2}$. And, consequently, $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\frac{a+3d}{mr^3}+\text{\&c.}=\frac{r}{m}\left\{\frac{a(r-1)+d}{(r-1)^2}\right\}$ = sum of the series, as required.

138. Required the sum of n terms of the series

$$\frac{r}{m(m+r)}+\frac{r}{(m+r)(m+2r)}+\frac{r}{(m+2r)(m+3r)}+\frac{r}{(m+3r)(m+4r)}$$

$+$&c. Here, leaving out the numerators, and the last factor of each denominator, let there be assumed

$\frac{1}{m}+\frac{1}{m+r}+\frac{1}{m+2r}+\ldots+\frac{1}{m+(n-1)r}+\frac{1}{m+nr}=s$. Then

$\frac{1}{m+r}+\frac{1}{m+2r}+\frac{1}{m+3r}+$&c. to n terms $+\frac{1}{m+nr}=s-\frac{1}{m}$.

Whence, by subtraction and substitution,

$\frac{r}{m(m+r)}+\frac{r}{(m+r)(m+2r)}+\frac{r}{(m+2r)(m+3r)}+$ &c. to n terms $=\frac{1}{m}-\frac{1}{m+nr}=$ sum required. Also $\frac{1}{m(m+r)}+\frac{1}{(m+r)(m+2r)}+\frac{1}{(m+2r)(m+3r)}+$ &c. to n terms $=\frac{1}{mr}-\frac{1}{mr+nr^2}$. Where, if the number of terms (n) be increased without limit, $\frac{1}{mr+nr^2}$ will vanish, and the sum of the series will be $\frac{1}{mr}$.

Or, which is the same, $=$ the reciprocal of the product of the two quantities m and r. In which case, if m and r be each taken $=1$, we shall have $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+$ &c. to n terms $=1-\frac{1}{1+n}=\frac{n}{1+n}$.

And by taking m and r of different values, the sum of n terms of a variety of other series may be found.

139. Required the sum of the infinite series

$$\frac{2r}{m(m+r)(m+2r)}+\frac{2r}{(m+r)(m+2r)(m+3r)}+$$

$\frac{2r}{(m+2r)(m+3r)(m+4r)}+$ &c. Here, leaving out the numerators, and the last factor of each denominator, as before, let there be assumed

$\frac{1}{m(m+r)}+\frac{1}{(m+r)(m+2r)}+\frac{1}{(m+2r)(m+3r)}+$&c. $=s$. Then

$\frac{1}{(m+r)(m+2r)}+\frac{1}{(m+2r)(m+3r)}+\frac{1}{(m+3r)(m+4r)}=s-\frac{1}{m(m+r)}$. Whence, by subtraction,

$\frac{2r}{m(m+r)(m+2r)}+\frac{2r}{(m+r)(m+2r)(m+3r)}+$&c. $=\frac{1}{m(m+r)}$.

Also $\frac{1}{m(m+r)(m+2r)}+\frac{1}{(m+r)(m+2r)(m+3r)}+\&c.=$ $\frac{1}{2rm(m+2r)}$. In which case, the sum of n terms of the series, determined as in the 139th will be $=$ $\frac{1}{2rm(m+r)}-\frac{1}{2r(m+nr)\{m+(n+1)r\}}$. So that if n and r be each taken $=1$, we shall have $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+$ &c. ad infinitum $=\frac{1}{4}$. And $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+$&c. to n terms, $=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$. And by taking m and r of different values, the sums of various other series of this kind may be found, as in the preceding example.

Questions without Solutions.

140. Sum 100 terms of the series 2, 5, 8, 11, 14, &c. and 50 terms of the series $1+2^2+3^2+4^2+5^2+$&c.

Answers, 15050, and 42925.

141. Required the sum of the infinite series $1+3x+6x^2+10x^3+15x^4+$&c. when x is less than 1.

142. Required the sum of the infinite series $1+4x^2+10^2+20x^3+35x^4+$&c. when x is less than 1.

143. Sum the infinite series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11}+$ &c.

Ans. $\frac{1}{(1-x)^3}$; $\frac{1}{(1-x)^4}$; and $\frac{5}{10}$, or $\frac{1}{2}$.

144. Sum 40 terms of the series $(1\times2)+(2\times3)+(3\times4)+(4\times5)+$ &c.

Ans. 22960.

145. Sum n terms of the series $\frac{2x-1}{2x}+\frac{2x-3}{2x}+\frac{2x-5}{2x}+\frac{2x-7}{2x}$, the infinite series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+$&c. and n terms of the series $1+8x+27x^2+64x^3+125x^4,+$ &c.

Answers, $n\left(\frac{2x-n}{2x}\right)$; $\frac{1}{18}$; and $\frac{1+4n+x^2}{(1-x)^4}$.

146. Sum n term of the series

$\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+\frac{5}{r^5}+\frac{6}{r^6}+$ &c. Ans. $\frac{r}{(r-1)^2}-\frac{1}{r^n}\left\{\frac{nr+r-n}{(r-1)^2}\right\}$

147. Required the sum of n terms of the series

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}\&c.+\frac{1}{n(3+n)}$$

Ans. $\frac{n}{3+3n}+\frac{n}{12+6n}+\frac{n}{27+9n}$.

148. Required the sum of the series $\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}+$&c... $+\frac{1}{2n(4+2n)}$. Ans. $\Sigma=\frac{3}{16}$, $s=\frac{5n+3n^2}{32+48n+16n^2}$.

The symbol Σ, in this and the following series, denotes the sum of an infinite number of terms, and s the sum of n terms.

149. Required sum of the series $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\frac{1}{12.20}$ $+$&c. . . $+\frac{1}{3n(4+4n)}$ Ans. $\Sigma=\frac{1}{12}$, $s=\frac{n}{12+12n}$.

150. Required the sum of the series $\frac{6}{2.7}+\frac{6}{7.12}+\frac{6}{12.17}+$ $\frac{6}{17.22}+$&c. . . $+\frac{5}{(5n-3).(5n+2)}$. Ans. $\Sigma=\frac{3}{5}$, $s=\frac{3n}{2+5n}$.

151. Required sum of the series $\frac{1}{4.8}-\frac{1}{6.10}+\frac{1}{8.12}-\frac{1}{10.14}$ $+$ &c. $\pm\frac{1}{(2+2n)(6+2n)}$. Ans. $\Sigma=\frac{1}{48}$, $s=\frac{n}{16+16n}-\frac{n}{36+24n}$

152. Sum the series $\frac{1}{8.18}+\frac{1}{10.21}+\frac{1}{12.24}+\frac{1}{14.27}+$&c.$+$ $\frac{1}{(6+2n).(15+3n)}$. Ans. $\Sigma=\frac{3}{80}$, $s=\frac{n}{24(8+2n)}+\frac{n}{30(10+2n)}$.

153. Sum the series $\frac{1}{3.6}-\frac{1}{6.8}+\frac{1}{9.10}-\frac{1}{12.12}+\frac{1}{15.14}-$&c. . . . $\pm\frac{1}{3n(4+2n)}$. Ans. $\Sigma=\frac{1}{24}$, $s=\frac{n}{2(3+6n)}-\frac{n}{4(6+6n)}$.

154. Sum n terms of the series $\frac{1}{1.2.4}+\frac{1}{2.3.5}+\frac{1}{3.4.6}+\frac{1}{4.5.7}+$ &c.$+\frac{1}{n(1+n)\cdot(3+n)}$. Ans. $\frac{n}{3(1+n)}-\frac{n}{12(2+n)}-\frac{n}{18(3+n)}$.

155. Required sum of the series, $\frac{2}{3.5}+\frac{3}{5.7}-\frac{4}{7.9}-\frac{5}{9.11}+$ &c. . . $\pm\frac{1+n}{(1+3n).(3+2n)}$. Ans. $\Sigma=\frac{1}{12}$, $s=\frac{1}{12}-\frac{1}{4(3+4n)}$.

156. Sum the series $\frac{1}{3}\left(\frac{2}{1.3}\right)+\frac{1}{9}\left(\frac{3}{3.5}\right)+\frac{1}{27}\left(\frac{4}{5.7}\right)+\frac{1}{81}\left(\frac{5}{7.9}\right)$ $+$ &c.$+\frac{1+n}{3^n(2n-1).(2n+1)}$. Ans. $\Sigma=\frac{1}{4}$, $s=\frac{1}{4}+\frac{1}{4.3^n(1+2n)}$

156. Required, sum of the series $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\frac{8}{4.5.6}$ $+$ &c. $+\frac{4+n}{n(1+n).(2+n)}$. Ans. $\Sigma=\frac{3}{2}$, $s=\frac{3}{2}-\frac{2}{1+n}+\frac{1}{2+n}$.

157. Given $\frac{2x+\sqrt{x}}{2x-\sqrt{x}}=3\frac{7}{15}-.3\frac{2x-\sqrt{x}}{2x+\sqrt{x}}$, to find the values of x by a simple equation. Ans. $x=4$.

158. Find the cube root of $4\sqrt{5}+8$. Ans. $\frac{\sqrt{5}+1}{\sqrt[3]{2}}$.

159. Given $\frac{4x-34}{17}-\frac{258-5x}{3}=\frac{69-x}{2}$; to find the value of x. $x=51$.

160. Given $2x-\frac{4x-2}{13}=\frac{2x+11}{5}\ \frac{7-8x}{7}$; Ans. $x=7$.

161. Let $ab^3x^2+(1+c)bd\sqrt{c}+cb^2x^2=\{b^3d\sqrt{c}+(ab+c)(1+c)\}x$. to find the values of x.

By tr. $b^2(ab+c)x^2-\{b^3d\sqrt{c}+(ab+c)(1+c)\}x=-(1+c)bd\sqrt{c}$.

Let $x=\frac{z}{b^2(ab+c)}$; $z^2-\{b^3d\sqrt{c}+(ab+c)(1+c)\}z=-b^3d(ab+c)(1+c)\sqrt{c}$. Let $m=b^3d\sqrt{c}$, and $n=(ab+c)(1+c)$, then will $z^2-(m+n)z=-mn$, by art. 70, case 4, I have $z=\sqrt{\{\frac{1}{4}(m^2-2mn+n^2)\}}\pm\frac{1}{2}(m+n)$; $b^2(ab+c)x$, or $z=m$, or n;

$\therefore x=\frac{b^3d\sqrt{c}}{b^2(ab+c)}=\frac{bd\sqrt{c}}{ab+c}$; and $x=\frac{(ab+c)(1+c)}{b^2(ab+c)}=\frac{1+c}{b^2}$. A

162. A merchant having mixed a certain number of gallons of brandy and water, found, that if he had mixed 6 gallons more of each, there would have been 7 gallons of brandy to every 6 gallons of water; but if he had mixed 6 gallons less of each, there would have been 6 gallons of brandy to every 5 gallons of water: how many of each did he mix?

Let $7x-6$, and $6x-6$ denote the number of gallons of brandy and water respectively; which assumption fulfils the first condition of the question. By the second condition we have $7x-12 : 6x-12 :: 6 : 5$, or $x : 6x-12 :: 1 : 5$, whence $5x=6x-12$, or $x=12$; and $7x-6=78$ gallons of brandy, $6x-6=66$ gallons of water.

163. Find two numbers, such, that their sum, product, and difference of their squares, shall be all equal.

The sum of two numbers differing by unity, is equal to the difference of their squares. Let, then, $x+1$ and x denote the numbers: then their sum must be equal to their product; that is, $2x+1=x^2+x$, or $x^2-x=1$ by case 2; $4x^2-4x+1=4+1=5$; extracting the root, $2x-1=\pm\sqrt{5}$, $\therefore x=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$; and $x+1=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$, the two numbers required.

164. Find two numbers, such, that their product may be equal

to the difference of their squares, and the sum of their squares equal to the difference of their cubes. If we assume ax and $(a+1)x$ for the numbers, a being $=\frac12+\frac12\sqrt5$, we shall fulfil the first condition of the question. (See Quest 163.) By the second condition, we must have $(2a^2+2a+1)x^2=(3a^2+3a+1)x^3$;

or $x=\frac{2a^2+2a+1}{3a^2+3a+1}=\frac{4a+3}{6a+4}$, (since $a^2=a+1$). Hence, $ax=$ $\frac{4a^2+3a}{6a+4}=\frac{7a+4}{6a+4}=\frac{\frac{15}{2}+\frac72\sqrt5}{7+3\sqrt5}=\frac12\times\frac{\sqrt5(3\sqrt5+7)}{3\sqrt5+7}=\frac12\sqrt5$,

and $(a+1)x=a^2x=ax\times a=\frac12\sqrt5(\frac12+\frac12\sqrt5)=\frac14(5+\sqrt5)$: that is, the numbers are $\frac12\sqrt5$, and $\frac14(5+\sqrt5)$.

165. Given $\frac{1}{11}(6x+8)-\frac12(5x+3)=\frac13(27-4x)-\frac12(3x+9)$,

166. Given $x+\frac{27-9x}{4}-\frac{5x+2}{6}=\frac{61}{12}-\frac{2x+5}{3}-\frac{29+4x}{12}$, to find the value of x. Answers, $x=6$, and $x=5$,

167. Given $\frac{7x-8}{11}+\frac{15x+8}{13}=3x-\frac{31-x}{2}$, to find x. $x=9$.

168. Given $\frac12(5x-1)-\frac{10}{11}(7x-2)=6\frac35-\frac12x$, to find x. $x=3$.

169. Given $\frac{5a+10ab^2}{9b^2-3a^2b^2}x^2-\left(\frac{5\sqrt{(a+b)}}{3b^3}+\frac{(1+2b^2)cd\sqrt c}{3-a^2}\right)x-\frac{cd}{ab}$ $\times\sqrt{(a+b)c}$, to find the values of x.

By tr. $\frac{5a(1+2b^2)}{3b^2(3-a^2)}x^2-\left(\frac{5\sqrt{(a+b)}}{3b^3}+\frac{(1+2b^2)cd\sqrt c}{3-a^2}\right)x=-$ $\frac{cd\sqrt c\sqrt{(a+b)}}{ab}$. Let $m=\frac{1+2b^2}{3-a^2}$; and $n=\frac{5\sqrt{(a+b)}}{3b^3}$, then will $\frac{5am}{3b^2}x^2-\{n+mcd\sqrt c\}x=-\frac{cd\sqrt c\sqrt{(a+b)}}{ab}$; let $x=\frac{3b^2z}{5am}$; then $z^2-(n+mcd\sqrt c)z=-\frac{5cdm\sqrt c\sqrt{(a+b)}}{3b^3}$; $\therefore \frac{5am}{3b^2}=z$, or z^2- $(n+mcd\sqrt c)z=-cdmn\sqrt c$. By case 4, art. 70, I shall have $\{z-\frac12(n+mcd\sqrt c)\}^2=\frac14(n+mcd\sqrt c)^2-cdmn\sqrt c$; or $\{z-\frac12(n+mcd\sqrt c)\}^2=\frac14n^2-\frac12mncd\sqrt c+\frac14(m^2c^2d^2c)$; root $=z-$ $\frac12(n+mcd\sqrt c)=\pm\frac12(n-mcd\sqrt c)$. Hence I have

$z=\frac12(n+mcd\sqrt c)\pm\frac12(n-mcd\sqrt c)$; $\frac{5a(1+2b^2)}{3b^2(3-a^2)}x$, or $z=n$, or $mcd\sqrt c$, $\therefore z=\frac{5\sqrt{(a+b)}}{3b^3}$; or $\frac{(1+2b^2)cd\sqrt c}{3-a^2}$; hence

$x=\frac{(3-a^2)\sqrt{(a+b)}}{ab(1+2b^2)}$; or $\frac{3b^2cd\sqrt c}{5a}$, Answer.

170. Find the cube root of $26+15\sqrt3$. Ans. $2+\sqrt3$.

171. Two mixtures are made of brandy and sherry; the quantities of brandy in each being as 4 to 3; and the difference of the quantities of sherry being greater by twenty-five gallons than the difference of the quantities of brandy. Also, if three times the quantity of brandy had been put into the first mixture, and twice the quantity into the second, the quantities of brandy would have been proportional to the quantities of sherry. But if the sherry in the second mixure had been mixed with the brandy in the first, and the sherry in the first with the brandy in the second, the whole mixtures would then have been in the ratio of 5 to 6. Required the quantities of brandy and sherry in each mixture.

Let $4x$ and $3x$ be the quantities of brandy, $\therefore$ $(12x : 6x : :)2 : 1$, the ratio of the quantities of sherry. Let $\therefore$ $2y$ and y be the quantity of sherry, $\therefore$ $y=x+25$. Also, $4x+y : 3x+2y : : 5 : 6$, and $24x+6y=15x+10y$, $\therefore$ $9x=4y=4x+100$, and $x=\frac{100}{5}=20$, therefore the quantities of brandy are 80 and 60, and that of sherry 90 and 45, Ans.

172. During a winter, when fuel was scarce, 2 men, A and B, went in quest of coals and turf, which they agreed to use in common. A met with three bushels of coals, and B two, at the same price per bushel, and also seven baskets of turf. A stipulated that he should consume twice as many coals as B. B assented, but demanded of him 2*s*. 10*d*. When this stock was exhausted, B purchased one bushel of coals, and A five, together with six baskets of turf, at the same rates respectively as before; but now B consumed three times as many coals as A, and paid him 18*s*. 6*d*. What was the price of a bushel of coals, and of a basket of turf; equal quantities of turf having been consumed by each person?

Let $x=$ the value of a bushel of coals in pence, and $y=$ the value of a basket of turf. Now in the first case A consumes $\frac{2}{3}$ of coals; $\therefore$ $\frac{2}{3}\cdot 5x=$ the value of coals consumed by him, and $\frac{7y}{2}=$ the value of turf consumed by him $\therefore$ $\frac{10}{13}x+\frac{7}{2}y=3x+34$ In the second case B consumed $\frac{3}{4}$ of coals, $\therefore$ $\frac{3}{4}\cdot 6x=$ the value of coals consumed by him, and $\frac{3}{4}\cdot 6x+\frac{6y}{2}=x+222$, $\therefore$ $7x+6y=444$. Hence $(7x+6y=444)-(7x+\frac{147}{2}y=714)=\frac{135}{2}y=270$, and $y=4$ and $x=60$. Ans. The price of a bushel of coal was 5*s*. and of a basket of turf 4*d*.

173. Two Spanish muleteers, A and B, were seated under a tree in order to dine; and on examining, found their stock of provisions to consist of five small loaves of bread, three of which were A's property, and a bottle of wine, which was B's. A stranger,

who happened to come up at the time, was invited to partake of their fare, which was just sufficient for three persons; and at parting, being pleased with their behavior, he gave them what Spanish money he had about him, which amounted to 6*s.* 5½*d.*, to be equitably shared between them. Now as many shillings as a loaf cost pence would, with four pence more, at the next town have bought six such loaves and four bottles of the same wine; and when the money was divided, B received 1*s.* 10½*d.* more than A. What was the price of each loaf, and a bottle of wine?

Ans. A loaf cost 7 pence, and a bottle of wine 11½ pence.

Let $x =$ the price of a loaf in pence, and $y =$ the price of a bottle of wine, $\therefore 3x =$ the price of A's loaves, and $2x =$ the price of B's, and $12x+4=6x+4y$, $\therefore y=\dfrac{3x+2}{2}$, they all eat equal portions; $\therefore$ each eat $\frac{1}{3}$ of 5 loaves. A, $\therefore$ eat $\frac{5}{3}$ and gave $\frac{4}{3}$ to the stranger; and B, having 2 loaves, gave $\frac{1}{3}$ to the stranger. But B had a bottle of wine, $\frac{1}{3}$ of which he gave to the stranger. Hence $\dfrac{4x-y}{2}$ is the price of the provisions A furnished to the stranger, and $\frac{1}{3}(x+2y) =$ the price of what B furnished. Now A receives $\frac{55}{2}$ pence, and B 50 pence, $\therefore \frac{1}{3}(4x-y) : \frac{1}{3}(x+2y) :: \frac{55}{2} : 50$, or $4x-y : x+2y :: 11 : 20$, $\therefore 3y-3x : x+2y :: 9 : 20$, or $y-x : x+2y :: 3 : 20$, $\therefore 20y-20x=3x+6y$, and $14y=23x$, $\therefore y=\dfrac{23x}{14}$ Hence $\dfrac{23x}{14}=\dfrac{3x+2}{2}$, and $46x=42x+28$, and therefore $x=\frac{28}{4}=7$D, therefore $y=11\frac{1}{2}$D.

174. Find numbers, which are in the proportion of 8 to 5, and whose product is equal to 360. Ans. ± 24, and ± 15.

Let $5x$ and $8x$ be the numbers, $\therefore 40x^2=360$, $x^2=9$, or $x=3$.

175. There are 2 numbers, whose sum is to their difference as 8 to 1, and the difference of whose squares is 128. What are the numbers? Ans. $\pm$ 18, and $\pm$ 14.

Let $8x =$ their sum; $\therefore x =$ their difference, whence $\frac{9}{2}x =$ the greater, and $\frac{7}{2}x =$ the less, $\therefore \frac{1}{4}(81x^2-49x^2)=\frac{1}{4}\cdot 32x^2=8x^2=128$, and $x^2=16$, $\therefore x=\pm 4$, and the numbers are ± 18 and ± 14.

176. In a court there are 2 square grass-plots; a side of one of which is ten yards longer than the side of the other; and their ares are as 25 to 9. What are the lengths of the sides?

Let $x =$ a side of the one, $\therefore x+10 =$ a side of the other, and $(x+10)^2 : x^2 :: 25 : 9$, $\therefore x+10 : x :: 5 : 3$, and $x=15$, and the sides are 15 and 25, Ans.

177. A person bought 2 pieces of linen, which together measured 36 yards. Each of them cost as many shillings per yard as here were yards in the piece; and their whole prices were in the

proportion of 4 to 1. What were the lengths of the pieces?

Let x, and $36-x$ be the lengths, $\therefore x^2 : (36-x)^2 :: 4 : 1$, and $x=24$, and the lengths are 24 and 12, Ans.

178. There are 2 numbers, whose sum is to the less as 5 to 2: and whose difference, multiplied by the difference of their squares, is 135. Required the numbers. Ans. 9, and 6.

Let $2x$ = the less, and $3x$ = the greater, and $x \times 5x^2=135$, and $x^3=27$, and $x=3$, $\therefore$ the numbers are 9 and 6, Ans.

179. There are two numbers, which are in the proportion of 3 to 2; the difference of whose fourth powers is to the sum of their cubes as 26 to 7. Required the numbers. Ans. 6, and 4.

Let $3x$ and $2x$ be the numbers, $\therefore 81x^4-16x^4 : 27x^3+8x^3 :: 26 : 7$, or $65x : 35 :: 26 : 7$, or $5x : 5 :: 2 : 1$, and $x=2$, $\therefore$ the numbers are 6 and 4.

180. There is a field in the form of a rectangular parellelogram, whose length is to its breadth in the proportion of 6 to 5. A part of this, equal to one-sixth of the whole, being planted, there remain for ploughing 625 square yards. What are the dimensions of the field? Ans. The sides are 30, and 25 yards.

Let $6x$ and $5x$ = the sides, $\therefore$ the area $=30x^2$, and $25x^2 = 625$, $\therefore 5x=25$, $x=5$, and the sides are 30 and 25.

181. Some gentlemen made an excursion; and every one took the same sum. Each gentleman had as many servants attending him as there were gentlemen, and the number of dollars which each had was double the number of all the servants; and the whole sum of money taken out was \$3456. How many gentlemen were there? Ans. 12.

Let x = the number, x^2 and $2x^2$ be the number of servants, and of dollars each took, $\therefore 2x^4=3456$, and $x^3=1728$, $\therefore x=12$, Ans.

182. Divide the number 49 into two such parts, that the quotient of the greater divided by the less may be to the quotient of the less divided by the greater as $\frac{4}{3}$ to $\frac{3}{4}$. Ans. 28, and 21.

Let x, and $49-x$ be the numbers, then by the question we have

$$\frac{x}{49-x} : \frac{49-x}{x} :: \frac{4}{3} : \frac{3}{4}, \text{ and } x^2 : (49-x)^2 :: 16 : 9,$$

or $x : 49-x :: 4 : 3$. Hence $x : 49 :: 4 : 7$, and $x= 28$, Ans.

183. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished four times as many men as there were companies in the regiment; but there being found to be insufficient, each company furnished three more men; when their number was found to be increased in the ratio of 17 to 16. How many companies were there in the regiment?

Let x = the number, $\therefore 4x^2$ = the number first finished, and $4x^2 : 3x :: 16 : 1$, and $x : 3 :: 4 : 1$, and $x=12$, Ans.

184. A charitable person distributed a certain sum amongst some poor men and women, the numbers of whom were in the proportion of 4 to 5. Each man received one-third of as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. Now the men received all together 18*s.* more than the women. How many were there of each? Ans. 12 men, and 15 women.

Let $4x$, and $5x$ be the number of men and women, and $3x$, and $2x =$ the sum each man and woman received, then $12x^2 = 18 + 10x^2$, and $2x^2 = 18$, $x^2 = 9$, and $x = 3$. Ans. 12 and 15 women.

185. A gentleman who had a certain number of horses, kept part of them at livery stables, for which he paid £4 10*s.* per week. The rest he kept at home, and their number was to the number kept at the livery stables as 7 to 3. He found that the expense of keeping 5 at home was just equal to that of keeping 4 at the stables; and the number of shillings that one horse cost him at home was to the number of horses kept at home as 6 to 7. How many horses had he?

Let $3x$ and $7x$ be the number at the stable and at home, and $6x$ = the number of shillings one at home cost, $\therefore \frac{15}{2}x =$ the price of one at the stables, and $3x \times \frac{15}{2}x = 90$, $45x^2 = 180$, $x^2 = 4$, and $x = 2$. Ans. 6 at the livery stables, and 14 at home.

186. A city barge, with chairs for the company and benches for the rowers, went a summer excursion, with two bargemen on every bench. The number of gentlemen on board was equal to the square of the number of bargemen, and the number of ladies was equal to the number of gentlemen, twice the number of bargemen, and one over. Among other provisions, there were a number of turtles equal to the square root of the number of ladies; and a number of bottles of wine less than the cube of the number of turtles by 361. The turtles in dressing consumed a great quantity of wine, and the party having staid out till the turtles were all eaten, and the wine all gone, it was computed, that supposing them all to have consumed an equal quantity, (viz. gentlemen, ladies, bargemen, and turtles,) each individual would have consumed as many bottles as there were benches in the barge. Required the number of turtles.

Let x, x^2, $(x+1)^2$, $x+1$, and $(x+1)^3 - 361 =$ the number of bargemen, gentlemen, ladies, turtles, and bottles of wine respectively, and $\frac{(x+1)^3 - 361}{2(x+1)^2} = \frac{x}{2}$, or $x^3 + 3x^2 + 3x + 1 = x^3 + 2x^2 + x - 361$, and $x^2 + 2x + 1 = 361$, $\therefore x + 1 = 19$, the number required.

187. From two towns, C and D, two travellers, A and B, set out to meet each other; and it appeared when they met, B had

gone 35 miles more than three-fifths of the distance that A had travelled; but from their rate of travelling, A expected to reach C in 20 hours and 50 minutes; and B to reach D in 30 hours. Required the distance of C from D. Ans. 275 miles.

Let $5x$, and $3x+35$ be the number of miles A and B each travels, and $3x+35 : 5x :: 20\frac{5}{6}$: the number of hours A has travelled $=\frac{625x}{6(3x+35)}$; and in the same way, the number B has travelled $=\frac{6(3x+35)}{x}$; $\therefore \frac{625x}{6(3x+35)}=\frac{6(3x+35)}{x}$, and $625x^2=36(3x+35)^2$, $\therefore 25x=6(3x+35)$ and $7x=6\times35$ and $x=30$,

188. A Farmer bought two flocks of sheep, the first of which contained 18 fewer than the second. If he had given for the first flock as many pounds as there were sheep in the second, and for the second as many pounds as there were sheep in the first, then the price of six sheep of the first flock would have been to the price of 7 sheep of the second in the proportion of 7 to 6. Required the numbers in each flock. Ans. 108, and 126.

Let x and $x+18$ be the numbers, then $x : x+18 :: 6$: the price of six of the first flock $\frac{6(x+18)}{x}$, and the price of 7 of the 2d $=\frac{7x}{x+18}$, $\therefore \frac{6(x+18)}{x} : \frac{7x}{x+18} :: 7 : 6$, and $36(x+18)^2=49x^2$ and $6(x+18)=7x$, and $x=108$, the numbers are 108, 126.

189. A Poulterer bought a number of ducks and turkeys, the number of ducks exceeding the number of turkeys by 8. For each duck he gave half as many shillings as there were turkeys, and for each turkey half as many shillings as there were ducks. He afterwards bought another small flock of turkeys containing 4 fewer than the number of turkeys he bought before; and having given for each of them as many shillings as there were turkeys in the flock, he found, that if his former purchase had cost 16 shillings more, it would have cost exactly four times as much as the present one. How many ducks and turkeys did he buy at first?

Let x, and $x+8$ be the number of turkeys and ducks, and $\frac{1}{2}x(x+8)$ = the prices of each set, $\therefore x^2+8x+16=4(x-4)^2$, and $x+4=2(x-4)$, and $x=12$, and the numbers were 12 and 20.

190. Two men, A and B, entered into partnership with stocks, which are in the proportion of 9 to 8; and after trading one year, A found his share of their gain to amount to one-third of his stock. They continued to trade for as many years as are equal to three-fourths of the number of dollars which B contributed to the stock, and found their whole gain amount to \$1666. What did each contribute to the stock; and how many years did they

trade? Ans. A contributed \$63, B \$56.

Let $9x$, and $8x$ be A's and B's stock, $\therefore 3x =$ A's gain, and $6x =$ the number of years. Also $9x : 3x :: 8x$: B's gain $= \frac{8}{3}x$, $\therefore 6x(3x+\frac{8}{3}x)=1666$, or $34x^2=1666$, and $x^2=49$, $x=7$, A contributed \$63, B 56, and the number of years is 42.

191. A person wishing to ascertain the area of a certain quadrilateral field, found that he could determine it the most readily by dividing it into two portions, one of which was of the form of a rectangular parallelogram, the shorter side of which measured 60 yards. The other was of the form of a right-angled triangle, whose shortest side was equal to the shorter side of the parellelogram, and the other side, containing the right angle, was equal to the diagonal of the parallelogram; and the area of the triangle was to the area of the parallelogram as 5 to 8. What was the area of the field? Ans. 7800 square yards.

Let x=the longer side of the parallelogram, $\therefore x^2 : \sqrt{(x^2+3600)}$ = diagonal, and $60x =$ the area of the parallelogram, and $30\sqrt{(x^2+3600)} =$ the area of the triangle, $\therefore 60x : 30\sqrt{(x^2+3600)}::8 : 5$, or $x : \sqrt{(x^2+3600)}::4 : 5$, $\therefore x^2 : 3600 :: 16 : 9$, $\therefore x : 60::4 : 3$, and $x : 20::4 : 1$, $\therefore x=80$, and the area $=60\times80+30\sqrt{(6400+3600)}=7800$.

192. A Merchant laid out a certain sum upon a speculation, and found at the end of a year that he had gained \$69. This he added to his stock, and at the end of another year found that he had gained exactly as much per cent. as in the year preceding. Proceeding in the same manner, and each year adding to his stock the gain of the year preceding, he found at the beginning of the fifth year that his stock was to the original stock as 81 to 16. What was the sum he first laid out? Ans. \$138.

Let $x =$ the sum, $\therefore x+69 =$ the stock at the beginning of the second year, and $x : 69::x+69$: the gain the 2d year $= \dfrac{69(x+69)}{x}$, $\therefore$ the stock the third year $= (x+69) + \dfrac{69}{x}(x+69)$ $= \dfrac{(x+69)^2}{x}$, and $\dfrac{69}{x}\times\dfrac{(x+69)^2}{x} =$ the gain the third year. Hence the stock at the beginning of the fifth year $= \dfrac{(x+69)^4}{x^3}$, $\therefore \dfrac{(x+69)^4}{x} : x :: 81 : 16$, and $x+69 : x::3 : 2$, $\therefore 69 : x::1 : 2$, and $x = 138$.

193. There is a number consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10. Required the number. Ans. 23.

Let $10x+y$ be the number, $(10x^2+xy=46)-(x^2+xy=10)=9x^2=36$, and $x=\frac{4}{2}=2$, and $2y=6$ and $y=3$, the number is 23.

194. From two towns, C and D, which were at the distance of 396 miles, two persons, A and B, set out at the same time, and meet each other, after travelling as many days as are equal to the difference of the number of miles they travelled per day; when it appears that A has travelled 216 miles. How many miles did each travel per day? Ans. A went 36, and B 30.

Let x and y be the number A and B each went, then by the question $(x^2-xy=216)-(xy-y^2=180)=x^2-2xy+y^2=36$, or $x-y=6$, and $x=36$, and $y=30$.

195. There are two numbers, whose sum is to the greater as 40 is to the less, and whose sum is to the less as 90 is to the greater. What are the numbers? Ans. 36, and 24.

Let x and y be the numbers, $\therefore x+y : 40 :: x : y$,

and $y : x+y :: x : 90$,

$\therefore$ ex. æquali $y : 40 :: x^2 :: 90y$, and $4x^2=9y^2$, $\therefore 2x=3y$, and $x+\frac{2}{3}x=\dfrac{40x}{\frac{2}{3}x}$ or $\dfrac{5x}{3}=60$, and $x=36$, $y=24$.

196. It is required to find 2 numbers such, that the product of the greater and the cube of the less may be to the product of the less and the cube of the greater as 4 to 9; and the sum of the cubes of the numbers may be 35. Ans. 3, and 2.

Let x and y = the numbers, $\therefore xy^3 : yx^3 :: 4 : 9$, or $y^2 : x^2 :: 4 : 9$, and $y=\frac{2}{3}x$. Also $x^3+y^3=35$, or $x^3+\frac{8}{27}x^3=35$, $\therefore 35x^3=27\times 35$, and $x^3=27$, or $x=3$, and $y=2$.

197. The paving of two square court-yards cost £255; a yard of each costing one-fourth of as many shillings as there were yards in a side of the other. And a side of the greater and less together measure 41 yards. Required the length of a side of each. Ans. 25, and 16 yards.

Let x and y = the lengths, $\therefore x^2y+xy^2=205\times 20\times 4$, and $x+y=41$, $\therefore 41xy=16400$, and $xy=400$. Hence $(x-y=9)\pm(x+y=41)=2x=50$, or $x=25$, and $2y=32$, and $y=16$.

198. A person bought a number of apples and pears, amounting together to 80. Now the apples cost twice as much as the pears; but had he bought as many apples as he did pears, and as many pears as he did apples, his apples would have cost 10*d.*, and his pears 3*s.* 9*d.* How many did he buy of each?

Let x and y be the number of apples and pears, then I have

$\dfrac{10x}{y}=2\cdot\dfrac{45y}{x}$, and $x^2=9y^2$, $\therefore x=3y$, whence $(3y+y)=80$, and $y=\frac{80}{4}=20$, and $x=60$. Ans. 60 apples, and 20 pears.

199. A person exchanged a quantity of brandy for a quantity of rum and £11.5*s.*; the brandy and rum being each valued at as many shillings per gallon as there were gallons of that liquor.

Now had the rum been worth as many shillings per gallon as the brandy was, the whole value of the rum and brandy would have been £56 5*s*. How many gallons were there of each ?

Let x and y be the number of gallons of brandy and rum, then x^2 and y^2 will equal the prices of the brandy and rum, $\therefore x^2 - y^2 = 225$, and $x^2+xy=1125$, hence $x=25$, and $y=20$.

200. There are two rectangular vats, the greater of which contains 20 solid feet more than the other. Their capacities are in the ratio of 4 to 5; and their bases are squares, a side of each of which is equal to the depth of the other. What are the depths ?

Let x, and y denote the depths, $\therefore x^2y - xy^2 = 20$, and $x^2y : xy^2 :: 5 : 4$, or $x : y :: 5 : 4$, $\therefore y=\frac{4}{5}x$. Hence $\frac{4}{5}x^3 - \frac{16}{25}x^3 = 20$, or $\frac{4}{25}x^3=20$, $\therefore x=5$, and $y=4$.

201. Bought two square carpets for £62 1*s*., for each of which I paid as many shillings per yard as there were yards in its side. Now had each of them cost as many shillings per yard as there were yards in a side of the other, I should have paid 17*s*. less. What was the size of each ?

Let x and y be the lengths of the sides, then $x^3+y^3=1241$, and $x^2y+xy^2 = 1224$; adding 3 times the second equation to the first, $x^3 + 3x^2y + 3xy^2 + y^3 = 4913$, and $\therefore x + y = 17$, and $xy = \frac{1224}{x+y}=72$, whence $x^2-2xy+y^2=1$, and $x-y=\pm 1$, but $x + y = 17$, and $x=9$ or 8, and $y=8$ or 9.

202. The number of men in both fronts of two columns of troops, A and B, when each consisted of as many ranks as it had men in front, was 84 ; but when the columns changed ground, and A was drawn up with the front B had, and B with the front A had, the number of ranks in both columns was 91. Required the number of men in each column. Ans. 2304, and 1296.

Let x^2 and y^2 = the numbers, $x+y=84$, $\frac{x^2}{y}+\frac{y^2}{x}=91$; whence $\{x^3+y^3)=91xy$, but $x^3+y^3+3xy(x+y)=84^3,\}$ $\therefore 91xy+3(84xy) = 84^3$, or $343xy=84^3$, and $xy = 12^3 = 1728$, and since $x+y=84$, $\therefore x-y=\pm 12$, and $x=48$ or 36, $y=36$ or 48, Ans.

203. A field in the form of a rectangular parallelogram was planted with trees placed at such distances as to have four on every square yard. The expense of planting was such, that every 40 trees cost one-third of as many shillings as there were yards in the diagonal of the parallelogram. But had they been planted at such a price as that every hundred should have cost as many shillings as there were yards in the shorter side of the parallelogram, the expense would have been less by £224. Now a square described upon the diagonal of the parallelogram would be equal

to $\frac{8}{3}$ of the square described on the less side, together with the square described on a line which is equal to the difference of the sides. Required the dimensions of the parallelogram.

Let x, y, and $4xy$ denote the less and greater sides, and the number of trees, As $40 : \frac{1}{3}\sqrt{(x^2+y^2)} :: 4xy$: the price of planting $=\frac{xy\sqrt{(x^2+y^2)}}{30}$, $\therefore \frac{xy\sqrt{(x^2+y^2)}}{30}-\frac{x^2y}{25}=4480$, and $5xy\sqrt{(x^2+y^2)}-6x^2y=150\times4480$, but $x^2+y^2=\frac{8}{3}x^2+(y-x)^2=x^2+y^2+\frac{8}{3}x^2-2xy$, $\therefore 2xy=\frac{8}{3}x^2$, and $y=\frac{4}{3}x$. $\therefore \frac{20}{3}x^2\times\frac{5}{3}x-6x^2\times\frac{4}{3}x=150\times4480$, $\therefore 28x^3=9\times150\times4480$, and $x^3=9\times150\times160=216000$, $\therefore x=60$ and $y=80$, Ans.

Questions for practice without their solutions. App. p. 576

204. Given $\frac{3x-3}{4}-\frac{3x-4}{3}=5\frac{1}{3}-\frac{27+4x}{9}$ to find x, $x=9$

205. Given $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$. Ans. $x=72$.

206. Given $\frac{7x+9}{8}-\frac{3x+1}{7}=\frac{9x-13}{4}-\frac{249-9x}{14}$. Ans. $x=9$.

207. Given $5-\frac{3x+25}{4}-\frac{17-6x}{9}=2\frac{1}{36}+x-\frac{9x+40}{8}$; $x=4$.

208. Given $\frac{7x-43}{12}+13\frac{1}{2}-\frac{5+4x}{6}=256-\frac{3x-12}{9}-\frac{5x+29}{8}-12x$, to find the value of x. Ans. $x=19$.

Given $4x+\frac{1}{10}-\frac{3x-13}{16}-\frac{12+7x}{9}=7x-33-\frac{9+5x}{10}-\frac{11x-17}{8}$ to find the value of x. Ans. $x=15$.

Given $\frac{31+4x}{3}-\frac{3x+47}{8}-\frac{3x-19}{16}=47\frac{3}{4}+\frac{16-10x}{11}-\frac{5x+20}{7}$, to find the value of x. Ans. $x=17$.

211. Given $\frac{3a+x}{x}-5=\frac{6}{x}$ to find x, Ans. $x=\frac{3a-6}{4}$

212. Given $\frac{4}{5}x\cdot ab+\frac{1}{5}ac-\frac{2}{3}cx=\frac{3}{4}ac+2ab-6cx$ to find x.

213. Given $\frac{a^2x}{bc}-\frac{d^2}{a}+bx=\frac{ex}{f}-b+(d+b)x$, to find x.

214. Given $\frac{ax}{b}+\frac{cx}{d}+\frac{ex}{f}-g=h$, to find the value of x.

Ans. $x=\frac{70ab-3ac}{320c}$; $x=\frac{(d^2-ab)bc}{a^3f-abce-abcd}$; $x=\frac{(g+h)bdf}{adf+bcf+bde}$

215. Given $\frac{a^2x}{b-c}-dc=bx-ac$, Ans. $x=\frac{c(b-c)\cdot(d-a)}{a^2-b^2+bc}$

216. Given $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$. Ans. $x=8$.

217. Given $\frac{20x+36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+\frac{86}{25}$. Ans. $x=4$.

218. Given $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}$. Ans. $x=4$.

219. Given $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}$. $x=7$.

220. Given $\frac{a.(b^2+x^2)}{bx}=ac+\frac{ax}{b}$. $x=\frac{b}{c}$.

221. Given $\frac{cx^m}{a+bx}=\frac{dx^m}{e+fx}$. $x=\frac{ad-ce}{cf-bd}$.

222. Given $\frac{a}{bx}+\frac{c}{dx}+\frac{e}{fx}+\frac{g}{hx}=k$, to find the value of x.

Ans. $x=\frac{adfh+bcfh+bdeh+bdfg}{bdfhk}$.

223. Given $(a+x).(b+x)-a.(b+c)=\frac{a^2c}{b}+x^2$. $x=\frac{ac}{b}$.

224. Given $\frac{10+x}{5}:\frac{4x-9}{7}::14:5$. $x=4$.

225. Given $\frac{17-4x}{4}:\frac{15+2x}{3}-2x::5:4$. $x=3$.

226. Given $16x+5:\frac{4y+14}{9x+31}::36x+10:1$, to find the value of x. Ans. $x=5$.

227. Given $\frac{4x+3}{6x-43}:1::2x+19:3x-19$. $x=8$.

228. Given $5x+\frac{7x+9}{4x+3}=9+\frac{10x^2-18}{2x+3}$, to find x. $x=3$.

229. Given $\sqrt[3]{(10x+35)}-1=4$, to find x. $x=9$.

230. Given $\sqrt[5]{(9x-4)}+6=8$, to find x. $x=4$.

231. Given $\sqrt{(x+16)}=2+\sqrt{(x)}$, to find x. $x=9$.

232. Given $\sqrt{(x-32)}=16-\sqrt{(x)}$, to find x. Ans. $x=81$.

233. Given $\sqrt{(4x+21)}=2\sqrt{(x)}+1$, to find x. $x=25$.

234. Given $a\sqrt[3]{(bx-c)}=d.\sqrt[3]{(ex+fx-g)}$ to find x.

235. Given $\sqrt[3]{(a^2+c)}=\sqrt[4]{\left(\frac{a^2+c}{d\cdot(x+b)}\right)}$, to find the value of x.

Answers, $x=\frac{a^3c-d^3g}{a^3b-d^3\cdot(e+f)}$, and $x=\frac{1}{d\sqrt[3]{(a^2+c)}}-b$.

236. Given $\sqrt[m]{(a+x)}=\sqrt[2m]{(x^2+5ax+b^2)}$.

237. Given $a+b\cdot\sqrt[m]{(x+d)}=c$, to find the value of x.

Answers $x=\dfrac{a^2-b^2}{3a}$, and $x=\left(\dfrac{c-a}{b}\right)^m-d$.

238. Given $\dfrac{\sqrt{(9x)}-4}{\sqrt{(x)}+2}=\dfrac{15+\sqrt{(9x)}}{\sqrt{(x)}+40}\Big\}$ Ans. $x=4$.

239. Given $\dfrac{\sqrt{(x)}+\sqrt{(b)}}{\sqrt{(x)}-\sqrt{(b)}}=\dfrac{a}{b}$. Ans $x=b\cdot\left(\dfrac{a+b}{a+b}\right)^2$.

240. Given $\dfrac{\sqrt{(6x)}-2}{\sqrt{(6x)}+2}=\dfrac{4\sqrt{(6x)}-9}{4\sqrt{(6x)}+6}\Big\}$, to find x. $x=6$.

241. Given $\dfrac{5x-9}{\sqrt{(5x)}+3}-1=\dfrac{\sqrt{(5x)}-3}{2}\Big\}$, to find x. $x=5$.

242. Given $\sqrt{\{1+x\sqrt{(x^2+12)}\}}=1+x$, to find x. $x=2$.

243. Given $\dfrac{ax}{b}\cdot\sqrt{(c^2x^2+d^2)}+\dfrac{acx^2}{b}=ex$. Ans. $x=\dfrac{b^2e^2-a^2d^2}{2abce}$.

244. Given $\{x^2+xy+y^2\}\dfrac{1}{x+y}=r$, and $\{x^2-xy+y^2\}\dfrac{1}{x-y}=s$, to find x and y.

$$x=r\cdot\frac{(s^2-r^2)^{\frac{1}{3}}+(s+r)^{\frac{2}{3}}}{(s+r)^{\frac{2}{3}}+(s^2-r^2)^{\frac{1}{3}}+(s-r)^{\frac{2}{3}}};\ y=r\cdot\frac{(s^2-r^2)^{\frac{1}{3}}+(s-r)^{\frac{2}{3}}}{(s+r)^{\frac{2}{3}}+(s^2-r^2)^{\frac{1}{3}}+(s-r)^{\frac{2}{3}}}$$

245. Given $\dfrac{7x-21}{6}+\dfrac{3y-x}{3}=4+\dfrac{3x-19}{2}$
and $\dfrac{2x+y}{2}-\dfrac{9x-7}{8}=\dfrac{3y+9}{4}-\dfrac{4x+5y}{16}\Bigg\}$, to find x and y.

Ans. $x=9$, and $x=4$.

246. Given $\dfrac{7x+6}{11}+\dfrac{4y-9}{3}=3x-\dfrac{13-x}{2}-\dfrac{3y-x}{5}$, and $3x+4:2y-3::5:3$, to find x and y. Ans. $x=7$, and $y=9$.

247. Given $\dfrac{a}{b+y}=\dfrac{b}{3a+x}$, and $ax+2by=c$, $\Big\}$, to find x and y.

Answers. $x=\dfrac{2b^2-6a^2+c}{3a}$; and $y=\dfrac{3a^2-b^2+c}{3b}$.

248. Given $\dfrac{5x+13}{2}-\dfrac{8y-3x-5}{6}=9+\dfrac{7x-3y+1}{3}$; and $\dfrac{x+7}{3}:\dfrac{3y-8}{4}+4x::4:21$, to find x and y. Ans. $x=5$; $y=4$.

249. Given $\dfrac{3x+4y+3}{10}-\dfrac{2x+7-y}{15}=5+\dfrac{y-8}{5}$; and $\dfrac{9y+5x-8}{12}-\dfrac{x+y}{4}=\dfrac{7x+6}{11}$, to find x and y. Ans. $x=7$; $y=9$.

250. Given $13x+\frac{4y-17+x}{12}-\frac{15-3x}{4}=\frac{12y+11}{3}-\frac{12x+7y+28}{6}$; and $\frac{9x+18}{4}-\frac{12+5y-6x}{5}=\frac{15x-3y-5}{8}-\frac{7x+y-10}{15}$, to find x and y. Ans. $x=2$, and $y=11$.

251. Given $3x+5y=\frac{(8a-2b)\cdot ab}{a^2-b^2}$; & $a^2x-\frac{acb^2}{a^2-b^2}+(a+b+c)\cdot by=b^2x+(a+2b)ab$, to find x and y.

Ans. $x=\frac{ab}{a+b}$; and $y=\frac{ab}{a-b}$.

252. Given $\frac{2x+y}{9}+\frac{7y+6x+11}{18}=9\frac{1}{2}-\frac{5x-17}{6}$, and $\frac{5x+3y+2}{7}:\frac{9y+6}{2}::1:3$, to find x and y. Ans. $x=7$; and $y=4$.

253. Given $\frac{3x-5y}{3}-\frac{2x-8y-9}{12}=\frac{y}{2}+\frac{1}{3}+\frac{1}{4}$; and $\frac{x}{7}+\frac{y}{4}+1\frac{1}{3}:4x-\frac{y}{8}-24::3\frac{1}{3}:3\frac{1}{2}$, to find x and y. Ans. $x=7$; and $y=4$.

254. Given $\sqrt{x}+\sqrt{(x-9)}=\frac{36}{\sqrt{(x-9)}}$, to find x. $x=25$.

255. Given $x+15y=53$, and $y+3x=27$, to find the values of x and y. Ans. $x=8$; and $y=3$.

256. Given $4x+9y=51$, and $8x-13y=9$, to find the values of x and y. Ans. $x=6$; and $y=3$.

257. Given $\frac{x}{6}+\frac{y}{4}=6$; and $\frac{x}{4}+\frac{y}{6}=5\frac{2}{3}$, to find x and y.

Ans. $x=12$; and $y=16$.

258. Given $\frac{x}{8}+8y=194$; and $\frac{y}{8}+8x=131$, to find x and y.

Ans. $x=16$, and $y=24$.

259. Given $\frac{3x-1}{5}+3y-4=15$; and $\frac{3y-5}{6}+2x-8=7\frac{2}{3}$, to find x and y. Ans. $x=7$; and $y=5$.

260. Given $9x+\frac{8y}{5}=70$; and $7y-\frac{13x}{3}=44$. $x=6$; $y=10$.

261. Given $\frac{7+x}{2}-\frac{2x-y}{4}=3y-5$; and $\frac{5y-7}{2}+\frac{4x-3}{6}=18-5x$, to find x and y. Ans. $x=3$; and $y=2$.

262. Given $x+1-\frac{3y+4x}{7}=7-\frac{9y+33}{14}$; and $y-3-\frac{5x-4y}{2}$

237. Given $a+b\cdot\sqrt[m]{(x+d)}=c$, to find the value of x.

Answers $x=\frac{a^2-b^2}{3a}$, and $x=\left(\frac{c-a}{b}\right)^m-d$.

238. Given $\left.\frac{\sqrt{(9x)}-4}{\sqrt{(x)}+2}=\frac{15+\sqrt{(9x)}}{\sqrt{(x)}+40}\right\}$ Ans. $x=4$.

239. Given $\frac{\sqrt{(x)}+\sqrt{(b)}}{\sqrt{(x)}-\sqrt{(b)}}=\frac{a}{b}$. Ans $x=b\cdot\left(\frac{a+b}{a+b}\right)^2$.

240. Given $\left.\frac{\sqrt{(6x)}-2}{\sqrt{(6x)}+2}=\frac{4\sqrt{(6x)}-9}{4\sqrt{(6x)}+6}\right\}$, to find x. $x=6$.

241. Given $\left.\frac{5x-9}{\sqrt{(5x)}+3}-1=\frac{\sqrt{(5x)}-3}{2}\right\}$, to find x. $x=5$.

242. Given $\sqrt{\{1+x\sqrt{(x^2+12)}\}}=1+x$, to find x. $x=2$.

243. Given $\frac{ax}{b}\cdot\sqrt{(c^2x^2+d^2)}+\frac{acx^2}{b}=ex$. Ans. $x=\frac{b^2e^2-a^2d^2}{2abce}$.

244. Given $\{x^2+xy+y^2\}\frac{1}{x+y}=r$, and $\{x^2-xy+y^2\}\frac{1}{x-y}=s$, to find x and y.

$$x=r\cdot\frac{(s^2-r^2)^{\frac{1}{3}}+(s+r)^{\frac{2}{3}}}{(s+r)^{\frac{2}{3}}+(s^2-r^2)^{\frac{1}{3}}+(s-r)^{\frac{2}{3}}};\ y=r\cdot\frac{(s^2-r^2)^{\frac{1}{3}}+(s-r)^{\frac{2}{3}}}{(s+r)^{\frac{2}{3}}+(s^2-r^2)^{\frac{1}{3}}+(s-r)^{\frac{2}{3}}}$$

245. Given $\frac{7x-21}{6}+\frac{3y-x}{3}=4+\frac{3x-19}{2}$ and $\frac{2x+y}{2}-\frac{9x-7}{8}=\frac{3y+9}{4}-\frac{4x+5y}{16}$, to find x and y.

Ans. $x=9$, and $x=4$.

246. Given $\frac{7x+6}{11}+\frac{4y-9}{3}=3x-\frac{13-x}{2}-\frac{3y-x}{5}$, and $3x+4:2y-3::5:3$, to find x and y. Ans. $x=7$, and $y=9$.

247. Given $\frac{a}{b+y}=\frac{b}{3a+x}$, and $ax+2by=c$, $\Big\}$, to find x and y.

Answers. $x=\frac{2b^2-6a^2+c}{3a}$; and $y=\frac{3a^2-b^2+c}{3b}$.

248. Given $\frac{5x+13}{2}-\frac{8y-3x-5}{6}=9+\frac{7x-3y+1}{3}$; and $\frac{x+7}{3}:\frac{3y-8}{4}+4x::4:21$, to find x and y. Ans. $x=5$; $y=4$.

249. Given $\frac{3x+4y+3}{10}-\frac{2x+7-y}{15}=5+\frac{y-8}{5}$; and $\frac{9y+5x-8}{12}-\frac{x+y}{4}=\frac{7x+6}{11}$, to find x and y. Ans. $x=7$; $y=9$.

250. Given $13x+\frac{4y-17+x}{12}-\frac{15-3x}{4}=\frac{12y+11}{3}-\frac{12x+7y+28}{6}$; and $\frac{9x+18}{4}-\frac{12+5y-6x}{5}=\frac{15x-3y-5}{8}-\frac{7x+y-10}{15}$, to find x and y. Ans. $x=2$, and $y=11$.

251. Given $3x+5y=\frac{(8a-2b)\cdot ab}{a^2-b^2}$; & $a^2x-\frac{acb^2}{a^2-b^2}+(a+b+c)\cdot by=b^2x+(a+2b)ab$, to find x and y.

Ans. $x=\frac{ab}{a+b}$; and $y=\frac{ab}{a-b}$.

252. Given $\frac{2x+y}{9}+\frac{7y+6x+11}{18}=9\frac{1}{2}\cdot\frac{5x-17}{6}$, and $\frac{5x+3y+2}{7}:\frac{9y+6}{2}::1:3$, to find x and y. Ans. $x=7$; and $y=4$.

253. Given $\frac{3x-5y}{3}-\frac{2x-8y-9}{12}=\frac{y}{2}+\frac{1}{3}+\frac{1}{4}$; and $\frac{x}{7}+\frac{y}{4}+1\frac{1}{2}:4x-\frac{y}{8}-24::3\frac{1}{2}:3\frac{1}{2}$, to find x and y. Ans. $x=7$; and $y=4$.

254. Given $\sqrt{x}+\sqrt{(x-9)}=\frac{36}{\sqrt{(x-9)}}$, to find x. $x=25$.

255. Given $x+15y=53$, and $y+3x=27$, to find the values of x and y. Ans. $x=8$; and $y=3$.

256. Given $4x+9y=51$, and $8x-13y=9$, to find the values of x and y. Ans. $x=6$; and $y=3$.

257. Given $\frac{x}{6}+\frac{y}{4}=6$; and $\frac{x}{4}+\frac{y}{6}=5\frac{2}{3}$, to find x and y.

Ans. $x=12$; and $y=16$.

258. Given $\frac{x}{8}+8y=194$; and $\frac{y}{8}+8x=131$, to find x and y.

Ans. $x=16$, and $y=24$.

259. Given $\frac{3x-1}{5}+3y-4=15$; and $\frac{3y-5}{6}+2x-8=7\frac{2}{3}$, to find x and y. Ans. $x=7$; and $y=5$.

260. Given $9x+\frac{8y}{5}=70$; and $7y-\frac{13x}{3}=44$. $x=6$; $y=10$.

261. Given $\frac{7+x}{2}-\frac{2x-y}{4}=3y-5$; and $\frac{5y-7}{2}+\frac{4x-3}{6}=18-5x$, to find x and y. Ans. $x=3$; and $y=2$.

262. Given $x+1-\frac{3y+4x}{7}=7-\frac{9y+33}{14}$; and $y-3-\frac{5x-4y}{2}$

291. Given $\frac{1}{1-\sqrt{(1-x^2)}} - \frac{1}{1+\sqrt{(1-x^2)}} = \frac{\sqrt{(3)}}{x^2}$, to find x and y. Ans. $x=\pm\frac{1}{2}$.

292. Given $x^2y+y^2=116$; and $xy^{\frac{1}{2}}+y=14$, to find x and y. Ans. $x=5$, or $2\sqrt{(\frac{2}{5})}$; and $y=4$, or 10.

293. Given $\sqrt[3]{x}+\sqrt[3]{y}=6$; and $x+y=72$, to find x and y. Ans. $x=64$, or 8; and $y=8$, or 64.

294. Given $4x^2+\frac{5}{2}=\frac{x^2}{y}+10y$; and $x^2+3y=55$, to find x and y. Ans. $x=\pm5$; and $y=10$.

295. Given $\frac{1}{x+\sqrt{(2-x^2)}}+\frac{1}{x-\sqrt{(2-x^2)}}=ax$, to find the values of x. Ans. $x=\pm\frac{a+1}{a}$.

296. Given $\frac{x}{\sqrt{\{a^2+x^2\}}-x}=b$. Ans. $x=\pm\frac{ab}{\sqrt{(2b+1)}}$.

297. Given $\sqrt{(y)}-\sqrt{(a-x)}=\sqrt{(y-x)}$; and $\sqrt{(y-x)}+\sqrt{(a-x)} : \sqrt{(a-x)} :: 5 : 2$, to find x and y. Ans. $x=\frac{4}{5}a$; and $y=\frac{5}{4}a$.

298. Given $x^{\frac{4}{3}}+y^{\frac{2}{5}}=20$; and $x^{\frac{2}{3}}+y^{\frac{1}{5}}=6$, to find x and y. Ans. $x=\pm8$, or $\pm\sqrt{8}$; and $y=32$, or 1024.

299. Given $x^4+2x^2x^2+y^4=1296-4xy\cdot(x^2+xy+y^2)$; and $x-y=4$, to find x and y. Ans. $x=5$, or 1; $y=1$, or 5.

300. Given $\frac{\sqrt{a}-\sqrt{(a-x)}}{\sqrt{a}+\sqrt{(a-x)}}=a$, to find x. Ans. $x=\frac{4a^2}{(a+1)^2}$.

301. Given $\frac{\sqrt{x}+\sqrt{(x-y)}}{\sqrt{x}-\sqrt{(x-y)}}=4$; and $\sqrt{x} : \sqrt{y} :: \sqrt{y} : 4$, to find x and y. Ans. $x=\frac{625}{16}$; and $y=25$.

302. Given $\frac{1}{y}-\frac{1}{x}=\frac{1}{4}$; and $x^2y-xy^2=16$, to find x and y. Ans. $x=4$, or -2; $y=2$, or -4.

303. Given $\frac{\sqrt{(4x+1)}+\sqrt{4x}}{\sqrt{(4x+1)}-\sqrt{4x}}$, to find x. Ans. $x=\frac{4}{9}$.

304. Given $\frac{a+x+\sqrt{(2ax+x^2)}}{a+x-\sqrt{(2ax+x^2)}}=b$. Ans. $x=\pm a\cdot\frac{(\sqrt{b}\mp1)^2}{2\sqrt{b}}$,

305. Given $x^4y^3-x^3y^4=216$; and $x^2y-xy^2=6$, to find x and y. Ans. $x=3$, or -2; and $y=2$, or -3.

306. Given $x^2+x\sqrt[3]{(xy^2)}=208$; & $y^2+y\sqrt[3]{(x^2y)}=1053$, to find x and y. Ans. $x=\pm8$; and $y=\pm27$.

307. Given $x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=1009$; and $x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3=$

582193, to find x and y. Ans. $x=81$, or 16; and $y=16$, or 81.

308. Given $x^2+y^2+xy.(x+y)=68$; and $x^3+y^3-3x^2=13+3y^2$ to find x and y. Ans. $x=4$, or 2; and $y=2$, or 4.

309. Given $xy.(x+y)=84$; and $x^2y^2.(x^2+y^2)=3600$, to find x and y. Ans. $x=4$, or 3; and $y=3$, or 4.

310. Given $\frac{x^2+xy+y^2}{x+y}=7$; and $\frac{x^2-xy+y^2}{x-y}=9$, to find x and y. Ans. $x=6$, and $y=3$.

311. Given $x^2+4x=140$, to find the values of x.

312. Given $x^2-6x+8=80$, to find x.
Ans. $x=10$, or -14, and $x=12$, or -6.

313. Given $x^2-10x+17=1$, to find x. Ans. $x=8$, or 2.

314. Given $x^2-x-40=170$ to find x. Ans. $x=15$, or -14.

315. Given $3x^2-9x-4=80$, to find x. Ans. $x=7$, or -4.

316. Given $7x^2-21x+13=293$, to find x. Ans. $x=8$, or -5.

317. Given $\frac{x^2}{3}+\frac{4x}{5}-19=15\frac{1}{5}$, to find x. Ans. $x=9$, or $-\frac{57}{5}$.

318. Given $\frac{2x^2}{3}+3\frac{1}{2}=\frac{x}{2}+8$, to find x. Ans. $x=6$, or $-\frac{3}{2}$.

319. Given $x+4+\frac{7x-8}{x}=13$, to find x. Ans. $x=4$, or -2.

320. Given $4x-\frac{36-x}{6}$, to find x Ans. $x=12$, or $-\frac{3}{4}$.

321. Given $\frac{x+3}{2}+\frac{16-2x}{2x-5}=5\frac{1}{5}$. Ans. $x=5$, or $-\frac{69}{10}$.

322. Given $14+4x-\frac{x+7}{x-7}=3x+\frac{9+4x}{3}$. Ans. $x=9$, or 28.

323. Given $\frac{x+4}{3}-\frac{7-x}{x-3}=\frac{4x+7}{9}-1$. Ans. $x=21$, or 5.

324. Given $\frac{15-x}{4}-\frac{12-3x}{4x-5}=7x-\frac{23x+60}{7}$. $x=3$, or $\frac{222}{148}$.

325. Given $\frac{x+11}{x}+\frac{9+4x}{x^2}=7$. Ans. $x=3$, or $=\frac{1}{2}$.

326. Given $\frac{2x+9}{9}+\frac{4x-3}{4x+3}=3+\frac{3x-16}{18}$. $x=6$, or $-\frac{12}{4}$.

327. Given $\frac{x}{x+60}=\frac{7}{3x-5}$. Ans. $x=14$, or -10.

328. Given $\frac{3x-7}{x}+\frac{4x-10}{x+5}=3\frac{1}{2}$. Ans. $x=7$, or $-\frac{10}{7}$.

329. Given $\frac{8x}{x+2}-6=\frac{20}{3x}$. Ans. $x=10$, or $-\frac{2}{3}$

330. Given $\frac{40}{x-5}+\frac{27}{x}=13$. Ans. $x=9$, or $\frac{15}{13}$.

331. Given $\frac{5x-12}{9}+\frac{3x-24}{4x-12}=9-\frac{7x-34}{15}$. $x=12$, or $\frac{477}{154}$.

332. Given $\frac{2x}{x-4}+\frac{2x-5}{x-3}=8\frac{1}{3}$. Ans. $x=6$, or $\frac{40}{13}$.

333. Given $\frac{2x+3}{10-x}=\frac{2x}{25-3x}-6\frac{1}{2}$. Ans. $x=8$, or $13\frac{22}{31}$.

334. Given $\frac{4x-5}{x}-\frac{3x-7}{3x+7}=\frac{9x+23}{13x}$. Ans. $x=2$, or $-\frac{154}{45}$.

335. Given $2x+18-\frac{8x^2+16}{4x+7}=27-\frac{12x-11}{2x-3}$. $x=8$, or 5.

336. Given $\frac{x}{x+9}+\frac{5}{2x+18}=\frac{x-\frac{x^2+20}{x+8}}{2}$. $x=4$, or $-\frac{55}{6}$.

337. Given $\frac{x+4}{x+6}+\frac{5}{2x+4}=\frac{3x+7}{3x+4}$. Ans. $x=8$, or $-\frac{3}{5}$.

338. Given $\frac{4}{2x+3}+\frac{3x+6}{5x+18}=\frac{3x+5}{5x}$. $x=6$, or $-\frac{45}{2}$.

339. Given $\frac{8}{9+5x}+\frac{8x-17}{2+4x}=\frac{4x+3}{2x+12}$. $x=3$, or $-\frac{283}{137}$.

340. Given $\frac{12}{5-x}+\frac{8}{4-x}=\frac{32}{x+2}$. Ans. $x=2$, or $\frac{58}{13}$.

341. Given $\frac{3}{6x-x^2}+\frac{6}{x^2+2x}=\frac{11}{5x}$. $x=3$, or $\frac{28}{11}$.

342. Given $\frac{4x^2+7x}{19}+\frac{5x-x^2}{3+x}=\frac{4x^2}{9}$. $x=3$, or $-\frac{87}{16}$.

343. Given $\frac{x^4+2x^3+8}{x^2+x-6}=x^2+x+8$. $x=4$, or $-\frac{14}{5}$.

344. Given $\frac{x+12}{x}+\frac{x}{x+12}=5\frac{3}{15}$. Ans. $x=3$, or -15.

345. Given $\sqrt{(4x+5)}\times\sqrt{(7x+1)}=30$. $x=5$, or $-\frac{179}{28}$.

346. Given $\frac{\sqrt{x}+9}{\sqrt{x}}=\frac{\sqrt{9x}-3\frac{4}{5}}{9-\sqrt{x}}$. Ans. $x=25$, or $\frac{6561}{400}$.

347. Given $\frac{x+\sqrt{x}}{x-\sqrt{x}}=\frac{x^2-x}{4}$. $x=4$, or 1, or $\frac{-3\pm\sqrt{(-7)}}{2}$.

348. Given $\frac{x-\sqrt{(x+1)}}{x+\sqrt{(x+1)}}=\frac{5}{11}$. $x=8$, or $-\frac{8}{9}$.

349. Given $5\cdot\frac{3x-1}{1+5\sqrt{x}}+\frac{\cdot 2}{\sqrt{x}}=3\sqrt{x}$. $x=4$, or $\frac{1}{5}$.

350. Given $\sqrt{x^5}-\dfrac{40}{\sqrt{x}}=3x$, to find x; $x=4$, or $\overline{-5}|^{\frac{2}{3}}$.

351. Given $x^{\frac{4}{3}}+7x^{\frac{2}{3}}=44$. Ans. $x=\pm 8$, or $\pm\overline{-11}|^{\frac{3}{2}}$.

352. Given $4x^{\frac{1}{3}}+x^{\frac{1}{6}}=39$. Ans. $x=729$, or $\overline{\frac{13}{4}}|^{6}$.

353. Given $3x^6+42x^3=3321$. $x=3$, or $-\sqrt[3]{41}$.

354. Given $\dfrac{8}{x^3}+2=\dfrac{17}{x^{\frac{3}{2}}}$. Ans. $x=4$, or $\sqrt[3]{\frac{1}{4}}$.

355. Given $x^{\frac{7}{3}}+\dfrac{41\sqrt[3]{x}}{x}=\dfrac{97}{\sqrt[3]{x^2}}+x^{\frac{5}{6}}$. $x=4$, or $\overline{-7}|^{\frac{2}{3}}$.

356. Given $\sqrt{\dfrac{1}{x^4}}+\sqrt[3]{\dfrac{1}{x}}=\dfrac{3-\sqrt[3]{x^2}}{x}$. $x=1$, or $-\frac{27}{8}$.

357. Given $3x^n\sqrt[3]{x^n}-\dfrac{4x^n}{\sqrt[3]{x^n}}=4$. $x=\overline{8}|^{\frac{1}{2n}}$, or $\overline{-\frac{8}{27}}|^{\frac{1}{2n}}$.

358. Given $adx-acx^2=bcx-bd$. $x=\dfrac{d}{c}$, or $-\dfrac{b}{a}$.

359. Given $\dfrac{a^2x^2}{b^2}-\dfrac{2ax}{c}+\dfrac{d^2}{c^2}=0$. $x=\dfrac{b}{a}\times\dfrac{b\pm\sqrt{(b^2-d^2)}}{c}$.

360. Given $9a^4b^4x^2-6a^3b^2x=b^2$. $x=\dfrac{a^2\pm\sqrt{(a^2+b^2)}}{3a^2b^2}$.

361. Given $(a+b).x^2=cx+\dfrac{ac}{a+b}$. $x=\dfrac{c\pm\sqrt{(c^2+4ac)}}{2.(a+b)}$.

362. Given $3\sqrt{(112-8x)}=19+\sqrt{(3x+7)}$. $x=6$, or $\frac{7398}{625}$.

Given $\sqrt{(2x+7)}+\sqrt{(3x-18)}=\sqrt{(7x+1)}$. $x=9$, or $-\frac{18}{5}$.

364. Given $7.\sqrt{(\frac{3x}{2}-5)}-\sqrt{(\frac{x}{5}+45)}=\frac{7}{4}\sqrt{(10x+56)}$.

Ans. $x=20$, or $\frac{145689880}{28746649}$.

365. Given $\dfrac{16-4\sqrt{x}}{8-3\sqrt{x}}=\dfrac{88+33\sqrt{x}}{4+\sqrt{x}}+\dfrac{x^2-5x+11}{(8-3\sqrt{x}).(4+\sqrt{x})}$ to find x. Ans. $x=93$, or 7.

366. Given $\dfrac{54-9\sqrt{x}}{x+2\sqrt{x}}=\dfrac{23x-46\sqrt{x}}{6+\sqrt{x}}+\dfrac{7x^2-3x+4}{(x+2\sqrt{x})\cdot(6+\sqrt{x})}$ to find x. $x=5$, or $-\frac{32}{15}$.

367. Given $x+\sqrt{x} : x-\sqrt{x} :: 3\sqrt{x}+6 : 2\sqrt{x}$. $x=9$, or 4.

368. Given $x^2+11+\sqrt{(x^2+11)}=42$. $x=\pm 5$, or $\pm\sqrt{38}$.

369. Given $\overline{x-5}|^3-3\cdot\overline{x-5}|^{\frac{3}{2}}=40$. $x=9$, or $\overline{-5}|^{\frac{2}{3}}+5$.

370. Given $x+\sqrt{(x+6)}=2+3\sqrt{(x+6)}$. $x=10$, or -2.

371. Given $(x^2+5)^2-4x^2=160$. $x=\pm 3$, or $\pm\sqrt{(-15.)}$

372. Given $x^2-7x+\sqrt{(x^2-7x+18)}=24$, to find x.

Ans. $x=9$, or -2, or $\frac{1}{2}(7\pm\sqrt{173})$.

373. Given $9x-4x^2+\sqrt{(4x^2-9x+11)}=5$, to find x.
Ans. $x=2$, or $\frac{1}{4}$, or $\frac{1}{8}\{9\pm\sqrt{(-31)}\}$.

374. Given $x^2+\sqrt{(5x+x^2)}=42-5x$, to find x.
Ans. $x=4$, or -9, or $\frac{1}{2}(-5\pm\sqrt{221}.)$

375. Given $\frac{2}{(x+2)^{\frac{3}{2}}}+\frac{\sqrt{(x+2)}}{2}=\frac{17}{4\sqrt{(x+2)}}$. $x=6$, or $-\frac{3}{2}$.

376. Given $\frac{x}{x+4}+\frac{4}{\sqrt{(x+4)}}=\frac{21}{x}$, to find x.
Ans. $x=12$, or -3, or $\frac{1}{2}(49\pm\sqrt{3185})$.

377. Given $\frac{3x+5}{3x-5}-\frac{3x-5}{3x+5}=\frac{135}{176}$. $x=9$, or $-\frac{25}{81}$.

378. Given $x+\sqrt{x}+2=\frac{x^2+x-4}{\sqrt{x}}$. $x=4$, or 1.

379. Given $\frac{x^2}{(x^2-4)^2}+\frac{6}{x^2-4}=\frac{351}{25x^2}$. $x=\pm3$, or $\pm\sqrt{\frac{39}{11}}$.

380. Given $\left(x+\frac{8}{x}\right)^2+x=42-\frac{8}{x}$; $x=4$, or 2, or $\frac{1}{2}(-7\pm\sqrt{17})$.

381. Given $x+4-2\sqrt{\left(\frac{x+4}{x-4}\right)}=\frac{3}{x-4}$; $x=\pm5$, or $\pm\sqrt{17}$.

382. Given $x^4\cdot\left(1+\frac{1}{3x}\right)^2-(3x^3+x)=70$, to find x.
Ans. $x=3$, or $-\frac{10}{3}$, or $\frac{1}{6}\{-1\pm\sqrt{(-251)}\}$.

383. Given $x^2-\frac{5x}{2}+15=\frac{25x^2}{16}-\frac{64}{x^2}$, to find x.
Ans. $x=4$, or -8, or $\frac{1}{8}\{-2\pm2\sqrt{(-71)}\}$.

384. Given $\frac{35\frac{5}{7}}{\sqrt{(x^4-9x^2)}}+\frac{\sqrt{(x^2-9)}}{7x}=\frac{19}{2x}$, to find x.
Ans. $x=\pm5$, or $\pm\frac{1}{2}(\sqrt{15661})$.

385. Given $3\{(x-1)^2-x\}^2+2x=341+2\cdot(x-1)^2$, to find x.
Ans. $x=5$, or -2; or $\frac{3\sqrt{3}\pm\sqrt{(-109)}}{2\sqrt{3}}$.

386. Given $x^4+\frac{13}{3}x-39x=81$, to find x.
Ans. $x=\pm3$, or $\frac{1}{6}\{-13\pm\sqrt{(-155)}\}$.

387. Given $4x^4+\frac{1}{2}x=4x^3+33$, to find x.
Ans. $x=2$, or $-\frac{3}{2}$; or $\frac{1}{4}\{1\pm\sqrt{(-43)}\}$.

388. Given $(x-2)^2-6x^{\frac{1}{2}}.(x-2)=24-14x+15x^{\frac{1}{2}}$, to find x.
Ans. $x=16$, or 1; or $\frac{1}{2}\{\pm3\sqrt{(-11)}-1\}$.

389. Given $(4x+1)^2+4x^{\frac{1}{2}}.(4x+1)=1912-(10x+3x^{\frac{1}{2}})$, to find x.
Ans. $x=9$, or $\frac{49}{4}$; or $\frac{1}{8}\{-90\mp\sqrt{(-181)}\}$.

390. Given $8x^2-13=\frac{1}{2}(3x)+\sqrt{(6x^3+52x^2)}$, to find x.
Ans. $x=2$, or $-\frac{13}{8}$; or $\frac{1}{64}(3\pm\sqrt{3337})$.

391. Given $4x^2+21x+8x^{\frac{1}{2}}\sqrt{(7x^2-5x)}=207-4x$.

Ans. $x=3$, or $\frac{207}{16}$, or $\frac{1}{32}\{-129\pm3\sqrt{(-2567)}\}$

392. Given $x+4y=14$, and $y^2+4x=2y+11$, to find x and y.

Ans. $x=-46$, or 2, $y=15$, or 3.

393. Given $2x+3y=118$, and $5x^2-7y^2=4333$, to find x and y.

Ans. $x=35$, or $-\frac{3899}{17}$, and $y=16$, or $\frac{3268}{17}$.

394. Given $\frac{2x+7y}{4x}=2y-\frac{51+2x}{10}$, and $\frac{4x+3y}{16}=y-2$.

Ans. $x=5$, or $-\frac{56}{27}$, and $y=4$, or $\frac{640}{351}$.

395. Given $\frac{4xy+3y-3}{5x}-1=\frac{4y+3x-2}{5}-\frac{18-x}{3}$, and $\frac{3x+y}{7}=\frac{3x-5y}{3}+2$. $x=6$, or $-\frac{3}{286}$, and $y=3$, or $\frac{2784}{2527}$.

396. Given $x^2-y^2=a^2$; and $(x+y+b)^2+(x-y+b)^2=2c^2$.
Ans. $x=\frac{1}{2}\{-b\pm\sqrt{(2a^2-b^2+2c^2)}\}$; and $y=\pm\sqrt{(\frac{1}{2}\{c^2-a^2\mp\sqrt{(2a^2-b^2+2c^2)}\})}$.

397. Given $x^2+y:x^2-y::9:7$, and $1+x^2:y+4::5y+7:3y$, to find x and y. $x=\pm4$, or $\pm4\sqrt{(-\frac{7}{13})}$, or $y=2$, or $-\frac{14}{13}$.

398. Given $x^2+2x^3y=441-x^4y^2$, and $xy=3+x$, to find x and y. Ans. $x=3$, or -7, or $-2\pm\sqrt{(-17,)}$ and $y=2$, or $\frac{4}{7}$, or $\frac{1}{7}\{5\mp\sqrt{(-17)}\}$.

399. Given $x^2+4y^2=256-4xy$, and $3y^2-x^2=39$.

Ans. $x=\pm6$, or ±102, and $y=\pm5$, or ±59.

400. Given $(x+y)^2-3y=28+3x$, and $2xy+3x=35$.
Ans. $x=5$, or $\frac{7}{2}$, or $\frac{1}{4}\{-5\pm\sqrt{(-255)}\}$, and $y=2$, or $\frac{7}{2}$, or $\frac{1}{4}\{-11\mp\sqrt{(-255)}\}$.

401. Given $x^2+10x+y=119-2\sqrt{y}\times(x+5)$, and $x+2y=13$.
Ans. $x=5$, or $\frac{17}{2}$, or $\frac{1}{4}(-69\mp\sqrt{241})$, and $y=4$, or $\frac{9}{4}$, or $\frac{1}{8}\{121\pm\sqrt{241}\}$.

402. Given $\frac{x^4}{y^2}+\frac{2x^2}{y}=9\frac{39}{49}$, and $x^2+y^2=65$. Ans. $x=\pm4$, or $\pm\frac{4\sqrt{(-65)}}{7}$; or $\pm\frac{\sqrt{\{-450\mp30\sqrt{3410}\}}}{\sqrt{7}}$, and $y=7$, or $-\frac{65}{7}$; or $\frac{1}{7}(15\pm\sqrt{3410})$.

403. Given $x+y+\sqrt{(x+y)}=6$, and $x^2+y^2=10$.
Ans. $x=3$, or 1; or $\frac{1}{2}\{9\pm\sqrt{(-61)}\}$, and $y=1$, or 3; or $\frac{1}{2}\{9\mp\sqrt{(-61)}\}$.

404. Given $x^2+4\sqrt{(x^2+3y+5)}=55-3y$, and $6x-7y=16$.
Ans. $x=5$, or $\frac{1}{7}(-51)$; or $\frac{1}{7}(-9\pm\sqrt{3895})$, and $y=2$, or $-\frac{46}{7}$; or $\frac{1}{49}(-70\pm\sqrt{3895}.)$

405. Given $x^2+3x+y=73-2xy$, and $y^2+3y+x=44$.
$x=4$, or 16; or $-12\mp\sqrt{58}$, and $y=5$, or -7; or $-1\pm\sqrt{58}$.

406. Given $\frac{y}{(x+y)^{\frac{3}{2}}}+\frac{\sqrt{(x+y)}}{y}=\frac{17}{4\sqrt{(x+y)}}$, and $x=y^2+2$.
Ans. $x=6$, or 3, or $\frac{1}{32}\{9\mp3\sqrt{(-119)}\}$, and $y=2$, or -1, or $\frac{1}{8}\{(-3\pm\sqrt{(-119)}\}$.

407. Given $y-y^{\frac{1}{2}}=16-x$, and $28-y=x+4x^{\frac{1}{2}}$.
Ans. $x=4$, or $\frac{59}{17}|^2$, and $y=16$, or [illegible].

408. Given $x^4+y^4=97$, and $x+y=5$. Ans. $x=3$, or 2; or $\frac{1}{2}\{5\pm\sqrt{(-151)}\}$, and $y=2$, or 3; or $\frac{1}{2}\{5\mp\sqrt{(-151)}\}$.

409. Given $\sqrt{(\frac{3x-2y}{2x})}+\sqrt{(\frac{2x}{3x-2y})}=2$, and $x^2-18=x\cdot(4y-9)$.
Ans. $x=6$, or 3, and $y=3$, or $\frac{3}{2}$.

410. Given $x+4\sqrt{x}+4y=21+8\sqrt{y}+4\sqrt{xy}$, and $\sqrt{x}+\sqrt{y}=6$, to find x and y. $x=25$, or $\frac{25}{9}$, and $y=1$, or $\frac{169}{9}$.

411. Given $x+y=5$, and $(x^2+y^2)\times(x^3+y^3)=455$. Ans. $x=3$, or 2; or $\frac{5}{2}\pm\frac{1}{2}\sqrt{(-\frac{103}{3})}$, and $y=2$, or 3; or $\frac{5}{2}\mp\frac{1}{2}\sqrt{(-\frac{103}{3}.)}$

412. Given $x+y-\sqrt{(\frac{x+y}{x-y})}=\frac{6}{x-y}$, and $x^2+y^2=41$.
$x=\pm5$, or $\pm3\sqrt{\frac{5}{2}}$, $y=\pm4$, or $\pm\sqrt{\frac{37}{2}}$.

413. Given $\frac{x^4}{y^2}+\frac{y^4}{x^2}=136\frac{1}{9}-2xy$, and $x+4=14-y$. $x=6$, or 4; or $5\pm5\sqrt{(-\frac{13}{11})}$, and $y=4$, or 5; or $5\mp5\sqrt{(-\frac{13}{11})}$.

414. Given $\frac{x+y}{x-y}-\frac{x-y}{x+y}=4\frac{4}{5}$, and $\sqrt{(\frac{x-y}{x^4})}+\frac{1}{x}=\frac{4}{9\sqrt{(x-y)}}$
$x=3$, or $\frac{45}{2}$; or $\frac{3}{16}$, or $\frac{45}{32}$, and $y=2$, or $-\frac{135}{4}$; or $\frac{1}{8}$, or $-\frac{135}{64}$.

415. Given $\sqrt{(6\sqrt{x}+6\sqrt{y})}+\frac{1}{2}\sqrt{x}=9-\frac{1}{2}\sqrt{y}$, and $x-y=12$.
Ans. $x=16$, or $\frac{59536}{81}$, and $y=4$, or $\frac{58564}{81}$.

416. Given $y^4-432=12xy^2$, and $y^2=12+2xy$. $x=2$; $y=6$.

417. Given $\frac{4}{y^2}+\frac{4+y}{y}=\frac{8+4y}{x}+\frac{12y^2}{x^2}$, and $4y^2-xy=x$.
Ans. $x=2$, or $-\frac{50}{3}$, and $y=1$, or $-\frac{5}{3}$.

418. Given $\sqrt{\{(1+x)^2+y^2\}}+\sqrt{\{(1-x)^2+y^2\}}=4$, and $(4-x^2)^2=18\pm4y^2$, to find x and y.
Ans. $x=\pm1$, or $\pm\sqrt{10}$, and $y=\pm\frac{3}{2}$, or $\pm3\sqrt{(-\frac{1}{2})}$.

419. Given $\frac{x+\sqrt{x+y}}{x-\sqrt{x+y}}\;\frac{\sqrt{x-x-y}}{\sqrt{x+x+y}}=2\frac{9}{40}$, and $y^2-\sqrt{(xy^2)}=\frac{4}{3}x$, to find x and y. Ans. $x=9$, or $\frac{196}{9}$, or $\frac{289}{9}$, or 16, and $y=4$, or $=\frac{14}{3}$, or $-\frac{68}{9}$, or $\frac{4}{3}$.

420. Given $\frac{x+\sqrt{(x^2-y^2)}}{x-\sqrt{(x^2-y^2)}}=4\frac{1}{4}-\frac{x-\sqrt{(x^2-y^2)}}{x+\sqrt{(x^2-y^2)}}$, and $x.(x+y)=52-\sqrt{(x^2+xy+4)}$.

Ans. $x=\pm5$, or $\pm\frac{10}{\sqrt{3}}$, and $y=\pm4$, or $\pm\frac{8}{\sqrt{3}}$.

421. Given $5y+\frac{\sqrt{(x^2-15y-14)}}{5}=\frac{x^2}{3}-36$, and $\frac{x^2}{8y}+\frac{2x}{3}=\sqrt{(\frac{x^3}{3y}+\frac{x^2}{4})}-\frac{y}{2}$. Ans. $x=12$, or $-\frac{12}{2}$; or $\frac{1}{20}(25\pm\sqrt{41569})$; or $\frac{1}{4}(45\pm\sqrt{3849})$, and $y=2$, or $-\frac{19}{12}$; or $\frac{1}{120}(25\pm\sqrt{41569})$; or $\frac{1}{8}(135\pm3\sqrt{3849})$.

422. Given $\sqrt{(\frac{x+y^2}{4x})}+\frac{y}{\sqrt{(y^2+x)}}=\frac{y^2}{4}\cdot\sqrt{(\frac{4x}{y^2+x})}$ and $\frac{\sqrt{x}+\sqrt{(x-y-1)}}{\sqrt{x}-\sqrt{(x-y-1)}}=y+1$. $x=4$, or $\frac{1}{4}$; or $\frac{1}{8}\{1\mp3\sqrt{(-7)}\}$; and $y=2$, or -1; or $\frac{1}{2}\{-7\pm\sqrt{(-7)}\}$.

423. Given $\frac{x+y+\sqrt{(x^2-y^2)}}{x+y-\sqrt{(x^2-y^2)}}=\frac{9}{8y}.(x+y)$, and $(x^2+y)^2+x-y=2x.(x^2+y)+506$. Ans. $x=5$, or $-\frac{23}{5}$; or $\frac{1}{5}\{1\pm\sqrt{(-1209)}\}$, and $y=3$, or $-\frac{69}{25}$; or $\frac{1}{25}\{3\pm3\sqrt{(-1209)}\}$.

424. Given $\frac{y}{x}\cdot\sqrt{\frac{x}{y}}+\frac{1}{2}\cdot\sqrt{\frac{x}{y}}\cdot\sqrt[4]{\frac{y^3}{x^3}}=5$; and $\frac{2x^2}{y}-\frac{x}{3\sqrt{y}}=\frac{1}{3}$; $x=4$, or $\frac{16}{9}$, or $\frac{625}{64}$, or $\frac{625}{144}$, and $y=64$, or $\frac{256}{9}$, or $\frac{625}{32}|^2$, or $=\frac{625}{48}|^2$.

425. Given $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{61}{\sqrt{(xy)}}+1$, and $\sqrt[4]{(x^3y)}+\sqrt[4]{(y^3x)}=78$. Ans. $x=81$, or 16; or $-\frac{371}{12}\pm3\sqrt{47}$, and $y=16$, or 81; or $-\frac{371}{12}\mp3\sqrt{47}$.

426. Given $\frac{y}{2x}+\frac{2}{3}\cdot\frac{y-\sqrt{(x-1)}}{y^2-2\sqrt{(x^2-1)}}=\frac{\sqrt{(x+1)}}{x}$, and $\frac{1}{4}y^4=y^3x-1$. $x=\frac{5}{4}$, or $-\frac{85}{36}$, and $y=\pm2$, or $\pm\sqrt{\{-\frac{85}{18}+\frac{\sqrt{5929}}{648}\}}$

427. Given $x^2-y^2=3$, & $(x^4+y^4)^2+x^2y^2.(x^2-y^2)^2+x^2-y^2=328$. Ans. $x=\pm2$, or $\pm\sqrt{(-1)}$, or $\pm\sqrt{(\frac{1}{2}\{3\pm2\sqrt{(-13)}\})}$, and $y=\pm1$, or $\pm2\sqrt{(-1)}$, or $\pm\sqrt{(\frac{1}{2}\{-3\mp2\sqrt{(-13)}\})}$.

428. Given $\frac{2y^2-8\sqrt{x}}{\sqrt{x}}+\sqrt{(4y^2-16\sqrt{x})}=\frac{3\sqrt{x}}{2}$, and $\sqrt{x}+\sqrt{\{8\cdot(y-\sqrt{x})-4\}}=y+1$, to find the value of x and y.

Ans. $x=4$, or $\frac{4}{9}$, or $\frac{4}{25}$; or $-\frac{4}{3}\mp16\sqrt{-\frac{13}{3}}$; or $\frac{788\pm24\sqrt{644}}{25}$ and $y=3$, or, $\frac{5}{3}$; or $\frac{7}{5}$; or -1, or $1\pm2\sqrt{-\frac{13}{3}}$; or $\frac{37\pm\sqrt{644}}{5}$

429. A person being asked what o'clock it was, answered, that it was between 8 and 9, and that the hour and minute hands were exactly together. What was the time?

Let x denote the number of minutes past 8 o'clock; then, since the minute hand was at 12, when the hour hand was at 8, the hour hand was 40' before the minute hand; therefore $x-40$ will be the distance moved by the hour hand in x minutes, and $12:1::x:x-40$; hence $x=12x-480$, or $x=\frac{480}{11}=43'38''\frac{2}{11}$, the time past 8.

430. What two numbers are those, whose difference is 12, and their squares equal to each other?

Let x and y denote the numbers sought; then, by the question, $x-y=12$, and $x^2=y^2$. Hence $x=12+y$, and $x^2=144+24y+2y^2$; therefore $144+24y+y^2=y^2$, or $144+24y=0$; hence $y=-\frac{144}{24}=-6$, and $x=12-6=6$.

431. There is a certain number, consisting of two places which is equal to the difference of the squares of its digits; and if 36 be added to it, the digits will be inverted: quære the number.

Let x and y be the digits in the units' and tens' place, respectively; then, by the question, $10y+x=x^2-y^2$, and $10y+x+36=10x+y$. From the latter equation $y=x-4$; and this being substituted for y in the first equation, it becomes $11x-40=8x-16$, from whence $x=\frac{1}{3}(40-16)=8$, and $y=8-4=4$; therefore 48 is the number required.

432. Given $x^3+y^2=31$, and $y^3+x^2=17$, to find x and y.

Put $a=31$, and $b=17$; then, from the second equation, $x=\sqrt{(b-y^3)}$, and this value of x substituted in the first equation, gives $(b-y^3)^{\frac{3}{2}}+y^2=a$, or $(b-y^3)^3=(a-y^2)^2$; whence, by involution and transposing, $y^9-3by^6+y^4+3b^2y^3-2ay^2=b^3-a^2$; or, in numbers, $y^9-51y^6+y^4+867y^3-62y^2=3952$; where, by a few trials, we find $y=2$, and then $x=\sqrt{(b-y^3)}=3$.

433. Given $y^3-xy=666$, and $x^3+xy=406$.

Put $a=666$, and $b=406$; then, from the 2d equation, $y=\frac{b-x^3}{x}$; and this value of y being substituted in the first equation, gives $\frac{b^3-3b^2x^3+3bx^6-x^9}{x^3}-b+x^3=a$; and multiplying by x^3, and transposing, I have $x^9-(3b+1)x^6+(3b^2+b+a^2)x^3=b^3$; or, in numbers, $x^9-1219x^6+937470x^3=406^3$; from whence, by the resolution of a cubic equation, I have $x^3=343$; $\therefore x=7$, and $y=\frac{b-x^3}{x}=\frac{406-343}{7}=\frac{63}{7}=9$.

434. Given the sum of three numbers, in harmonical proportion, $=26$, and their continued product $=576$, to find the numbers.

Let x, y, and z, be the numbers; then, by the question, $x+y+z=26$, $x-y : y-z :: x : z$, and $xyz=576$. From the 2d equation $zx-zy=xy-xz$, or $xz=\frac{1}{2}y(x+z)$; and, from the first equation, $x+z=26-y$; hence $xz=\frac{1}{2}y(26-y)$, and $xyz=\frac{1}{2}y^2(26-y)=576$, or $26y^2-y^3=1152$. By resolving this equation, I find $y=8$; then $x+z=26-8=18$, and $xz=\frac{8}{2}\times 18=72$; hence $(x+x)^2-4xz=(x-z)^2=18^2-288=36$, and $x-z=6$; $\therefore x=\frac{1}{2}(18+6)=12$, and $z=\frac{1}{2}(18-6)=6$, and the numbers are 12, 8, 6.

435. What two numbers are those, whose difference, sum, and product, are to each other as the numbers 2, 3, and 5, respectively?

Let x and y denote the numbers; then, by the question, $2(x+y) = 3(x-y)$, or $2x+2y=3x-3y$; hence $x=5y$, $x+y = 6y$, and $xy = 5y^2$; therefore, by the question, $6y : 5y^2 :: 3 : 5$; hence $3y^2=6y$, or $y=2$, and $x=5y=10$.

436. To find that number whose cube being subtracted from its square, shall leave the greatest remainder possible.

Let x be the number; then, by the question, x^2-x^3 is to be the greatest possible. Suppose that x is increased by a small quantity, e, then $x^2-x^3=(x+e)^2-(x+e)^3 = x^2 - x^3+(2x-3x^2)e + (1 - 3x)e^2 - e^3$. Let $x^2 - x^3$ be taken from both sides of the equation, and let the remainder be divided by e, then we shall have $2x-3x^2+(1-3x)e-e^2= 0$; and, if we leave out the terms effected with e, on account of their smallness, then $2x-3x^2=0$, or $x=\frac{2}{3}$, the Answer.

437. It is required to find the least 3 whole numbers, so that $\frac{3}{8}$ of the first, $\frac{5}{14}$ of the second, and $\frac{7}{20}$ of the third, shall be all equal to each other.

Let x be the first number; then will $\frac{3}{8}x \times \frac{14}{5} = \dfrac{21x}{20}$, be the second; and $\dfrac{3}{8}x \times \dfrac{20}{7} = \dfrac{15x}{14}$, the third. Hence $\dfrac{x}{8}, \dfrac{x}{20}$, and $\dfrac{x}{14}$, must be whole numbers, or x must be divisible by 8, 20, and 14; but since the product of 20 and 14 is divisible by 8, it is necessary only that x be divisible by 20 and 14; that is, $\dfrac{x}{20}$ and $\dfrac{x}{14}$ must be whole numbers, which they evidently will be when $x=14\times20 = 280$; and then $\dfrac{21x}{20}=294$, $\dfrac{15x}{14}=300$, and the three numbers are 280, 294, and 300.

438. Given $zx^3+xz^3=290$, and $x^4+z^4=641$, to find x and z.

Put $a=290$, and $b = 641$; then the first equation, divided by xz, becomes $x^2+z^2=\dfrac{a}{xz}$; and if from the square of this equation we subtract the second equation, we have $2x^2z^2=\dfrac{a^2}{x^2z^2}-b$, or $2x^4z^4 + bx^2z^2=a^2$. This quadratic resolved gives $x^2z^2 = 100$, or $xz=10$; hence $x^2+z^2=\dfrac{a}{10}=29$, $x^2+z^2+2xz=(x+z)^2 = 29+20 = 49$, and $x^2+z^2-2xz=(x-z)^2=29-20=9$; whence $x+z=7$, and $x-z=3$; consequently $x=\dfrac{7+3}{2}=5$, and $z=\dfrac{7-3}{2}=2$.

239. Given the sum of 3 numbers in continued geometrical progression $= 39$, and the sum of their squares $= 819$, to find the numbers.

Put $a=39$, and $b=819$; then, by the question, $x+y+z=a$, $x^2+y^2+z^2=b$, and $xz=y^2$. From the first equation $x+z=a-y$; and, if from the square of this, the second equation be subtracted, there will remain $2xz-y^2=a^2-2ay+y^2-b$; but since $xz=y^2$, $2xz-y^2$ is $=y^2$; $\therefore$ $y^2=a^2-2ay+y^2-b$, or $2ay=a^2-b$, and $y=\frac{a^2-b}{2a}=\frac{1521-819}{78}=9$. Hence $x+z=a-9=30$, $xz=81$, and $x^2+z^2=b-81=738$; therefore $(x-z)^2=x^2+z^2-2xz=738-162=576$, and $x-z=24$; whence $x=\frac{1}{2}(30+24)=27$, and $z=3$; and the three numbers are 27, 9, and 3.

440. Required the least number of weights, and the weight of each, that will weigh from 1 pound to 2900 weight.

Let 1, a, b, c, d, e, &c. be the weights; then it is evident, that with the weights 1 and a, we can weigh 1, $a-1$, a, $a+1$ pounds; and since the least number of weights are to be employed, $a-1$ must be $=2$, or $a=3$; hence with the weights 1 and 3 we can weigh 1, 2, 3, & 4 lbs. Again, with the weights 1, a, & b, we can weigh, besides the weights already enumerated, $b-(a+1)$, $b-a$, $b-(a-1)$, $b-1$, b, $b+1$, $b+(a-1)$, $b+a$, and $b+(a+1)$, the least of which, or $b-(a+1)$, must be $=a+2$, or 5; hence $b=2a+3=3a=9$; and with the weights 1, 3, and 9, we can weigh from 1 pound to 13. Again, with the weights 1, a, b, and c, we can weigh, exclusive of the weights from 1 to 13, $c-(b+a+1)$, $c-(b+a)$, $c-(b+a-1)$, &c. to $c+b+a+1$; and $c-(b+a+1)$ must be $=b+(a+2)=14$, or $c=2b+2a+3=9a=27$.

By reasoning in the same manner, we shall find $d=2c+2b+2a+3=27a=81$, $e=2d+2c+2b+2a+3=81a=243$, $f=2e+2d+2c+2b+2a+3=243a=729$, &c. &c., so that the weights form the geometrical progression 1, 3, 9, 27 81, 243, 729, 2187 &c.; and as many of these weights must be used as will make up the given weight, or 3248 lbs. Now the sum of 8 terms of the above series is 3280, therefore 8 is the number of weights that must be used.

441. Required two numbers such, that their sum shall be equal both to their product and the difference of their squares.

Ans. 2.618034, and 1.618034.

442. Find the least 4 affirmative integers such, that the square of the greatest may be equal to the sum of the squares of the other three.

Let a, b, x, y, be four numbers, such, that $a^2+b^2+x^2=y^2$. Assume $y=x+1$; then $a^2+b^2+x^2=x^2+2x+1$, or $a^2+b^2=2x+1$; hence $x=\frac{1}{2}(a^2+b^2-1)$, where a and b may be taken at pleasure,

provided the one be an even and the other an odd number.

If $a=2$, and $b=3$, then $x=6$, and $y=7$; hence 2, 3, 6, and 7, are the least whole numbers that will satisfy the conditions of the question. If $a=3$, and $b=4$, then $x=12$, and $y=13$, the numbers in the book.

443. If money be lent at three per cent.
To those who choose to borrow,
In what time shall I be worth a pound,
If I lend a crown to-morrow?

Put $p=5s$, $r=1{\cdot}03$, the amount of £1 for 1 year; then, by the nature of compound interest, pr is the amount at the end of 1 year, pr^2 the amount at the end of 2 years, pr^3 the amount at the end of 3 years, and pr^x the amount at the end of x years; whence we have $pr^x=4p$, or $r^x=4$; therefore $x\times\log. r=\log. 4$, or $x=\frac{\log. 4}{\log. r}=\frac{.60206}{.0128372}=46.89963$; to which, if one day be added, we have 46.90236 years for the time required.

444. There are three numbers in geometrical proportion such, that, if the mean be subtracted from the sum of the two extremes, the remainder multiplied by the sum of the said two extremes will be $9\frac{1}{3}$; but, if that remainder be multiplied by the sum of all the three numbers, the product will be 133. It is required to find the three numbers by a simple equation.

Put $a=91$, and $b=133$, and let x, y, and z denote the three numbers. Put $x+z=s$, and $y=ns$; then, by the question, $(s-ns)s=a$, and $(s-ns)(s+ns)=b$. From the first we have $s^2=\frac{a}{1-r}$, and from the second, $s^2=\frac{b}{1-n^2}$; hence $\frac{a}{1-n}=\frac{b}{1-n^2}$ or $a(1-n^2)=b(1-n)$; and, dividing by $1-n$, we have $a+an=b$, or $n=\frac{b-a}{a}=\frac{42}{91}=\frac{6}{13}$, and $s=\sqrt{\left(\frac{a}{1-n}\right)}=13$. Hence $x+z=13$, and $y=6$; therefore $xz=y^2=36$, and $(z-x)^2=(x+z)^2-4xz=169-144=25$, or $z-x=5$; hence $x=\frac{1}{2}(13-5)=4$, and $z=\frac{1}{2}(13+5)=9$.

445. To determine two numbers whose sum shall be a cube, but their product and quotients squares.

Let x^2 and y^2 be the numbers; then, by the question, $x^2+y^2=$ a cube, $x^2y^2=$ a square, and $\frac{x^2}{y^2}=\square$. Now it is evident that the two last will be squares, whatever values be assigned to x and y; we have, therefore, only to make $x^2+y^2=$ a cube. Put $x=rz$, and $y=sz$; then $x^2+y^2=r^2z^2+s^2z^2=$ a cube. Assume its side $=\frac{z}{v}$; then $r^2z^2+s^2z^2=\frac{z^3}{v^3}$, or $r^2+s^2=\frac{z}{v^3}$, and $z=v^3(r^2+s^2)$. Hence

$x=rv^3(r^2+s^2)$, and $y=sv^3(r^2+s^2)$, where r, s, and v may be taken at pleasure.

If $v=1$, $r=1$, and $s=1$, then $x=2$, and $y=2$; and the numbers are 4 and 14. If $v=1$, $r=2$, and $s=1$, then $x=10$, and $y=5$, and the numbers are 100 and 25.

446. Required that arithmetical progression whose number of terms is 10, sum of the terms 185, and the sum of the cubes of the terms 104525.

Put $a=185\div10$, $b=104525$; and let $a-9z, a-7z, a-3z, a-5z, a-z, x+z, a+3z, a+5z, a+7z$, and $a+9z$, be the ten terms of the progression; then their sum is $=$ to $10a$, or 185. Also the sum of the cubes of the extremes $=2a^3+6\times9^2.az^2$; of the second and 9th terms $=2a^3+6\times7^2.az^2$; of the third and 8th, $=2a^3+6\times5^2.az^2$; of the 4th and 7th, $=2a^3+6\times3^2.az^2$; of the 5th and 6th, $=2a^3+6\times1^2.az^2$; and of all the terms $=10a^3+6az^2(9^2+7^2+5^2+3^2+1)=b$, whence $z^2=\frac{b-10a^3}{6a(9^2+7^2+5^2+3^2+1^2)}=\frac{41208\frac{3}{4}}{18315}=\frac{9}{4}$, and $z=\frac{3}{2}$. Hence $a-9z=\frac{185}{10}-\frac{27}{2}=5$, the first term; and by continually adding $2z$, or 3, we have 5, 8, 11, 14, 17, 20, 23, 26, 29, and 32, for the numbers.

447. To divide a given number (N) into 4 such parts that if any other number (n) be added to the first part, deducted from the 2d, multiplied by the third, and the 4th part divided by it, the sum, difference, product and quotient, shall be all equal to each other.

Let $x-n$, $x+n$, $\frac{x}{n}$, and nx, denote the numbers; then $x-n+x+n+\frac{x}{n}+nx=N$, or $2x+\frac{x}{n}+nx=N$; and, multiplying by n, $2nx+x+n^2x=nN$; whence $x=\frac{nN}{(1+n)^2}$, $x-n=\frac{nN}{(1+n)^2}-n$, $x+n=\frac{nN}{(1+n)^2}+n$, $\frac{x}{n}=\frac{N}{(1+n)^2}$, and $nx=\frac{n^2N}{(1+n)^2}$.

448. Given $x^3y+y^3x=512500$, and $x^2y-y^2x=5500$, to find x and y.

Put $a=512500$, and $b=2500$. Then, from the first equation multiplied by xy, take the square of the second, and there will remain $2x^2y^2=axy-b^2$; whence, by the resolution of a cubic equation, we get $xy=500$. But, from the second equation, $x-y=\frac{b}{xy}=\frac{2500}{500}=5$; hence $(x-y)^2+4xy=(x+y)^2=5^2+2000=2025$, and $x+y=45$; therefore $x=\frac{45+5}{2}=25$, and $y=\frac{45-5}{2}=20$.

449. Given $x+y+z=6$, $xy+xz+yz=11$, and $xyz=6$; to find x, y, and z.

From the second equation multiplied by z, let the third equation be subtracted, and there will remain $xz^2+yz^2=11z-6$, or $(x+y)z^2=11z-6$. But, from the first equation, $x+y=6-z$; hence, by substitution, $(6-z)z^2=11z-6$, or $z^3-6z^2+11z=6$. This cubic equation resolved gives $z=2$; whence $x+y=6-2=4$, and $xy=\frac{6}{2}=3$, therefore $(x-y)^2=(x+y)^2-4xy=16-12=4$, or $x-y=2$; hence $x=\frac{1}{2}(4+2)=3$, and $y=\frac{1}{2}(4-2)=1$.

450. To find two numbers in the ratio of 5 to 7, which, being respectively divided by 9 and 13, shall leave 3 and 8.

Let $5x$ and $7x$ be the numbers; then $\frac{1}{9}(5x-3)$ and $\frac{1}{13}(7x-8)$ must be whole numbers. First, $\frac{1}{9}(5x-3)\times2=\frac{1}{9}(10x-6)=x+\frac{1}{9}(x-6)=wh.$ or $\frac{1}{9}(x-6)=wh.=p$, and $x=9p+6$. Hence $\frac{1}{13}(7x-8)=\frac{1}{13}(63p+34)=4p+2+\frac{1}{13}(11p+8)=wh.$ or $\frac{1}{13}(11p+8)=wh.$; therefore $\frac{1}{13}(11p+8)\times6=\frac{1}{13}(66p+48)=5p+3+\frac{1}{13}(p+9)=wh.$ or $\frac{1}{13}(p+9)=wh.=r$, and $p=13r-9$; hence $x=117r-75$, where r may be taken at pleasure.

If $r=1$, then $x=117-75=42$, $5x=210$, and $7x=294$.

451. A, B, and C are to share \$100,000 between them, in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, respectively; but C's part being lost by death, it is required to divide the whole sum properly between the other two.

Since the shares of A and B are in proportion as $\frac{1}{3}$ to $\frac{1}{4}$, if $\frac{1}{3}x$ be put for the share of A, then will $\frac{1}{4}x$ be the share of B, and, by the question, $\frac{1}{3}x+\frac{1}{4}x=\$100000$; whence $\frac{1}{12}7x=100000$, or $x=\frac{1}{7}1200000=171428\frac{4}{7}$; therefore $\frac{1}{3}x=57142\frac{6}{7}$, and $\frac{1}{4}x=42857\frac{1}{7}$.

452. To find four numbers, x, y, z, and w, having the product of every three given, viz. $xyz=231$, $xyw=420$, $yzw=1540$, and $xzw=660$.

Put $a=231$, $b=420$, $c=1540$, and $d=660$; then $xyz=a$, $xyw=b$, $yzw=c$, $xzw=d$, by the question. By multiplying these equations together, and extracting the cube root, we have $xyzw=\sqrt[3]{(abcd)}=e=4620$; and if this equation be divided by each of the given ones, we shall have $w=\frac{e}{a}=20$, $z=\frac{e}{b}=11$, $x=\frac{e}{c}=3$, and $y=\frac{e}{d}=7$.

453. To find 3 numbers, x, y, and z, when the product of each by the sum of the other two are given; viz. $x\times(y+z)=48$, $y\times(x+z)=39$, and $z\times(x+y)=63$.

By subtracting the second equation from the first, we get $xz-xy=9$; and, if this be added to the third equation, we have $2xz=72$; hence $xz=36$, $xy=48-36=12$, and $yz=63-36=27$.

The three last equations being multiplied together, and the square root extracted, there results $xyz=\sqrt{(36\times12\times27)}=108$; and, if this equation be divided by each of those above, we have $y=\frac{108}{36}=3$, $z=\frac{108}{12}=9$, and $x=\frac{108}{27}=4$.

454. What number is that which, being any how divided, the square of one part, when added to the other part, shall always be a square number ?

Let x be the numbers, and y one of the parts ; then $x-y$ will be the other part ; and, by the question, y^2+x-y must be a □, and y, at the same time, indeterminate. Now it is evident that this can only be the case when x is $=\frac{1}{4}$, for then y^2+x-y is $=y^2-y+\frac{1}{4}=(y-\frac{1}{2})^2=(\frac{1}{2}-y)^2$. Hence $\frac{1}{4}$ is the number.

455. Given $y^3+z=127$, $y^3+x=135$, and $x^3+y^3+z^3=1133$, to find x, y, and z.

By subtracting the 1st equation from the 2d, we get $x-z=8$, or $x=8+z$; and, by the first equation, $y^3=127-z$; whence $x^3+y^3+z^3=(8+z)^3+127-z+z^3=1133$, or $2z^3+24z^2+191z=494$; and, by resolving this equation, we find $z=2$; hence $x=8+2=10$, and $y=\sqrt[3]{125}=5$.

456. Given $x^2+xy=108$, $y^2+yz=69$, and $z^2-xz=580$, to find x, y, and z.

Put $a=108$, $b=69$, and $c=580$; then, by the question, $x^2+xy=a$, $y^2+yz=b$, and $z^2+xz=c$. Put $my=x$, and $ny=z$; then we have $y^2(m^2+m)=a$, $y^2(1+n)=b$, and $y^2(n^2+nm)=c$; hence $y^2=\dfrac{a}{m^2+m}=\dfrac{b}{1+n}=\dfrac{c}{n^2+nm}$, or $a(1+n)=b(m^2+m)$, and $an(n+m)=c(m^2+m)$. From the first of these equations we have $n=\dfrac{b(m^2+m)-a}{a}$; and this value of n being substituted in the second equation, it becomes $\dfrac{(b(m^2+m)-a)(bm^2+(b+a)m-a)}{a}=c(m^2+m)$; or, by reduction, $b^2m^4+(ab+2b^2)m^3+(b^2-ab-ac)m^2-a(a+2b+c)m^2+a^2=0$. Or, in numbers, $m^4+\dfrac{82}{23}m^3-\dfrac{65331}{4761}m^2-\dfrac{89208}{4761}m-\dfrac{11664}{4761}=0$, from whence, by a few trials, we find $m=3$; then $n=\dfrac{b(m^2+m)-a}{a}=\dfrac{20}{3}$, $y=\sqrt{\left(\dfrac{a}{m^2+m}\right)}$; hence $x=9$, and $x=20$.

457. Find two mean proportionals between any two given numbers a and b.

Let x be the ratio of the terms ; then $a : ax : : ax^2 : b$, or ax^3

$=b$; whence $x=\sqrt[3]{\frac{b}{a}}$, $ax=a\sqrt[3]{\frac{b}{a}}=\sqrt[3]{a^2b}$, and $ax^2=a\sqrt[3]{\frac{b^2}{a^2}}=\sqrt[3]{b^2a}$.

458. Given $x+yz=384$, $y+xz=237$, and $z+xy=192$; to find x, y, and z.

Put $a=384$, $b=237$, and $c=192$; then by the second equation, $z=\frac{b-y}{x}$, and, by the third, $z=c-xy$; hence $\frac{b-y}{x}=c-xy$, or $b-y=cx-cx^2y$, and $y=\frac{cx-b}{x^2-1}$; therefore $z=c-xy=c-\frac{x(cx-b)}{x^2-1}$, and $yz=\frac{c(cx-b)}{x^2-1}-\frac{x(cx-b)^2}{(x^2-1)^2}$. This value of yz being substituted in the first equation, $x+yz=a$, it becomes $x+\frac{c(cx-b)}{x^2-1}-\frac{x(cx-b)^2}{(x^2-1)^2}=a$; or, $(x-a)(x^2-1)^2+c(cx-b)(x^2-1)-x(cx-b)^2=0$; from whence, by multiplication, we get $x^5-ax^4-2x^3+(2a+cb)x^2-(b^2+c^2-1)x=a-cb$; or, in numbers, $x^5-384-2x^3+46272x^2-43032x=-45120$; where, by a few trials, x is found $=10$; then $y=\frac{cx-b}{x^2-1}=17$, and $z=c-xy=22$.

459. To find three numbers such, that the sum or difference of any two of them shall be square numbers.

Put x, y, and z, to denote the numbers, and assume $x+y=a^2$, $x+z=b^2$, and $y+z=c^2$; then, by subtraction, $x-z=a^2-c^2=\square$, $x-y=b^2-c^2=\square$, and $y-z=a^2-b^2=\square$, and we have $x=\frac{a^2+b^2-c^2}{2}$, $y=\frac{a^2+c^2-b^2}{2}$, and $z=\frac{b^2+c^2-a^2}{2}$; therefore all the conditions of the question will be satisfied if we find three squares, a^2, b^2, and c^2, such, that the difference of each two may be a $\square$. Now I have found three such squares in the solution to the 27th question in the Diophantine Problems, viz. $a^2=485809$, $b^2=34225$, and $c^2=23409$; and, taking these numbers, I find $x=\frac{1}{2}(a^2+b^2-c^2)=248312\frac{1}{2}$; $y=\frac{1}{2}(a^2+c^2-b^2)=237496\frac{1}{2}$; and $z=\frac{1}{2}(b^2+c^2-a^2)=-214087\frac{1}{2}$; or, multiplying by 4, in order to have whole numbers, I get $x=993250$, $y=949986$, and $z=856350$, three numbers that will answer the conditions of the question.

By assigning different values to n in the solution to the 27th question, I find as many values of a, b, and c, as I please, and from thence deduce an indefinite number of answers. When a is $=2165$, $b=2067$, and $c=2040$, I find $x=2399057$, $y=2288168$, and $z=1873432$, the numbers.

460. Find two square numbers such, that their sum may be a

square, and their difference a cube, and the side of the said square and cube equal to each other.

Let $2rsx$ and $(r^2-s^2)x$ be the roots of the two squares; then $(2rsx)^2+(r^2-s^2)^2x^2=(r^2+s^2)^2x^2=$ a square; and, by the question, $(2rsx^2)-(r^2-s^2)^2x^2=(r^2+s^2)^3x^3$; or, dividing by x^2, $(2rs)^2-(r^2-s^2)^2=(r^2+s^2)^3x$, and $x=\frac{(2rs)^2-(r^2-s^2)s}{(r^2+s^2)^3}$, where r and s may be taken at pleasure.

If $r=2$, and $s=1$, then $x=\frac{4^2-3^2}{125}=\frac{7}{125}$, $2rsx=\frac{28}{125}$, and $(r^2-s^2)x=\frac{21}{125}$; hence $\frac{28^2}{125^2}=\frac{784}{15625}$, and $\frac{21^2}{125^2}=\frac{441}{15625}$.

462. Given the rates a and b of two ingredients, and the rate c of the compound m, to find what portions x and y of each must be taken to compose the mixture.

Here $ax+by=mc$, and $x+y=m$. From the second equation multiplied by a, let the first equation be subtracted, and there will remain $ay-by=(a-c)m$; hence

$$y=\frac{m(a-c)}{a-b}, \text{ and } x=m-y=m\left(1-\frac{a-c}{a-b}\right)=\frac{m(b-c)}{a-b}.$$

463. Given $x^2+xy+y^2=1087$, and $x^4+x^2y^2+y^4=4577295$.

Put $a=1087$, and $b=4577295$; then, from the first equation, $x^2+y^2=a-xy$; and, by squaring this equation, and transposing, we have $x^4+y^4=a^2-2axy-x^2y^2$. But, from the second equation, $x^4+y^4=b-x^3y^3$; whence $a^2-2axy-x^2y^2=b-x^3y^3$; or, by transposing, $x^3y^3-x^2y^2-2axy-2axy=b-a^2$, and by the resolution of a cubic equation, we find $xy=357$. Now let xy be added to the first equation; then we have $(x+y)^2=1087+357=1444$, or $x+y=38$; and, if $3xy$ be subtracted from the first equation, then $(x-y)^2=1087-1071=16$, and $x-y=4$. Hence $x=\frac{1}{2}(38+4)=21$, and $y=\frac{1}{2}(38-4)=17$.

464. Given $x+y+z=78$, $x^2+y^2+z^2=2546$, and $xy-xz-yz=527$, to find x, y, and z.

To the second equation add twice the third; then $x^2+y^2+z^2+2xy-2xz-2xy=3600$, the square root of which is $x+y-z=60$; and this being subtracted from the first equation, there remains $2z=78-60=18$; hence $z=9$, $x+y=60+9=69$, and $xz+yz=9\times69=621$. This last being added to the third equation, gives $xy=527+621=1148$; hence $(x+y)^2-4xy=(x-y)^2=4761-4592=169$, and $x-y=13$; therefore $x=\frac{1}{2}(69+13)=\frac{82}{2}=41$, and $y=\frac{1}{2}(69-13)=\frac{56}{2}=28$.

465. Given $x+y=152$, and $(x-y)^{\frac{2}{3}}\times(x-y)^{\frac{3}{2}}=8192$, to find x and y.

First, $(x-y)^{\frac{2}{3}}\times(x-y)^{\frac{3}{2}}=(x-y)^{\frac{13}{6}}=8192$; hence $(x-y)^{\frac{1}{6}}=\sqrt[13]{8192}=2$, and $(x-y)=2^6=64$; but $x+y=152$; hence $x=\frac{1}{2}(152+64)=108$, and $y=\frac{1}{2}(152-64)=44$.

466. Let the number of cards in a pack (p) be distributed into any number of heaps (n), by laying as many cards upon the bottom heap as are sufficient to make up its number q; then, by having the number of cards remaining in the dealer's hand (r), and the number of heaps (n) given, it is required to find the sum of all the bottom cards.

Let a, b, c, d, &c. be the number of the bottom card in the several heaps, and $x=a+b+c$, &c. Then it is evident that $q-(a+1)$ will be the number of cards in the first heap, $q-(b+1)$ the number in the second heap, $q-(c+1)$ the number in the third heap, and so on; therefore the number of cards in the n heaps will be $nq-n-x$. But these, with the cards that remain in the dealer's hand, must make up the whole pack; hence $nq-n-x+r=p$, or $x=(q-1)n+r-p$, the sum.

467. Find 3 numbers such, that if each be subtracted from the cube of their sum, the remainder shall be cubes.

Put $(r^6-a^3)x^3$, $(r^6-b^3)x^3$, and $(r^6-c^3)x^3$, for the three numbers, and r^2x for their sum; then, if each number be taken from the cube of the sum, there will remain the three cubes a^3x^3, b^3x^3, and c^3x^3; and all the conditions of the question will be satisfied if the sum of the three numbers be equal to the assumed sum r^2x, that is, if $\{(r^6-a^3)+(r^6-b^3)+(r^6-c^3)\}x^3$ be equal to r^2x, or $\{(r^6-a^3)+(r^6-b^3)+(r^6-c^3)\}x^2=r^2$; and it is evident that x will be a rational number when the coefficient of x^2 is a square. Now, since r may be assumed at pleasure, we have only to find 3 cubes such, that each of them being taken from the given squared cube r^6, the sum of the three remainders may be a square. This problem admits of an indefinite number of answers; but, to obtain the numbers in the book, let r be taken $=3$, and then r^6 is $=729=9^3$, and such values of a, b, and c must be found as will make the expression $3\times9^3-a^3-b^3-c^3$ a square. This will evidently be accomplished by taking $a=2$, $b=3$, and $c=6$; for then $3\times9^3-2^3-3^3-6^3=1936=44^2$, and we have $44^2x^2=r^2=9$, or $x=\frac{3}{44}$; whence $(r^6-a^3)x^3=721\times(\frac{3}{44})^3=\frac{19467}{85184}$; $(r^6-b^3)x^3=702\times(\frac{3}{44})^3=\frac{18954}{85184}$; $(r^6-c^3)x^3=513\times(\frac{3}{44})^3=\frac{13851}{85184}$. In this example, the roots a, b, and c, being small whole numbers, were easily found by inspection; but, as there are few cases in which this will be practicable, we shall show how they may be determined generally for any value of r. Let the two cube numbers, m^3 and n^3, be found, such, that if each of them be subtracted from the given squared cube r^6, the sum of the remainders may be a □ (s^2); and

let $m+z$, r^2-z, and n, be assumed for the roots of the three cubes a^3, b^3, and c^3. Then, if the cube of each number be subtracted from r^6, and the remainders added together, we have $(r^6-a^3)+(r^6-b^3)+(r^6-c^3)=2r^6-m^3-n^3-3(m+r^2)z^2+3(r^4-m^2)z=s^2-3(m+r^2)z^2+3(r^4-m^2)z=$ a square. Assume its root $=s+vz$; then, by reduction, we find $z=\dfrac{3(r^4-m^2)-2sv}{v^2+3(m+r^2)}$; where v may be taken at pleasure, provided $2sv$ be less than $3(r^4-m^2)$. From this value of z an indefinite number of answers may be deduced; but it will be necessary, in the first place, to show how the two cubes m^3 and n^3 may be found for any given value of r. Let n be assumed equal to r^2-m; then, if the cubes of m and n be taken from r^6, there will remain $r^2(r^4+3r^2m-3m^2)$, which is to be a $\square$, and $\therefore$ $r^4+3r^2m-3m^2$ must be a $\square$. Assume its root $=r^2+em$, then, by reduction, we obtain $m=r^2-\dfrac{3-2e}{3+e^2}$, and $n=r^2-m=r^2-\dfrac{e^2+2e}{3+e^2}$, where e may be any number less than $\dfrac{3}{2}$. To illustrate what has been done by an example, suppose $r=1$, and let e be also taken $=1$; then m is $=\frac{1}{4}$, and $n=\frac{3}{4}$; hence $r^6-m^3=1-\frac{1}{64}=\frac{63}{64}$, $r^6-n^3=1-\frac{27}{64}=\frac{37}{64}$, $s^2=\frac{63}{64}+\frac{37}{64}=\frac{100}{64}=\frac{25}{16}$, and $s=\frac{5}{4}$, $\therefore z=\dfrac{45-40v}{16v^2+60}$; and if v be taken $=\frac{1}{2}$, we have $z=\dfrac{25}{64}$, $a=(m+z)=\frac{41}{64}$, $b=r^2-z=\frac{39}{64}$, $c=n=\frac{3}{4}=\frac{48}{64}$, and $s+vz=\frac{185}{128}$, the root of the coefficient of x^2, when $x=\frac{128}{185}$, and $(r^6-a^3)x^3=\dfrac{193223\times 8}{185^3}=\dfrac{1545784}{6331625}$; $(r^6-b^3)x^3=\dfrac{202825\times 8}{185^3}=\dfrac{1622600}{6331625}$; $(r^6-c^3)x^3=\dfrac{151552\times 8}{185^3}=\dfrac{1212416}{6331625}$. If $n+z$, r^2-z, and m, be put for the roots instead of $m+z$, r^2-z, and n, we have $z=\dfrac{3(r^4-n^2)-2sv}{v^2+3(n+r^2)}=$(taking the same numbers as before) $\dfrac{21-40v}{16v^2+84}=\frac{1}{88}$. Hence $a=\frac{67}{88}$, $b=\frac{87}{88}$, $c=\frac{1}{4}=\frac{22}{88}$, and $s+vz=\frac{5}{4}+\frac{1}{176}=\frac{221}{176}$, the root of the coefficient of x^2; hence $x=\frac{176}{221}$, and $(v^6-a^3)x^3=\dfrac{380709}{88^3}\times\dfrac{2^3.88^3}{221^3}=\dfrac{3045672}{10793861}$; $(v^6-b^3)x^3=\dfrac{22969}{88^3}\times\dfrac{8\times 88^3}{221^3}=\dfrac{183752}{10793861}$; $(v^6-c^3)x^3=\dfrac{670824}{88^3}\times\dfrac{8\times 88^3}{221^3}=\dfrac{5366592}{10793861}$

468. Find 3 cube numbers such, that their sum shall be both a square and a cube number; and if that sum be squared it shall be a cube, and if it be cubed, it shall be a square.

Let x^3a^6, y^3a^6, and z^3a^6, be the numbers; then their sum is $=$

$(x^3+y^3+z^3)a^6$; and it is evident that all the conditions of the question will be answered if the sum of the cubes x^3, y^3, z^3, be both a square and a cube number. But it appears from the solution to the 62d question, following, that $\frac{1}{8}$, $\frac{8}{27}$, and $\frac{125}{216}$, are three cubes having that property. Hence $\frac{1}{8}a^6$, $\frac{8}{27}a^6$, and $\frac{125}{216}a^6$, are the numbers, where a may be taken at pleasure.

469. Given $xyz^{x^{\frac{1}{2}}}=z^{x^{\frac{3}{2}}}=100$, to find x and y.

Let X denote the log. of x, Z the log. of z, and l that of 100; then $z^{\frac{1}{2}}\times X=x^{\frac{3}{2}}\times z=l$, by the question and the nature of logarithms; whence $z=\frac{l}{x^{\frac{3}{2}}}$ Assume $x=47.5$, then $x^{\frac{3}{2}}=13.115$, and

$\frac{2.00000}{13.115}=z=0\cdot152421$, the log. of $1\cdot4215=z$, according to this

assumption; but $1\cdot4215^{\frac{1}{2}}\times X$ is $=1\cdot9986=$ log. of 99.69, which is too little by 0.31. If x be taken $=47\cdot6$ for a second assumption, then z comes out $=1\cdot42$, and the second error is $=0\cdot16$; whence $0\cdot5 : 0\cdot1 :: 0\cdot16 : 0\cdot106$, which, added to the last assumed value of x, gives 47.706 for its true value nearly, and then z is found $=1\cdot42$, &c.

470. Given $x^{3x}-21x^{2x}+147x^x=316$, to find x.

Put $x^x=y$, then $y^3-21y^2+147y=316$; and, by resolving this equation, we find $y=4$; hence $x^x=4$, and $x=2$.

471. Given $44000x^2+1=z^2$, to find x and z in whole numbers.

First, let $\frac{1}{20}y$ be substituted in the proposed equation, $4400x^2+1=z^2$, instead of x; then will $110y^2+1$ be $=z^2$. Now, z being greater than $10y$, assume $z=10y+b$, and we shall, by substitution and reduction, have $10y^2-20by=b^2-1$. In this last equation y is greater than $2b$; let, therefore, $y=2b+c$, and we shall have $b^2-20cb=10c^2+1$, from whence $b=10c+\sqrt{(110c^2+1)}$. Suppose now that c is $=0$, then we shall have $b=1$, $y=2$, $z=21$, and $x=\frac{2}{20}$; which last not being an integer, we must seek farther. To that end, take $c=2$, (the value of y just found,) then will $\sqrt{(110c^2+1)}$ be $=21$, (the above-mentioned value of z,) and $b=41$, $y=84$, $z=881$, and $x=\frac{84}{20}$; which not being an integer, we must proceed farther, by taking $c=84$, the last value of y; then will $\sqrt{(110c^2+1)}$ be $=881$, (the last value of z), and b will be $=1721$, $y=3526$, $z=36981$, and $x=\frac{3526}{20}$; which not being an integer, we must proceed yet farther, by taking $c=3526$, &c. At length, by proceeding in that manner, we find $x=40482981221781$, and $z=8491781781142001$.

453. Find three whole numbers such, that the excess of the greatest above the middle number shall be to the excess of the

middle number above the least, as 3 to 1; and also that the sum of every two of these shall be squares.

Let $\frac{1}{2}x^2+2y$, $\frac{1}{2}x^2-y$, and $\frac{1}{2}x^2-2y$, be the numbers; then the excess of the greatest above the middle number is to the excess of the middle number above the least, as 3 to 1; also the sum of the first and third is a square: we have, therefore, only to make the sum of the first and second, or x^2+y; and the sum of the second and third, or x^2-3y, into squares. Let $x+r$ be the side of the first, and $x-rs$ the side of the second, of these squares; then we shall have $x^2+y=x^2+2xr+r^2$, and $x^2-3y=x^2-2xrs+r^2s^2$. From the first equation $y=2xr+r^2$, and from the second, $y=\frac{1}{3}(2rsx-r^2s^2)$; hence $2xr+r^2=\frac{1}{3}(2rsx-r^2s^2)$, and $x=\dfrac{3r+rs^2}{2s-6}$, where r & s may be any numbers taken at pleasure, provided that s be greater than 3.

If s be taken $=4$, and $r=4$, then $x=38$, and $y=320$; hence $\frac{1}{2}x^2+2y=1362$, $\frac{1}{2}x^2-y=402$, and $\frac{1}{2}x^2-2y=82$; $\therefore$ 1362, 402, and 82, are three numbers that will satisfy the conditions of the question; that is to say, the sum of every two of them is a $\square$; and as this will also be the case when they are multiplied by any square number whatever, we may take $4^n\times 1362$, $4^n\times 402$, and $4^n\times 82$, for the three numbers, 4^n being always a square number.

473. Given $x+y=a(2)$, and $x^9+y^9=h$, to find x and y by quadratics.

Let $x+y=s$, and $xy=p$; then the sum of the nth powers of x and y is $s^n-ns^{n-2}p+n\cdot\frac{1}{2}(n-3)\cdot s^{n-4}p^2-n\frac{1}{3}(n-4)\cdot s^{n-6}p^3+n\frac{1}{2}(n=5)\cdot\frac{1}{3}(n-6)\cdot\frac{1}{4}(n-7)\cdot s^{n-8}p^4$, &c. (see page 281); $\therefore$, when $n=9$, we have $s^9-9s^7p+27s^5p^2-30s^3p^3+9sp^4=x^9+y^9=32$, and, by taking $s=2$, we get $3p^4-40p^3+144p^2-192p+80=0$; which being divided by p^2-4p+4, gives $3p-28p+20=0$; whence $p=\frac{1}{3}(14-2\sqrt{34})$; therefore $(x-y)^3=(x+y)^2-4p=4-4p$, or $x-y=2\sqrt{(1-p)}$, and $x=\frac{1}{2}\{2+2\sqrt{(1-p)}\}=1+\sqrt{(1-p)}=1.4697175$, and $y=\frac{1}{2}\{2-2\sqrt{(1-p)}\}=1-\sqrt{(1-p)}=\cdot5302824$.

474. Given $x^y=5000$, and $y^x=3000$, to find x and y.

Let $x^y=5000$, and $y^x=3000$; or, $y\times\log.x=\log.5000$, and $x\times\log.y=3000$. Now, by a few trials, x is found to be greater than 4, and less than 5, but rather nearer to the latter; let, therefore, 4·7 be taken for the assumed value of x; then $y=\dfrac{\log.5000}{\log.4\cdot7}=\dfrac{3\cdot698970}{\cdot673097}=5\cdot503617$, and $y^x=x\times\log.y=4\cdot7\times\log.5\cdot503617=4\cdot7\times\cdot7406483=3\cdot481047=\log.3027\cdot24$; hence $3027\cdot24-3000=27\cdot24$ = 1st error too great. Again, let 4·69 be assumed for a

second value of x; then $y = \dfrac{\text{log. } 5000}{\text{log. } 4·69} = \dfrac{3·698970}{·6711728} = 5·5112$, and $y^x = x \times \text{log.}y = 4·69 \times \text{log.}5·5112 = 4·69 \times ·7412462 = 3·4764446 = 2995·33$; hence $3000 - 2995·33 = 4·67$, the 2d error too little. Therefore $\dfrac{4·67 \times ·01}{27·24 + 4·67} = \dfrac{·0467}{31·91} = ·00146$, and $x = 4·69146$; and hence $y = 5·5102$, the two values extremely near. If the numbers be taken 500 and 300, then, by proceeding exactly as above, we shall find $x = 3·624$, and $y = 4·8265$.

475. Given $xy \times (x+z)^2 = 300$, $xz \times (y+z)^2 = 1296$, and $yz \times (x+y)^2 = 432$, to find x, y, and z.

Put $a = 300$, $b = 1296$, $c = 432$, $r = \dfrac{c}{b}$, $s = \dfrac{a}{c}$, $nx = y$, and $mx = z$; then, by the question, $nx^2 \times x^2(1+m)^2 = a$; $mx^2 \times x^2(n+m)^2 = b$; $nmx^2 \times x^2(1+n)^2 = c$. Hence $x^4 = \dfrac{a}{n(1+m)^2} = \dfrac{b}{m(n+m)^2} = \dfrac{c}{nm(1+n)^2}$; or, $cm(n+m)^2 = bnm(1+n)^2$, and $anm(1+n)^2 = cn(1+m)^2$. From the first of these equations we have $r(n+m)^2 = n(1+n)^2$; or, by extracting the root, and transposing, $m = (1+n)\sqrt{rn} - n$. Again, from the second equation, $sm(1+n)^2 = (1+m)^2$, or, by extracting the root, $(1+n)\sqrt{sm} = 1+m$; and if in this equation we substitute for m, its value found above, we shall have $(1+n)\sqrt{\{s(1+n)\sqrt{rn} - sn\}} = 1 + n\sqrt{rn} - n + 1$; from whence, by a few trials, we easily find $n = 3$; then $m = 1 + n\sqrt{rn} - n = 9$, $x = \sqrt[4]{\left(\dfrac{a}{n(1+m)}\right)} = 1$, $y = nx = 3$, and $z = mx = 9$.

476. Given $w^3 + x + y + z = 57$, $w + x^3 + y + z = 2763$, $w + x + y^3 + z = 1353$, and $w + x + y + z^3 = 153$, to find x, y, z, and w.

By subtracting the 1st equation from the 4th, we have $z^3 - z = 96 + w^3 - w$. Now, as w must be less than 4 by the first equation, and z less than 6 by the 4th, by a few trials we find $w = 3$, and $z = 5$. Again, by subtracting the first from the second, we have $x^3 - x = 2706 + w^3 - w = 2706 + 27 - 3 = 2710$, whence $x = 14$, and then $y^3 = 1353 - x - z - w = 1331$, or $y = 11$.

477. Given $x + y = 1750$, $xz + yv = 22708$, $xv + yz = 12292$, and $xzv + vzy = 159252$, to find x, y, z, and v.

Put $a = 1750$, $b = 22708$, $c = 12292$, and $d = 159252$; then, if the second and third equations be added together, we shall have $xz + yv + xv + yz$, or $(x+y)z + (x+y)v = b + c$; whence $z + v = \dfrac{b+c}{x+y} = \dfrac{b+c}{a} = 20$. Also, from the fourth equation, $zv(x+y) = d$, or $zv = \dfrac{d}{x+y} = \dfrac{d}{a} = 91$; hence $(z-v)^2 = (z+v)^2 - 4zv = 400 - 364$

$=36$, and $z-v=6$; therefore $z=\frac{1}{2}(20+6)=13$, and $v=\frac{1}{2}(20-6)=7$. Hence the third equation becomes $7x+13y=c$, and the first equation $7x+7y=7a$; whence, by subtraction, $6y=c-7a$, or $y=\frac{1}{6}(c-7a)=7$; therefore $x=1743$.

478. Find a square number such, that the sum of all its aliquot parts shall be a square number.

Let x^4 be the number required; then its aliquot parts are 1, x, x^2, and x^3; and their sum, $1+x+x^2+x^3$, must be a square. Now $1+x+x^2+x^3=(1+x)(1+x^2)$; and to find when this expression becomes a square, let $r^2(1+x)=s^2(1+x^2)$; then we have $x^2-\frac{r^2}{s^2}x=\frac{r^2-s^2}{s^2}$, and if $\left(\frac{r^2}{2s^2}\right)^2$ be added to both sides of the equation, then $x^2-\frac{r^2}{s^2}x+\frac{r^4}{4s^4}=\frac{r^2-s^2}{s^2}+\frac{r^4}{4s^4}$, or $\left(x^2-\frac{r^2}{2s^2}\right)^2=\frac{r^4+4r^2s^2-4s^4}{4s^4}$; therefore $r^4+4r^2s^2-4s^4$ must be a square. Put $r=s+v$, then the above expression will become $s^4+12s^3v+10s^2v^2+4sv^3+v^4$, which must be a square. Assume its side $=s^2+6sv-v^2$, then the square $=s^4+12s^3v+34s^2v^2-12sv^3+v^4$, and we have $10s^2v^2+4sv^3=34s^2v^2-12sv^3$, or $5s+2v=17s-6v$; hence $v=\frac{12}{8}s=\frac{3}{2}s$; where s may be taken at pleasure.

If s be taken $=2$, then $v=3$, and $r=s+v=5$: Also, $\frac{r^2}{2s^2}=\frac{25}{8}$, and $\frac{r^4+4r^2s^2-4s^4}{4s^4}=\frac{961}{64}$; $\therefore\ x-\frac{25}{8}=\sqrt{(\frac{961}{64})}=\frac{31}{8}$, or $x=7$, and $x^4=2401$, the numbers required.

479. Find two square numbers such, that either of them, when added to its aliquot parts, shall make the same sum.

Let $4x^2$ and $121y^2$ be the numbers required; then the aliquot parts of the first are $4x^2$, $2x^2$, x^2, $4x$, $2x$, x, 4, 2, and 1, and their sum $=7x^2+7x+7$. Also, the aliquot parts of the second are $121y^2+11y^2+y^2+121y+11y+y+121+11+1$, and their sum $=135y^2+133y+133$; whence, by the question, $7x^2+7x+7=133y^2+133y+133$, or $x^2+x=19y^2+19y+18$; and adding $\frac{1}{4}$ to both sides of the equation, we have $(x+\frac{1}{2})^2=19y^2+19y+\frac{73}{4}$; therefore $19y^2+19y+\frac{73}{4}$ must be a square, or $76y^2+76y+73=\square$. Put $y=1+v$, then the above expression becomes $225+228v+76v^2=\square$. Assume its side $=15-rv$; then we have $225+228v+76v^2=225-30rv+r^2v^2$, or $228v+76v^2=-30rv+r^2v^2$; from whence we have $v=\frac{228+30r}{r^2-76}$, where r may be taken at pleasure, provided its square be greater than 76.

Take $r=\frac{19}{2}$; then we have $v=\frac{2052}{57}=36$, and $y=37$; hence $x+\frac{1}{2}=\sqrt{(19y^2+19y+\frac{73}{4})}=163\frac{1}{2}$, or $x=163$; whence $4x^2=106276$, and $121y^2=165649$, the numbers required.

480. Find four whole numbers such, that the difference of every two shall be a square number.

Let w, x, y, and z, be the numbers; then, by the question, $w-x$, $w-y$, $w-z$, $x-y$, $x-z$, and $y-z$, are to be squares; and if we suppose $w=x+y+z$, then the three first become $y+z$, $x+z$, and $x+y$; so that we have to find three numbers, x, y, and z, such, that the sum and difference of every two of them may be a square. But we have already found three such numbers in the solution to the 459th question, viz. 1873432, 2288168, and 2399057; and therefore, if w be taken = to the sum of these three numbers, =6560657, we shall have four numbers that will answer the conditions of the question.

481. Find three numbers such, that if their sum be multiplied by the first, it shall be a triangular number, by the second a square, and by the third a cube.

Let x^3, y^3, and z^3, be the numbers, and let $x^3+y^3+z^3=v^6$, or $y^3+z^3=v^6-x^3$. Assume $y=\frac{v^4}{x^2}r-x$, and $z=v^2-r$; then y^3+z^3 $=\frac{v^{12}r^3}{x^6}-\frac{3v^8r^2}{x^3}+3v^2r^2-r^3-x^3+v^6=v^6-x^3$; or $v^{12}r-x^6r=(v^8-v^2x^3)\times 3x^2$, and $r=\frac{v^6-x^3}{v^{12}-x^6}\times 3v^2x^2=\frac{3x^2v^2}{v^6+x^3}$, where v and x may be any numbers taken at pleasure.

If $x=\frac{1}{2}$, and $v=1$, then $r=\frac{1}{3}$, $y=\frac{5}{6}$, $z=\frac{2}{3}$, and the three cubes are $\frac{1}{8}$, $\frac{8}{27}$, and $\frac{125}{216}$. If $x=1$, and $v=2$, then $r=\frac{12}{65}$, $y=\frac{127}{65}$, $z=\frac{248}{65}$, and the numbers are 1, $(\frac{127}{65})^3$, and $(\frac{248}{65})^3$.

482. Find three biquadrate numbers, the sum of which shall be a square.

Let $\frac{x}{v}$, $\frac{y^2}{v}$, and $\frac{z^3}{v}$, be the numbers, where x denotes a triangular number. Then it is evident that all the conditions of the question will be satisfied if $\frac{x}{v}+\frac{y^2}{v}+\frac{z^3}{v}=v$, or $x+y^2+z^3=v^2$; for then $\frac{x}{v}\times\left(\frac{x}{v}+\frac{y^2}{v}+\frac{z^3}{v}\right)=\frac{x}{v}\times v=x$, a triangular number; $\frac{y^2}{v}\times$ $\left(\frac{x}{v}+\frac{y^2}{v}+\frac{z^3}{v}\right)=\frac{y^2}{v}\times v=y^2$ a square; and $\frac{z^3}{v}\times\left(\frac{x}{y}+\frac{y^2}{v}+\frac{z^3}{v}\right)=$ $\frac{z^3}{v}\times v=z^3$, a cube. To find when $x+y^2+z^3$ will be a square, assume its side $=y+r$; then $x+y^2+z^3=y^2+2yr+r^2=(v^2)$, and $y=\frac{x+z^3-r^2}{2r}$, where x, z, and r may be taken at pleasure, provided that x be a triangular number, and r^2 less than $x+z^3$. If

$x=3$, $s=2$, and $r=1$, then $y=\frac{1}{2}(3+8-1)=5$, and $y+r=5+1=6$; hence $\frac{3}{6}$, $\frac{25}{6}$, and $\frac{8}{6}$, are the numbers.

483. Find a right-angled triangle such, that its perimeter shall be a cube, and the perimeter together with the area a square.

It appears from the solution to the 261st example of the Diophantine Problems, that r^2-s^2, $2rs$, and $2rs\frac{(r^2-s^2)}{r^2+s^2}$ are the roots of three biquadratics whose sum is a square; and if we multiply each of them by r^2+s^2, we shall have $(r^2-s^2)(r^2+s^2)=r^4-s^4$, $2rs(r^2+s^2)$ and $2rs(r^2-s^2)$ for the roots required; where r and s may be any two unequal numbers taken at pleasure. If $r=2$, and $s=1$, then $r^4-s^4=15$, $2rs(r^2+s^2)=20$, and $2rs(r^2-s^2)=12$; therefore 20^4, 15^4, and 12^4, are the numbers.

484. Determine the number of fifteens that can be made out of a common pack of 52 cards.

Cards.	*Combinations.*		*Cards.*	*Combinations.*	
10,5	16.4	64	7,3,2,2,1	4.4.6.4	384
10,4,1	16.4.4	256	7,3,2,1,1,1	4.4.4.4	256
10,3,2	16.4.4	256	7,2,2,2,2		4
10,3,1,1	16.4.6	384	7,2,2,2,1,1	4.4.6	96
10,2,2,1	16.6.4	384	7,2,2,1,1,1,1	4.6	24
10,2,1,1,1	16.4.4	256	6,6,3	6.4	24
9,6	4.4	16	6,6,2,1	6.4.4	96
9,5,1	4.4.4	64	6,6,1,1,1	6.4	24
9,4,2	4.4.4	64	6,5,4	4.4.4	64
9,4,1,1	4.4.6	96	6,5,3,1	4.4.4.4	256
9,3,3	4.6	24	6,5,2,2	4.4.6	96
9,3,2,1	4.4.4.4	256	6,5,2,1,1	4.4.4.6	384
9,3,1,1,1	4.4.4	64	6,5,1,1,1,1	4.4	16
9,2,2,2	4.4	16	6,4,4,1	4.6.4	96
9,2,2,1,1,	4.6.6	144	6,4,3,2	4.4.4.4	256
9,2,1,1,1,1	4.4	16	9,4,3,1,1	4.4.4.6	384
8,7	4.4	16	6,4,2,2,1	4.4.6.4	384
8,6,1	4.4.4	64	6,4,2,1,1,1	4.4.4.4	256
8,5,2	4.4.4	64	6,3,3,3	4.4	16
8,5,1,1	4.4.6	96	6,3,3,2,1	4.6.4.4	384
8,4,3	4.4.4	64	6,3,3,1,1,1	4.6.4	96
8,4,2,1	4.4.4.4	256	6,3,2,2,2	4.4.4	64
8,4,1,1,1	4.4.4	64	6,3,2,2,1,1	4.4.6.6	576
8,3,3,1	4.6.4	96	6,3,2,1,1,1,1	4.4.4	64
8,3,2,2	4.4.6	96	6,2,2,2,2,1	4.4	16
8,3,2,1,1	4.4.4.6	384	6,2,2,2,1,1.1	4.4.4	64
8,3,1,1,1,1	4.4	16	5,5,5		4
8,2,2,2,1	4.4.4	64	5,5,4,1	6.4.4	96

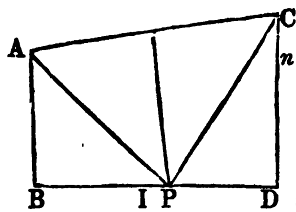
Cards.	Combinations	
8,2,2,1,1,1	4.6.4	96
7,7,1	6.4	24
7,6,2	4.4.4	64
7,6,1,1	4.4.6	96
7,5,3	4.4.4	64
7,5,2,1	4.4.4.4	256
7,5,1,1,1	4.4.4	64
7,4,4	4.6	24
7,4,3,1	4.4.4.4	256
7,4,2,2	4.4.6	96
7,4,2,1,1	4.4.4.6	384
7,4,1,1,1,1	4.4	16
7,3,3,2	4.6.4	96
7,3,3,1,1	4.6.6	444
5,3,3,2,1,1	4.6.4.6	576
5,3,3,1,1,1,1	4.6	24
5,3,2,2,2,1	4.4.4.4	256
5,3,2,2,1,1,1	4.4.6.4	384
5,2,2,2,2,1,1	4.6	24
5,2,2,2,1,1,1,1	4.4	16
4,4,4,3	4.4	16
4,4,4,2,1	4.4.4	64
4,4,4,1,1,1	4.4	16
4,4,3,1,3,1	6.6.4	144
4,4,3,2,2	6.4.6	144
4,4,3,2,1,1	6.4.4.6	576
4,4,3,1,1,1,1	6.4	24
4,4,2,2,2,1	6.4.4	96
4,4,2,2,1,1,1	6.6.4	144
4,3,3,3,2	4.4.4	64
4,3,3,3,1,1	4.4.6	96
5,5,3,2	6.4.4	96
5,5,3,1,1	6.4.6	441
5,5,2,2,1	6.6.4	144
5,5,2,1,1,1	6.4.4	96
5,4,4,2	4.6.4	96
5,4,4,1,1	4.6.6	144
5,4,3,3	4.4.6	96
5,4,3,2,1	4.4.4.4.4	1024
5,4,3,1,1,1	4.4.4.4	256
5,4,2,2,2	4.4.4	64
5,4,2,2,1,1	4.4.6.6	576
5,4,2,1,1,1,1	4.4.4	64
5,3,3,3,1	4.4.4	64
5,3,3,2,2	4.6.6	144
4,3,3,2,2,1	4.6.6.4	576
4,3,3,2,1,1,1	4.6.4.4	384
4,3,2,2,2,2	4.4	16
4,3,2,2,2,1,1	4.4.4.6	384
4,3,2,2,1,1,1,1	4.4.6	96
4 2,2,2,2,1,1,1	4.4	16
3,3,3,3,2,1	4.4	16
3,3,3,3,1,1,1		4
3,3,3,2,2,2	4.4	16
3,3,3,2,2,1,1	4.6.6	144
3,3,3,2,1,1,1,1	4.4	16
3,3,2,2,2,2,1	6.4	24
3,3,2,2,2,1,1,1	6.4.4	96
3,2,2,2,2,1,1,1,1		4

The number sought 17264.

If $(1+x)^4\times(1+x^2)^4\times(1+x^3)^4(9)\times(1+x^{10})^{16}$ be actually involved the coefficient of that power of x whose exponent is 15 will be the No. sought, to find the coefficient with facility, observe that the above expression is $=(1+x)^4\cdot(1+x^2)^4\times(1+x^4)^4\times(1+x^8)^4\times(1+x^3)^4\times(1+x^6)^4\times(1+x^5)^4\times(1+x^7)^4\times(1+x^9)\times(1+x^{10})^{16}=(1+x+x^2+x^3\ldots x^{15})^4\times(1+x^3+x^6+x^9)^4\times(1+x^5)^4\times(1+x^7)^4\times(1+x^9)^4\times(1+x^{10})^{16}$. Now $(1+x+x^2+x^3\ldots x^{15})^4$ being $=\left(\frac{1+x^{16}}{1-x}\right)^4=(1-4x^{16}+\frac{4.3}{1.2}x^{32}$ &c.$)\times(1+4x+\frac{4.5}{1.2}x^2+\frac{4.5.6}{1.2.3}x^3$, &c.) it appears, that in the value of $(1+x+x^2+x^3\ldots\ldots x^{15})^4$, the terms wherein the exponents of the powers of x are not greater than 15, are $1+4x+$

$\frac{4.5}{1.2}x^2+\frac{4.5.6}{1.2.3}x^3$, (16) $=\frac{16.17.18}{1.2.3}x^{15}+\frac{15.16.17}{1.2.3}x^{14}+\frac{14.15.16}{1.2.3}x^{13}$ ....$4x+1$. It plainly appears, also, by involution, that in the value of $(1+x^3+x^6+x^9)^4\times(1+x^5)^4\times(1+x^7)^4\times(1+x^9)^4\times(1+x^{10})^{16}$ the terms, wherein the exponents of the powers of x are not greater than 15, are $1+4x^3+4x^5+10x^6+4x^7+16x^8+24x^9+38x^{10}+40x^{11}+63x^{12}+128x^{13}+102x^{14}+212x^{15}$. It is obvious ∴ that the coefficient sought is $=1\times\frac{16.17.18}{1.2.3}+4\times\frac{13.14.15}{1.2.3}+4\times\frac{11.12.13}{1.2.3}+10\times\frac{10.11.12}{1.2.3}$; &c. (13)$=816+1820+1144+2200+660+1920+2016+2128+1400+1260+1280+408+212=17264$, as before, or 33528 holes in the cards or spots.

485. Two trees standing on an horizontal plane are 1352 feet asunder; the height of the higher tree = 185 feet, and that of the lower 153. Find that point in the plane which is equally distant from the top of each tree. See Index.



Let CD be the higher, and AB the lower of the 2 trees. Join AC, and bisect it in Q, Draw QP perpendicular to AC, meeting the plane in P, which is the point required. For since AQ=QC and QP is common, also the angle AQP=angle CQP, ∴ base AP = base CP. (Prop. 4, Euclid.) Suppose BD$=a$, BA$=b$, DC $=c$, and BP$=x$. Then AP$=\{b^2+x^2\}^{\frac{1}{2}}$ and CP $=\{(a-x)^2+c^2\}^{\frac{1}{2}}$. Now by the nature of the question AP= CP, ∴ $\{b^2+x^2\}^{\frac{1}{2}}=\{(a-x)^2+c^2\}^{\frac{1}{2}}$. Or $b^2+x^2=a^2-2ax+x^2+c^2$; ∴ $2ax=a^2+c^2-b^2$; ∴ $x=\frac{a^2+c^2-b^2}{2a}$=BP, which determines the point P to be 680 feet from B. and PD = 672 feet.

Rule. Divide the difference of the squares of the pillars by twice the line of distance; to the quotient add ½ the line of distance, which will give the distance from the lesser pillar or tree, to where the ladder is to be fixed; then (by Eu. b. 47,) the length of the ladder is found. Let $a=1352$; $b=185$; $c=153$; then $185^2=34225$, and $153^2=23409$, $1352\times2=2704$, Divisor. or $\frac{b^2-c^2}{2a}$ and $\frac{185^2-153^2}{2\times1352}=\frac{34225-23409}{2704}=\frac{10816}{2704}=4$, ∴ $\frac{1352}{2}\pm4=680$, or 672 and $680^2=462400$, $672^2=451584$, and $\sqrt{(451584+34225)}=\sqrt{485809}=697=\sqrt{(462400+23409)}=\sqrt{485809}=697$, the length of the ladder, Ans.

486. A man buys some ells of cloth for 70 dollars, and finds, that if he had four ells more, he had then bought every ell 2 dollars cheaper. How many ells did he buy? Ans. 10.

Let x denote the number of ells; then $\frac{70}{x}=\frac{70}{x+4}+2$, and by reduction $2(x^2+4x)+70x=70x+280$, or $x^2+4x=140$, by art. 70, case1, I have $x=12\pm2=10$, Ans.

487. To divide the number 21 into two parts, so that if the greater be divided by the lesser, and again the lesser by the greater, and then the first quotient being multiplied by 4, and the latter by 25, the numbers produced may be equal.

Let x denote one of the numbers; then $21-x$, will be the other part, and $\frac{x}{21-x} : \frac{21-x}{x} :: 25 : 4$; by the question, $\therefore$ $\frac{4x}{21-x}=\frac{25(21-x)}{x}$, or $4x^2=25\times(21-x)^2$, or $2x=5(21-x)=105-5x$, and $x=15$, and $21-15=6$, Ans.

488. Two country-women, A and B, carry 100 eggs together to market, and in the sale of them, one took as much money as the other; but A (who had the largest, and consequently the best eggs,) says to B, had I carried as many eggs as you, I should have had 18 cents for them; B replies, if I had brought as many eggs as you, I should have had but 8 cents for them. How many eggs had each?

Let x and y denote the number A and B each had; then $\frac{18}{y}=$ the price of one egg of A's in cents, and $\frac{8}{x}=$ the price of one of B's; $\therefore$ $\frac{18x}{y}=\frac{8y}{x}$, and $9x^2=4y^2$; $\therefore 3x=\pm2y$ the negative value of which will not answer the conditions of the question. $(x+y)=x+\frac{3}{2}x=100$; $\therefore (2x+3x)=5x=200$, and $x=40, y=60$.

489. Two merchants have a parcel of silk, the first 40 ells, the second 90. The first sells for a dollar $\frac{1}{3}$ of an ell more than the second. When the sale was over, they had taken between them 42 dollars. How many ells did each of them sell for a dollar?

Let x, $x+\frac{1}{3}$, denote the number B and A each sold for a dollar, and $x : 90 :: 1 :$ the price of 90 ells, and $x+\frac{1}{3} : 40 :: 1 :$ the price of 40 ells. $\therefore 42=\frac{90}{x}+\frac{120}{3x+1}$; or $7=\frac{15}{x}+\frac{20}{3x+1}$ whence, $21x^2+7x=45x+15+20x$; $\therefore$ by $21x^2-58x=15$.

By art. 70, case 3, $x=\pm\frac{34}{21}+\frac{29}{21}=3$, or $-\frac{5}{21}$; whence B sold 3 ells, and A $3\frac{1}{3}$, for a dollar.

490. A post sets out from A towards B, who travels 8 miles a day. After he had gone 27 miles, another sets out from B to meet

him, who goes every day $\frac{1}{20}$ of the whole journey or distance of the places A and B, and meets the first post after so many days as is $\frac{1}{20}$ of the said distance. Required the distance of A and B.

Let x denote the distance, then $\frac{1}{20}x =$ the number of miles B travelled per day, and also $=$ the number of days he travelled before he met A; $\therefore \frac{x^2}{400}+27+\frac{8x}{20}=x$, or $\frac{x^2}{400}-\frac{12x}{20}=-27$, or $\frac{x^2}{400}-\frac{12x}{20}+36=36-27=9$, or $\frac{x}{20}-6=\pm 3$, $\frac{x}{20}=9$, or 3, and $x=$ 180 or 60 miles. Both of these values answer the conditions of the problem; the distance, therefore, was 180, or 60 miles.

491. A man buys 120 pounds of pepper, and as many of ginger, and received for a dollar one pound of ginger more than of pepper, so that the whole price of the pepper came to 6 dollars more than the price of the ginger. How many pounds of each did he buy for a dollar?

Let x denote the number of pounds of pepper which he has for a dollar, and the number of pounds of ginger will be equal $\frac{120}{x}$ dollars, and that of the ginger $\frac{120}{x+1}$; $\therefore \frac{120}{x}-\frac{120}{x+1}=6$; or $x^2+6x=120x+120-120x$, and $x^2+x=20$, $x=4.5\pm 5=4$, and 5 pounds of ginger.

492. A certain linen-draper buys two sorts of linen for 30 dollars, one finer, the other coarser. An ell of the finest cost as many dollars as he had ells: And also 28 ells of the coarsest at (which was the whole quantity) were at such price, that 8 ells cost as many dollars as one ell of the finest. How many ells of the finest linen did he buy, and what price did he give for them both?

Let x denote the number of ells of the finer; $\therefore x^2 =$ the price of the finer in dollars, and $8 : 28 :: x :$ the price of the coarser $= \frac{7}{2}x$, $\therefore x^2+\frac{7}{2}x=30$; by art. 70, case 1, I have $x^2+\frac{7}{2}x+\frac{49}{16}=30+\frac{49}{16}=\frac{529}{16}$, or $x=\pm\frac{23}{4}-\frac{7}{4}=4$, or $-\frac{15}{2}$, and $\therefore$ the price of the finer $=$ \$16, and of the coarser, \$14.

493. A man buys 18 ells of cloth of different sorts and colors, suppose red and black; what he bought of each cost 40 dollars; and he pays for every ell of red cloth one dollar more than for the black. How many ells of each did he buy?

Let x and $18-x$ denote the number of black and red respectively; then by the question I have $\frac{40}{x}=\frac{40}{18-x}-1$; therefore $x^2+62x=720$, and $x=\pm 41-31=10$ yards of black, and $18-x=8=$ the red, Ans.

494. A certain traveller goes 9 miles a day, three days after ano-

ther follows him, who the first day travels 4 miles, the second 5, the third 6, and so on, gaining a mile every day. In what time will he overtake the former?

Put $a=9$, the distance travelled by A or the first per day, $b=27$, the distance gone by A before B set out, $c=4$, the miles travelled by B in the first day, and $d=1$ mile, the common difference or excess, and $x=$ the number of days required; then it is evident that the distance travelled by B in the last day will be $c+(x-1)d$, and that $\{2c+(x-1)d\}\frac{x}{2}$ will express the whole distance travelled by B in x days. But the distance travelled by A in x days being ax, the whole distance travelled by A will therefore be $ax+b$, which being equal to that of B, I have $\{2c+(x-1)d\}\frac{x}{2}=ax+b$, or $2cx+dx^2-dx=2ax+2b$, or $x^2+\frac{2c-2a-d}{d}x=\frac{2b}{d}$; or put $\frac{2c-2a-d}{d}=n$, then $x=\sqrt{\left(\frac{2b}{d}+\frac{n^2}{4}\right)}-\frac{n}{2}=$ 14.6877 days; or, $x^2+11x=54$; $\therefore x=\left(\sqrt{\frac{337}{4}}+5.5\right)=$ $9.17877985+5.5=14.6877$ &c. days, Ans.

495. Again: Two travellers set out at the same time from two cities, the one from A, and the other from B, which are 120 miles distant from one another; the first goes 5 miles a day, and the other 3 miles less than the number of days in which they meet. When will they meet?

Let x, and $120-x$, denote the number of miles A and B each went, and they met at the end of $\frac{1}{5}x$ days, (for A went 5 miles a day. Then B went $\frac{1}{5}x-3$ miles a day, and $\therefore$ B went in all $\frac{1}{5}x(\frac{1}{5}x-3)$ miles; $\therefore 120-x=\frac{1}{25}x^2-\frac{3}{5}x$, or $x^2+10x=3000$, and $x=55-5=50=$A's, and $120-50=70=$ B travelled, Ans.

496. A and B set out from Boston at the same time to go round the world, 23661 miles, one east, the other west. A goes 1 mile the first day, 2 the second, and so on. B goes 20 miles in a day. In how many days will they meet; and how many miles will be travelled by each?

Let $x=$ the number of days; then $(x+1)\frac{1}{2}x=$ the number of miles A goes, and $20x=$ the number of miles B goes; $\therefore \frac{1}{2}(x^2+x)+20x=23661$, or $x^2+41x=47322$, by art. 70, case 1, $x=\pm\frac{437}{2}-\frac{41}{2}=198$, or -239. Hence they travel 198 days. A goes 19701. and B 3960 miles.

497. Two men, A and B, set out at the same time; A travels 8 miles a day, and B travels the first day 1 mile, the second day 2 miles, the third day 3 miles, &c. In how many days will B overtake A?

If x be the number of days required, then A will travel $8x$

miles. Also, the sum of an arithmetical progression whose first term is 1, common difference 1, and number of terms x, is $\frac{1}{2}\{(x+1)\times x\}=\frac{1}{2}(x^2+x)$. Therefore B will travel $\frac{1}{2}(x^2+x)$ miles. But $\frac{1}{2}(x^2+x)=8x$ by question; therefore $\frac{1}{2}(x+1)=8$; and $x=15$.

498. Two post-boys, A and B, set out at the same time from two cities, which are 360 miles asunder, in order to meet each other; A rides 40 miles the first day, 38 the second, 36 the third, and so on, decreasing 2 miles every day; but B goes 20 miles the first day, 22 the second, 24 the third, &c. increasing two miles every day. In what number of days will they meet? See Index.

If $x=$ the time of their meeting; then x terms of $(40+38+36$, &c.$)=40x-\frac{1}{2}(x^2-x)\times 2$, and x terms of $(20+22+24$, &c.$)=20x+\frac{1}{2}(x^2-x)\times 2$. Now $40x-x^2+x+20x+x^2-x=360$. That is $(40x+20x)=60x=360$; $x=6$. days Ans.

499. Two travellers, A and B, set out together from the same place. A goes 8 miles the first day, 12 the second, 16 the third, and so on, increasing 4 miles every day; but B goes one mile the first day, 4 the second, 9 the third, and so on, according to the squares of the number of days. How many days must they travel before B overtakes A?

Suppose B will overtake A in x days; then x terms of $8+12+16$, &c. $=8x+\frac{1}{2}\{x\times(x-1)\}4$; and x terms of $1+4+9$, &c. $=\frac{x\times(x+1)\times(2x+1)}{1\times 2\times 3}$. $\therefore$ by the question $\frac{x\times(x+1)\times(2x+1)}{1\times 2\times 3}=8x+\frac{x\times(x-1)}{2}\times 4$. Or, by division, $\frac{(x+1)\times(2x+1)}{2\times 3}=8+(x-1)\frac{4}{2}$. Or $2x^2+3x+1=48+12x-12$; then $x^2-\frac{9}{2}x=\frac{35}{2}$; but $x^2-\frac{9}{2}x+(\frac{9}{4})^2=(\frac{81}{16}+\frac{35}{2})=\frac{361}{16}$; therefore $x-\frac{9}{4}=\frac{19}{4}$, and $x=(\frac{19+9}{4})=7$ days Ans. See Index.

500. There are two places 462 miles asunder, from which two persons, A and B, set out at the same time to meet each other. A goes 1 mile the first day, and increases each succeeding day's journey by 1 mile; and B travels each day the cube of the miles that A travels. In what time will they meet?

Suppose $x=$ the time of their meeting; then x terms of $1+2+3+4$, &c. $=\frac{1}{2}\{(x+1)\times x\}$, and x terms of $1+8+27+64$, &c. $=\frac{(x+1)^2\times x^2}{2\times 2}$. Now by question, $\frac{(x+1)^2\times x^2}{2\times 2}+\frac{(x+1)\times x}{2}=462$; and (putting $\frac{1}{2}\{(x+1)\times x\}=y$); $y^2+y=462$; but $y^2+y+\frac{1}{4}=(462+\frac{1}{4})=\frac{1849}{4}$; therefore $y+\frac{1}{2}=\frac{43}{2}$; and $y=(\frac{43-1}{2})=21$: but $y=\frac{1}{2}\{(x+1)\times x\}=\frac{1}{2}(x^2+x)=21$; therefore $x^2+x=42$; but $x+x^2+\frac{1}{4}=(42+\frac{1}{4})=\frac{169}{4}$; then $x+\frac{1}{2}=\frac{13}{2}$; and $x=(\frac{13-1}{2})=6$.

501. The discounting of two notes, whose value taken together was \$$308\frac{1}{3}$, cost, $8\frac{1}{3}$ one of them was due at the end of 6, the other

at the end of 8 months; and the interest of the two sums of money (each for its respective time) is $\frac{17}{72}$ more than the discount; the value of each note is required?

Let x = value of the note due at six months end, and r = the interest of \$1 for 1 year. Then $\frac{925}{3}-x$ = value of the note due at 8 month's end, $\frac{1}{2}(rx)$ = interest of the first for 6 months, $\frac{2}{3}\times\frac{1}{3}(925r)-\frac{2}{3}(rx)$ = interest of the 2d for 8 months, their sum $\frac{1}{9}(1850r)-\frac{1}{6}(rx)$ = sum of interests: consequenly I have, $\left(\frac{\frac{1}{2}rx}{1+\frac{1}{2}r}\text{ or }\frac{rx}{2+r}=\text{discount of 1st,}\right)\text{and}\left(\frac{2}{3}\times\frac{925r}{3}-\frac{2rx}{3}\right)$; or $\frac{\frac{1}{3}\cdot1850r-2rx}{3+2r}$ = discount of 2d. their sum $\frac{3700r-3rx+1850r^2}{3\times(6+7r+2r^2)}$ $=\frac{25}{3}=$ sum of discounts. Then. $\frac{3525r+1800r^2-150}{3r}=x$.

Also, $\frac{1850r}{9}-\frac{rx}{6}=\frac{25}{3}+\frac{17}{72}$. Th. $\frac{14800r-617}{12}=x$. Now $\frac{3525r+1800r^2-150}{3r}=\frac{14800r-617}{12r}$. Th. $r^2-\frac{7}{72}r=-\frac{17}{7200}$;

And $\left(\frac{7}{144}+\frac{1}{720}\right)=\frac{36}{720}=\frac{1}{20}=r$. Now

$$x=\left(\frac{14800\times\frac{1}{20}-617}{12\times\frac{1}{20}}\right)=\frac{14800-12340}{12}=\frac{2460}{12}=205.\text{ Ans.}$$

502. A man lent 186 dollars for x months, and gained thereby 31 dollars, and at the same rate of interest lending 360 dollars for y months, he gained 90 dollars. Required the values of x and y, when $x+y=20$. Solution: Let r = the interest of 1 dollar for 1 year; then $186r$ and $360r$ will represent the interest for 1 year of each sum, (viz. 186 and 360.) Then $\frac{186}{12}rx$ = interest for 186 dollars for x months, and $\frac{360}{12}ry$ = interest for 360 for y months; $\therefore\ \frac{186}{12}rx=31$; or $\frac{1}{12}r=\frac{31}{186x}=\frac{1}{6x}$; and $\frac{360}{12}ry=90$; or $\frac{r}{12}$ $=\frac{90}{360y}=\frac{1}{4y}$; $\frac{1}{6x}=\frac{1}{4y}$; or $4y=6x$; but $x+y=20$, then $x=20-y$, whence $4y=6(20-y)=120-6y$, or $10y=120$, then $y=12$. and $x=8$ Ans.

503. **A and B hired a pasture, into which A put 4 horses, and B as many as cost him 18 dollars a week; afterwards put in 2 additional horses, and found that he must pay 20 dollars a week. At what rate was the pasture hired?**

Let x denote the number of B's horses at first, $\frac{18}{x}$; $\frac{72}{x}$ and $\frac{72}{x}$

$+18$ be $=$ the pay of each per week in dollars $=$ what A paid $=$ the price of the pasture respectively, and $x+6=$ the whole number of horses in the second case; then $\therefore x+6 : x+2 : \frac{72}{x}+18 : 20 :: 72+18x : 20x$; or $20x^2+120x=18x^2+108x+144$; or $2x^2+12x=144$, or $x^2+6x=72$, and $x=6$ or -12; $\therefore$ B had 6 horses in the pasture at first, and $\frac{72}{x}+18=30$ dollars per week, was the price of the pasture.

4. A and B set out from 2 towns which were at the distance of 247 miles, and travelled the direct road till they met. A went 9 miles a day; and the number of days at the end of which they met, was greater by 3 than the number of miles which B went in a day. How many miles did each go?

Let $x=$ the number of days they travelled; $\therefore 9x=$ the number of miles A went, and $247-9x=$ the number B went, and the number B went per day was $\frac{247-9x}{x}$; $\therefore \frac{247-9x}{x}=x-3$; $\therefore x^2-3x=247-9x$, or $x^2+6x=247$. By Art. 70, Case 1, $x=\pm 16-3=13$, or -19, and therefore A went 117, and B 130 miles, Ans.

505. A and B set out to meet each other, A leaving the town C at the time that B left D, and on meeting it appeared that A had travelled 18 miles more than B, and that A could have gone B's distance in $15\frac{3}{4}$ days, but B would have been 28 days in going A's distance. Required the distance between C and D.

Let x, and $x-18$, denote the number of miles A and B each travelled, then $x-18 : x :: 15\frac{3}{4}$: the number of days A travelled, also $x : x-18 :: 28$: the number of days B travelled; $\therefore \frac{x-18}{15\frac{3}{4}}$ $=$ A's, and $\frac{x}{28}=$ B's daily progress respectively; $\therefore x : x-18 :: \frac{x-18}{15\frac{3}{4}} :: \frac{x}{28}$, or $\frac{28(x-18)}{x}=\frac{63x}{4(x-18)}$, or $16(x-18)^2=9x^2$, or $4(x-18)=\pm 3x$, and $x=72$, or $10\frac{2}{7}$; whence A travelled 72 miles and B 54 miles, and $\therefore$ the whole distance C D $=126$ miles.

6. A and B lay out some money on speculation. A disposes of his bargain for 11 dollars, and gains as much per cent. as he lays out. B's gain is 36 dollars; and it appears that A gains 4 times as much per cent. as B. Find the capital of each.

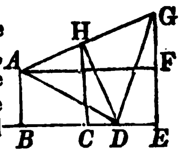
Let $4x=$ B's capital, and $\therefore$ A's gain per cent.; then $x=$ B's gain per cent., and $100 : 4x :: x : 36$, $\therefore 4x^2=3600$, and $x^2=9\times 100$; $\therefore x=\pm 30$, and $\therefore$ B's capital $=120$, and $220 : 10 :: 11$: A's capital $=\frac{11\times 10}{22}=5$.

507. A coach set out from Taunton to Boston with a certain number of passengers, 4 more being on the outside than within; 7 outside passengers could travel at \$2 less expense than 4 inside. The fares of the whole amounted to \$180; but at the end of half the journey, it took up 3 more outside and 1 more inside passengers, in consequence of which the fare of the whole became increased in the proportion of 17 to 15. Required the number of passengers, and the fare of the inside and outside.

Let x and $x+4$ denote the number of inside and outside passengers, and $y=$ the fare of an outside passenger; $\therefore \frac{1}{4}(7y+2)=$ fare of an inside passenger, and $\frac{1}{4}(xy+2x)+y(x+4)=\$180$. Also, $\frac{3}{2}y+\frac{1}{8}(7y+2)=$the fare of the passengers taken up half way $=\frac{1}{8}(19y+2)$; $\therefore \frac{1}{8}(19y+2) : 180 :: 2 : 15$, or $\frac{1}{8}(19y+2) : 12 :: 2 : 1$; $\therefore \frac{1}{8}(19y+2)=24$, and $19y+2=192$, or $y=\frac{190}{19}=10$; and from the first equa. $\frac{1}{4}(70x+2x)+10(x+4)=180$, or $28x=140$, and $x=5$; $\therefore$ there were 5 inside and 9 outside passengers, and the fares were 18 and 10 dollars respectively.

8. On a horizontal plane is a garden which was surrounded by a very high wall; but the owner, thinking the wall too high, had its top cut off, by a plane oblique to the horizon, so as to leave the highest point 16, and the lowest only 8 feet high: now the horizontal distance of these two parts is 20 yards; and if the foot of a ladder, of a certain length, be placed in a certain point in the garden, its top will just reach the top of the wall all around. Required the length of the ladder, and the area of the garden.

Draw the horizontal line BE $=60$ feet, and $\perp$ to it draw BA $=8$, and EG $=16$; join A, G, and draw AF $\parallel$ BE; bisect AG in H, and draw HC $\perp$ BE, and HD $\perp$ AG; and lastly, join D, A, and D, G. Then will D be the point where the foot of the ladder is to be placed, and DA $=$ DG the length of it; moreover, the form of the garden is an ellipse, whose conjugate axe is $=$ AF, and its transverse axe AG: all which is too evident to need any demonstration.



Calculation. Because H bisects AG, and HC is $\parallel$ EG, also CI $=$ BA $=$ EF, therefore HC $=\frac{1}{2}$BA $+\frac{1}{2}$EG $=12$; and by sim. $\triangle$S, AF : FG :: HC : CD $=12\times 8\div 60=1{\cdot}6$; hence ED $=\frac{1}{2}$BE $-$ CD $=28.4$, and then (by 47, Euc. 1) DG $=\sqrt{(DE^2+EG^2)}=32{\cdot}597$ feet, the length of the ladder; also AG $=\sqrt{(AF^2+FG^2)}=60{\cdot}531$ feet; whence $\cdot 7854$ AF $\times$ AG $=2852{\cdot}462$ square feet $=316{\cdot}94$ square yards, the area of the garden.

509. The sum of \$27 was to be raised by subscription by A, B, and C; the sums to be subscribed by them respectively forming an arithmetical progression. But C dying before the money was paid, the whole fell to A and B; and C's share was raised between them

in the proportion of 3 : 2, when it appeared that the whole sum subscribed by A was to the whole sum subscribed by B :: 4 : 5. Find the original subscription of A, B, and C.

Let $x-y$, x, $x+y$, be the respective subscriptions of A, B, and C; then $3x=\$27$, and $x=9$. Now 5 : 2 :: (C's share)$=9+y$: the part paid by B$=\frac{2}{5}(9+y)$, and 5 : 3 :: $9+y$: the part paid by A$=\frac{3}{5}(9+y)$, and ∴ A paid upon the whole $9-y+\frac{3}{5}(9+y)=\frac{1}{5}(72-2y)$; also B paid upon the whole $9+\frac{2}{5}(9+y)$ equal to $\frac{1}{5}(63+2y)$; $72-2y : 63+2y :: 4 : 5$, or $135 : 63+2y :: 9 : 5$, or $15 : 63+2y :: 1 : 5$, or $63+2y=75$, or $2y=12$ and $y=6$; ∴ the sums to be subscribed originally were \$3, \$9, and \$15.

510. Divide the number a into 2 such parts, that one may be to the other as m to n.

Let x and $a-x$ denote the two parts; and $x : a-x :: m : n$; or $nx=am-mx$, or, by transposition, $(m+n)x=am$. Dividing by $(m+n)$; $x=\frac{am}{m+n}=$ one part; ∴ $a-x=a-\frac{am}{m+n}=\frac{an}{m+n}$ = the other part.

511. Divide the number a into 2 such parts, that the sum of the quotients, which it contains, when one part is divided by m, and the other by n, may be $=b$.

Let x and $a-x$ denote the parts, and $\frac{x}{m}+\frac{a-x}{n}=b$; multiply by mn, $nx+am-mx=bmn$, or, by trans., $(n-m)x=m(bn-a)$, or dividing by $n-m$; one part will be $x=\frac{m(bn-a)}{n-m}$, and ∴ $a-x=a-\frac{m(bn-a)}{n-m}=\frac{n(a-bm)}{n-m}$. the other part.

512. The sum of two numbers is a, and their difference b; what are those two numbers?

Let x and y denote the numbers; then $x+y=a$, and $a-y=b$; subtract the one from the other, $2y=a-b$, or $y=\frac{1}{2}(a-b)$, and $x=a-y=a-\frac{1}{2}(a-b)=\frac{1}{2}(a+b)$.

513. What 2 numbers are those, whose sum is 28, and the sum of their squares 400?

Here put $a=28$, and $b=400$; then x, and $a-x$, will denote the two numbers; and per question $x^2+a^2-2ax+x^2=b$, or $2x^2-2ax=b-a^2$, or $x^2-ax=\frac{1}{2}(b-a^2)$, by art. 70, case 3; $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{(2b-a^2)}=16$, and $a-x=28-16=12$, the less number.

Otherwise, let $14+x$, and $14-x$, denote the two numbers; their sum is evidently 28.

$(14+x)^2=196+28x+x^2$, the square of the greater.
$(14-x)^2=196-28x+x^2$, the square of the less.

Sum of squares, $392 \quad * \quad +2x^2=400$, per question; by trans-

position and division I have $2x^2=8$, or $x^2=4$, and $x=2$; therefore $14+x=14+2=16$, and $14-x=14-2=12$, as before.

514. Required two numbers which are to each other as m to n, and the sum of their squares $=b$.

Let x and y denote the numbers; then I have $x:y::m:n$ and $x^2+y^2=b$, per question, from the first equation, $my=nx$, or $y=\frac{nx}{m}$, by substituting this in the 2d equation, I have $x^2+\frac{n^2x^2}{m^2}=b$, or $m^2x^2+n^2x^2=bm^2$, or $x^2=\frac{bm^2}{m^2+n^2}$; or $x=\frac{m\sqrt{b}}{\sqrt{(m^2+n^2)}}$; and $y=\frac{nx}{n}=\frac{nm\sqrt{b}}{m\sqrt{(m^2+n^2)}}=\frac{n\sqrt{b}}{\sqrt{(m^2+n^2)}}$. The preceding equation may be solved much shorter, as follows; but the shortest process may not always be the best model or pattern for a learner.

Let mx and nx denote the numbers; $\therefore m^2x^2+n^2x^2=b$; dividing by m^2+n^2; $x^2=\frac{b}{m^2+n^2}$, or $x=\sqrt{(\frac{b}{m^2+n^2})}$;

Therefore $mx=\frac{m\sqrt{b}}{\sqrt{(m^2+n^2)}}$, and $nx=\frac{n\sqrt{b}}{\sqrt{(m^2+n^2)}}$

515. A son asks his father how old he was? His father replies thus: If you take 4 from my age, the remainder will be three times the number of your years. But if you take 1 from your age, half the remainder will be the square root of my age. Required the age of the father and son.

Let x and y denote the father's and son's age; then $x-4=3y$, (1) and $y-1=2\sqrt{x}$, (2); by (1), $x=3y+4$, and by squaring (2) I have $y^2-2y+1=4x$; $\therefore$ by substitution $4(3y+4)=12y+16$, and $y^2-14y=15$, $\therefore y=15=$ son's age, and $49=$ the father's age.

516. Three towns, A, B, C, are obliged to furnish a given number (a) of soldiers. The number which A must provide : the number which B must provide $::m:n$; and the number of men which B must provide : the number which C must provide $::p:q$. Required the number from each town.

Let x denote A's part; then the 2d or B's part is $m:n::x:\frac{nx}{m}$, and $p:q::\frac{nx}{m}:\frac{nqx}{mp}=$ the third or C's part; $\therefore x+\frac{nx}{m}+\frac{nqx}{mp}=a$; or $mpx+npx+nqx=amp$, or $x=\frac{amp}{mp+np+nq}=$ A.

Now $\frac{nx}{m}=\frac{n(amp)}{m(mp+np+nq)}=\frac{anp}{mp+np+nq}=$ B's share. 2d part.

And $\frac{nqx}{mp}=\frac{nq(amp)}{mp(mp+np+nq)}=\frac{anp}{mp+np+nq}=$ C's part.

517. Divide a into 3 such parts that the first may be to the second as m to n, and the second to the third as p to q.

$$\begin{array}{ccc} \text{A's number} & \text{B's number} & \text{C's number} \\ \dfrac{amp}{mp+np+nq}; & \dfrac{anp}{mp+np+nq}; & \dfrac{anq}{mp+np+np}. \end{array}$$

518. Four places are situated in the order of the letters A, B, C, D. The distance from A to D is 34 miles. The distance from A to B is to the distance from C to D as 2 to 3. And $\frac{1}{4}$ of the distance from A to B, added to half the distance from C to D, is three times the distance from B to C. What are the respective distances?

Let $2=$ the distance from A to B, and $3x=$ the distance from C to D, and $(\frac{1}{2}x+\frac{3}{2}x)=2x=3.\text{BC}$, $\therefore$ $\text{BC}=\frac{2}{3}x$, and $(2x+3x+\frac{2}{3}x)=\frac{17}{3}x=34$, $\therefore$ $\frac{1}{3}x=2$, and $x=6$.

Ans. From A to $\text{B}=12$; from B to $\text{C}=4$; from C to $\text{D}=18$.

519. Divide 10,000 in 2 such parts, that when each is divided by the other, the sum of both quotients will make 5.

Let x and y denote the two numbers, s their sum, and q the sum of their alternate quotients; then $x+y=s$, and $\frac{x}{y}+\frac{y}{x}=q$. Hence $x^2+y^2=qxy$, and $x^2+2xy+y^2=s^2$, or, by substitution, $qxy+2xy=s^2$; $\therefore$ $4xy=\frac{4s^2}{q+2}$; and by subtraction $x^2-2xy+y^2=s^2-\frac{4s^2}{q+2}=\frac{q-2}{q+2}\times s^2$; or $x-y=s\sqrt{\left(\frac{q-2}{q+2}\right)}$, $\therefore$ $x=\frac{1}{2}s+\frac{1}{2}s\sqrt{\left(\frac{q-2}{q+2}\right)}$, and $y=\frac{1}{2}s-\frac{1}{2}s\sqrt{\left(\frac{q-2}{q+2}\right)}$. Ans. $\left\{\begin{array}{l}1726.73164\\8273.26835\end{array}\right.$

520. Out of annuity of \$1000 per annum, for 10 years, the first payment being due 1 year hence, the owner desires to know how much he may spend a year, so that his annual savings, with the simple interest arising therefrom, at 7 per cent. per annum, may at the expiration of this annuity, amount to a sum whose interest, at 7 per cent. per annum, shall be equal to the yearly expenditure.

Put $a=$ the annuity $=\$1000$, $t=$ the time of continuance $=$ 10 years, $r=$ the interest of \$1 for a year $=.07$, and $a-x=$ the expenditures required; then, by the question,
$\frac{(t-1)r+2}{2}\times trx=a-x$, and $x=\frac{2a=\$2000}{(t-1)tr^2+2tr+2}=\$520,6977$, and $a-x=479.3023$ dollars, the annual expenditure sought.

521. Borrowed \$5000, at 7 per cent. per annum, but I am, including the interest, to pay the lender annually 8 per cent. on the original sum. When will such payments discharge this debt?

Put $a=5000$, $p\times .08=400=a$, $r=\cdot07$, and $x=$ time required; then, from the question, $r=$ the rate of interest, $a=$ annuity, and $p=$ its present worth, are given to find the time of

continuance $x = \dfrac{\text{Log. } a - \text{Log. } (a-pr)}{\text{Log. } (r+1)} = 30.734$ years, the time.

522. The principal, time, and rate of interest being given, to find the amount, or money due at the end of that time, at simple interest; Let p = principal, t = time, r = rate of \$1 for a certain time, as a year, &c. s = sum of all the arrears.

then $1 : r :: p : rp$, the interest of p for one year;

$1 : rp :: t : prt$, the interest for the time t.

and $p + prt$ = the whole arrear at the end of the time t, and $p + prt = s$, the arrear sought. See pages 316-17, or 215-16.

Cor. 1. Hence $p = \dfrac{s}{rt+1}$; when s, r, and t are given.

Cor. 2. $t = \dfrac{s-p}{pr}$; where s, p, and r are given.

Cor. 3. $r = \dfrac{s-p}{pt}$; when s, p, and t are given.

523. The annuity, time, and rate of interest being given, to find the arrear, at the end of that time, at simple interest.

Put a = annuity or yearly rent; t = time of forbearance; r = interest of \$1 for a year; and s = the whole arrear; $\{0$ = interest due at one year's end; ra = interest at 2 years' end; $2ra$ = interest at 3 years' end; $3ra$ = interest for 4 years; $(t-1)ra$ = interest for t years$\}$; ta = rents due at the end of t years $= (0 + 1+2+3 \ldots\ldots \text{to } t-1 \times \text{ into } ra + ta = s)$. By Arith. Progression, $0+1+2+3 \ldots\ldots t-1 = \frac{1}{2}\{t(t-1)\}$, and $\dfrac{t(t-1)}{2}ra + ta = s$, or $\dfrac{(t-1)r+2}{2}ta = s$, and $a = \dfrac{2s}{\{(t-1)r+2\} \times t}$; and $t = \sqrt{\left\{\dfrac{2s}{ar} + \left(\dfrac{2-r}{2r}\right)^2\right\}} - \dfrac{2-r}{2r}$, and $r = \dfrac{2s-ta}{(t-1)\times ta}$.

524. Find the present worth of an annuity, to continue a given time, at a given rate of simple interest.

Let p = present worth, a = annuity, t = time, r = interest of \$1; then $p + prt = s$, and $\dfrac{(t-1)r+2}{2}ta = s = p + prt$, or $p = \dfrac{(t-1)r+2}{2rt+2}ta = \dfrac{\frac{1}{2}(t-1)r+1}{rt+1}ta$, $\therefore a = \dfrac{rt+1}{\frac{1}{2}(t-1)r+1} \times \dfrac{p}{t}$, $t^2 + \left(\dfrac{2}{r} - \dfrac{2p}{a} - 1\right)t = \dfrac{2p}{ra}$, $\therefore r = \dfrac{2ta - 2p}{\{2p-(t-1)a\}t}$, where t may be found, by art. 70 case 1.

525. The principal, time, and rate of interest being given, to find the amount at the end of that time, at compound interest.

Let $p =$ principal, $t =$ time, $r =$ interest of \$1; $R = 1 + r$, the amount of \$1 and its interest, $s =$ sum of money due at the end of that time; then $1 + r$ or $R =$ money due at 1 year's end; then, as before treated, pages 316-17, or 215-16, I have
$1 : R :: R : R^2 =$ money due at 2 years' end;
$1 : R :: R^2 : R^3 =$ money due at 3 years' end, and $R^t =$ money due at t year's end; $1 : R^t :: p : pR^t =$ the amount of p for the time t, and $pR^t = s$; then I have, cor. 1, 2, and 3, $p = \frac{s}{R^t}$; $R^t = \frac{s}{p}$, or $t = \frac{\text{Log.}\,s - \text{Log.}\,p}{\text{Log. R}}$; $R = \sqrt[t]{\frac{s}{p}}$, or $\text{Log. R} = \frac{\text{Log.}\,s - \text{Log.}\,p}{t}$.

526. The annuity, time, and rate of interest being given, to find the arrears due at the end of that time, at compound interest.

Let $a =$ annuity or yearly rent, $t =$ time of forbearance, $r =$ interest for 1 dollar for a year, &c., $R = 1 + r$, $s =$ sum of all the arrears, $a =$ money due at one year's end. $2a + ra = a + Ra =$ arrears at 2 years' end, $a + aR + aR^2 =$ arrear in 3 years; $a + aR + aR^2 + aR^3 =$ arrears for 4 years, and $a + aR + aR^2 + aR^3 \ldots$ to $a.R^{t-1} =$ arrears for t years; then, by pages 192-3, Geometrical Progression, $1 + R + R^2 + R^3 \ldots$ to $R^{t-1} = \frac{R(R^{t-1} - 1)}{R - 1} = \frac{R^t - 1}{r}$; and $\frac{R^t - 1}{r}a =$ money due at the end of t years; $\frac{R^t - 1}{r}a = s$; $a = \frac{rs}{R^t - 1}$; $\therefore R^t = \frac{rs}{a} + 1$, or $t = \frac{\text{Log.}\,(\frac{rs}{a} + 1)}{\text{Log. R}}$, and cor. 3

$\frac{s}{a}R - R^t = \frac{s - a}{a}$, where R may be found, and then r.

527. Find the present worth of an annuity, to continue a given time, at a given rate of compound interest.

Let $p =$ present worth, $a =$ the annuity, $t =$ the time, $r =$ interest of 1 dollar; $R = 1 + r$, and I have found $pR^t = s$, $\frac{R^t - 1}{r}a = s$; $pR^t = \frac{R^t - 1}{r}a$, and $p = \frac{R^t - 1}{rR^t}a = \frac{1 - \frac{1}{R^t}}{r}a$;

Cor. 1. $a = \frac{pr}{1 - \frac{1}{R^t}}$. 2. $R^t = \frac{a}{a - pr}$, or $t = \frac{\text{Log.}\,a - \text{Log.}\,(a - pr)}{\text{Log. R}}$.

Cor. 3. $R^t + \frac{a}{p}R^t - R^{t+1} = \frac{a}{p}$, $\therefore$ R and r will be found.

528. To find the value of an annuity to continue forever, at a given rate of compound interest.

Let $p =$ present worth, $a =$ annuity, $r =$ interest for 1 dollar, and $R = 1 + r$; then $p = \frac{R^t - 1}{rR^t}a$, or $p = \left(\frac{R^t}{rR^t}a\right) = \frac{a}{r}$, or $r = \frac{a}{p}$

and $a = pr$; but in the first value of p. where t is infinite, R^t is infinitely greater than 1, where $R^t - 1 = R^t$, and $p =$ as before.

529. At what rate of interest will \$100 amount to \$200 in $9\frac{3}{4}$ years, at compound interest?

Put $r =$ rate of \$1; $R = 1 + r$, $t = 9\frac{3}{4}$; then by 525, I have $100R^{\frac{39}{4}} = 200$ per question; $\therefore R^{\frac{39}{4}} = 2$, or $R^{39} = 16$, and $R = \sqrt[39]{16} = 1.0737$ by logarithms; $R - 1 = r = .0737$, and then $100 \times .0737 = 7.37 =$ the rate per cent.

530. If a principal, x, be put out at compound interest, for x years, at x per cent., to find the time, x, in which it will gain x, I have found $pR^t = s$, and $p = x$, $r = \frac{x}{100}$, $R = 1 + \frac{x}{100}$, $t = x$, and $s = 2x$, $\therefore x \times \left(1 + \frac{x}{100}\right)^x = 2x$, or $\left(1 + \frac{x}{100}\right)^x = 2$, (by Log.;) $x \times \left(1 + \frac{x}{100}\right)M = .3010300$, and by page 301-4, I have $x \times \left(\frac{x}{100} - \frac{x^2}{20000} + \frac{x^3}{3000000}\right)$, &c., $= \frac{.3010300}{M}$; $\therefore \frac{x^2}{100} - \frac{x^3}{20000} + \frac{x^4}{3000000}$, &c. $= ,693147$, and $x = 8.49824$ years.

531. Given the rate per cent. for 1 year \$5, to find what the amount of any sum, \$100, will be at the year's end, at compound interest, supposing it to arise from the principal and interest due every day, &c.

Let $r =$ interest of \$ for a year, $n = 365$, the parts of a year, $\frac{r}{n} =$ interest for one day; $1 + \frac{r}{n} =$ money due at one day's end; and $\left(1 + \frac{r}{n}\right)^n =$ money due at the year's end, (by Log's, p. 301;) $n \times \log. \left(1 + \frac{r}{n}\right) =$ log. amount for a year $= .0215694$, and $1.0509 =$ amount for a year; $105.09 =$ amount of \$100; $\left(1 + \frac{r}{n}\right)^n = 1 + r + \frac{n(n-1)}{2n^2}r^2 + \frac{n(n-1)(n-2)}{2.3n^3}r^3$, &c., the amount for a year.

If the interest is supposed to gain interest every moment, by becoming part of the principal, then n is infinite, and $\left(1 + \frac{r}{n}\right)^n = 1 + r + \frac{r^2}{2} + \frac{r^3}{2.3} + \frac{r^4}{2.3.4}$, &c., the amount at the year's end; but this series is the number belonging to the hyperbolic logarithm r, whence the number belonging to the log. $.43429448r =$ amount of \$1 for a year $= 1.0513$; and from 100

= 105,13 to gain interest continually. If the interest for a day be required, so that it may amount to $1+r$ at the year's end, at compound interest; then the amount at 1 day's end will be $\sqrt[n]{(1+r)}$; which is something less than $1+\frac{r}{n}$.

532. A man puts out a sum of money, at 6 per cent., to continue forty years, and then both principal and interest is to sink; what is that per cent to continue forever? Or, if $100 be paid for annuity of $6 a year for forty years, what is that per cent.?

Put $a=6$, $p=100$, $t=40$, $r=$ rate of $1, $R=1+r$; then $t=\frac{\text{Log. }a-\text{Log. }(a-pr)}{\text{Log. R}}$; $\text{Log. R}=\frac{\text{Log. }a-\text{Log. }(a-pr)}{t}=\frac{\text{Log. }6-\text{Log. }(6-100r)}{40}$. $\left\{\begin{array}{l}\text{Suppose R}=1.05\text{; then }r=.05\text{, and}\\ \text{Log. }r=.019454\text{; whence R}=1.046,\end{array}\right.$ which is too little. Let $R=1.053$; then $r=.053$, and Log. R $=.023324$, and $R=1.055$, too large; then, by p. 272, art. 111, I find $R=1.052$, and the rate $=5.2$ per cent.

533. If $200 be due 3 years hence; $80, 5 years hence; in what time must both be paid together, at 5 per cent.?

Let $t=$ the time; then $\frac{200}{(1.05)^5}+\frac{80}{(1.05)^5}=(172.76+62.68)$ $=\$235.44=$ the whole present worth of both sums; and then $t=\frac{\text{Log. }280-\text{Log. }235.44}{\text{Log. }1.05}=3.5527$ years, Ans.

534. What must I pay for an annuity of $70, to begin 6 years hence, and then to continue 21 years, at 5 per cent.?

Let $a=70$, $t=21$, $R=1.05$, $x=6$; then $\frac{s}{R^x}=\frac{1-\frac{1}{R^t}}{rR^x}a=\frac{R^t-1}{rR^{t+x}}a=\$669.704$; $\frac{1-\frac{1}{R^t}}{r}a=$ the present worth of the annuity 7 years hence $=s$, and the present worth of s, 7 years hence, $=669.704$, the present worth of the annuity in reversion.

535. The discount of a bill came to $112. Had the rate per cent. been $1 more, it would have cost $126, and if the rate per cent. had been $1 less, only $96. Required the value of the bill, time when due, and the rate of interest.

Let $p=$ principal, $n=$ time, and $r=$ interest of $100 in one year, $t=$ rate of interest, and $a=112$, $b=126$, and $c=96$; then $\frac{rtp}{100+rt}=a$; $p=\frac{100a+art}{rt}$, and $\frac{(r+1)tp}{100+(r+1)t}=b$, $\therefore$ $p=\frac{100b+bt(r+1)}{(r+1)t}$; $\frac{(r-1)tp}{100+(r-1)t}=c$, $\therefore p=\frac{100c+ct(r-1)}{(r-1)\times t}$; then $\frac{100a+art}{r}=\frac{100b+bt(r+1)}{r+1}$, or $t=\frac{(r+1)100a-100br}{(b-a)r\times(r+1)}$.

Also $\frac{100a+art}{r}=\frac{100c+ct\times(r-1)}{r-1}$, or $t=\frac{100cr-(r-1)\times100a}{(a-c)\times r\times(r-1)}$; then $t=t$. By ques. $\frac{(r+1)100a-100br}{(b-a)\times r\times(r+1)}=\frac{100cr-(r-1)\times100a}{(a-c)r\times(r-1)}$; by reduction, $(b+c)ar-2brc=(b-c)a$, or $r=\frac{(b-c)\times a}{(b+c)a-2bc}=$

$$\frac{(126-96)112}{(126+96)112-(2\times112\times96)}=\frac{39\times112=3360}{24864-24192}=\frac{3360}{672}=5.$$

536. If I lend \$150 on condition that \$12 per annum be paid until the principal and its interest, at 5 per cent,. be satisfied; and that at every payment the interest then due shall be discharged, and the remainder, by which such payment exceeds that interest, applied to reduce the principal, how many years must the said payment continue? See Index.

Let $p=\$150$, $a=12$, $r=.05$, and $n=$ time required; then $p+pr=$ the sum due before first payment, $p-a+pr=$ the sum due after first payment, and $(p-a)r+pr^2=$ interest due at second payment, $(2p-a)r+pr^2=$ sum due after second payment. Also, $(p-2a)r+(2p-a)r^2+pr^3=$ the interest due at 3d payment; $p-3+(3p-3a)r+(3p-a)r^2+pr^3=$ the sum due after third payment. Again, $(p-3a)r+(3p-3a)r^2+(3p-a)r^3+pr^4$ $=$ the interest due at the fourth payment, $p-4a+(4p-6a)r+(6p-4a)r^2+(4p-a)r^3+pr^4=$ sum due after fourth payment; then I shall have $p+npr+\frac{n(n-1)}{1.2}pr^2+\frac{n(n-1)(n-2)}{1.2.3}pr^3$, &c.

$-na-\frac{n(n-1)}{2}ar-\frac{n(n-1)(n-2)}{1.2.3}ar^2-\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}ar^3$, &c., will be the sum due after the nth payment; which, by the question, is nothing. Then $p\times1+nr+\frac{n(n-1)}{1.2}r^2$, &c. $=$ $a\times n+\frac{n(n-1)}{1.2}r+\frac{n(n-1)(n-2)}{1.2.3}r^2$, &c., or $p(1+r)^n=$ $a\times\frac{(1+r)^n-1}{r}$, or $\frac{pr\times(1+r)^n}{a}=(1+r)^n-1$; or $1=\left(1-\frac{pr}{a}\right)$ $\times(1+r)^n$, or $1=\frac{a-pr}{a}\times(1+r)^n$, then $\frac{a}{a-pr}=(1+r)^n$, and $\frac{A-\text{Log.}(a-pr)}{\text{Log.}(1+r)}=n.$

537. The duties of certain goods amounted to \$2460, out of which a discount of $2\frac{1}{2}$ per cent. was allowed, in consideration of prompt payment on the sum actually paid. What did the discount amount to?

Put \$2460 $= s$, $\frac{2\frac{1}{2}}{100} = \frac{1}{40} = r$; $1 = t$, and $d =$ discount. Then $s - d = p$, and $s - d = \frac{s}{1+tr}$; then $(s - \frac{s}{1+tr} = \frac{str}{1+tr} = d = \frac{2460 \times \frac{1}{40}}{1+\frac{1}{40}} = \frac{2460}{41} = 60$, Ans.

38. A person has now due to him \$320, and at the end of 5 years \$96 more will be due from the same debtor; they agree, that the whole shall be discharged at one payment, at that time when the simple interest of the \$320 shall be equal to the discount of the \$96, both being calculated at 5 per cent. per annum. The time of payment is required.

Let $320 = p$, $t = 5$, \$96 $= s$, $\frac{5}{100} = \frac{1}{20} = r$, and $t - x =$ time; then $(t - x) \times pr =$ interest of p for $t - x$, and $\frac{srx}{1+xr} =$ the discount of s for x; then $(t-x)pr = tpr - xpr = \frac{srx}{1+xr}$, or $tp - xp = \frac{sx}{1+xr}$, or $x^2 + \frac{s+p-tpr}{pr}x = \frac{tp}{pr} = \frac{t}{r}$; by art. 70, case 1, $x = \sqrt{\left\{\frac{t}{r} + \left(\frac{s+p-tpr}{2pr}\right)^2\right\}} - \frac{s+p-tpr}{2pr} = 4$, and then $5 - 4 = 1$, the time required is one year, or, expressed in numbers, $\frac{96 + 320 - 5(320 \times \frac{1}{20})}{320 \times \frac{1}{20}} = 21$, and $x = \sqrt{\left\{(5 \times 20) + \frac{21^2}{4}\right\}} - \frac{21}{2} = 4$, as before.

39. Bought as many books, at 4*s.* and 6*s.* apiece, to pay in 9 and 6 months respectively, as cost £33 2*s*, but in consideration of prompt payment am allowed the just discount, which is £1 2*s.*, and which is $9\frac{3}{10}d.$ less than the interest of the price of the said two parcels, each being calculated for its respective time. Required the number of each sort of books, and the rate of interest.

Let $x =$ the number at 4 shillings, $z =$ the number at 6 shillings, and $r =$ the rate of interest; then $4x + 6z = 662$ shillings; then $\frac{rx}{50} + \frac{9rz}{200} = 22\frac{31}{40}s. =$ sum of the interest, and $\frac{4rx}{200+r} + \frac{18rz}{400+3r} = 22s. =$ sum of the discount. By the first equation, $x = \frac{331-3z}{2}$, which substituted for it in the other two, we have $\frac{662r+3rz}{200} = 22\frac{31}{40}$, and $\frac{662r-6rz}{200+r} + \frac{18rz}{400+3r} = 22$; and from the first of these two we have $z = \frac{4555-662r}{3r}$; which substituted for it in the other, and reduced, it is $24r^2 - 275r = -775$, from which $r = 5$, then $z = 83$, and $x = 41$. Otherwise,

Let $x =$ the price of that parcel at 4 shillings apiece, and $r =$ the interest of one pound for a year; then $\frac{1}{2}rx =$ interest of the price of that parcel at 4*s*. apiece, and $\frac{3}{4}r \times (\frac{331}{10} - x) =$ the interest of the price of that parcel at 6*s*. apiece; the sum of both is $\frac{993r}{40} - \frac{rx}{4} = \frac{993r - 10rx}{40}$; also $\frac{\frac{1}{2}rx}{1+\frac{1}{2}r} = \frac{rx}{2+r} =$ discount of the first parcel, and $\frac{3}{4} \times r \times \frac{\frac{331}{10} - x}{1+\frac{3}{4}r} = \frac{933 - 30x}{40+30r} \times r =$ discount of the second parcel; their sum is $\frac{993 - 30x}{40+30r} \times r + \frac{rx}{2+r}$ $= 22s. = £1\frac{1}{10}$ by the question, which equation reduced, we have $x = \frac{240r^2 + 469r - 22}{5r}$. Again, by the question, $\frac{933 - 10x}{40} \times r$ $= £1\ 2s. + 9\frac{3}{10}d. = £\frac{911}{800}$, and this equation reduced, we have $x = \frac{19860r - 911}{200r}$, consequently $\frac{240r^2 + 469r - 22}{5r} = \frac{19860r - 911}{200r}$; $\therefore r^2 - \frac{11}{96}r = -\frac{31}{9600}$; by art. 70, $r = \frac{1}{20}$, hence $x = \frac{164}{20} = £8$ 4*s*., and $\frac{164}{20} \div \frac{1}{5}(4s.) = 41 =$ the number of books at 4*s*. apiece; whence will be found $83 =$ the number at 6*s*. apiece.

540. Let as before, art. 137, &c. Put $P =$ principal or present worth, $a =$ the annuity, $r =$ rate of interest of one dollar for one year, $n =$ number of years or times, $m =$ the amount; then if p, r, and n are given, to find m. Now the amount of one dollar in one year will equal $1 + r$, $\therefore p = p(1 + r)$, and $p(1 + r)$ will amount to $p(1+r)^2$ in 2 years, and so on; $p(1+r)^2$ will $= p(1+r)^3$ in 3d year; and in general $p(1 + r)^{n-1}$ will amount $p(1 + r)^n =$ the amount of (p) at the end of (n) years or times; therefore $p(1+ r)^n = m$. In logarithms, $P + nL(1 + r) = M$, and cor. 1, $n = \frac{M - P}{L(1+r)}$, when p, r, and m are given; cor. 2, when p, n, and m are given, to find r, then $L(1 + r) = \frac{M - P}{n}$; and when r, n, and m are given, to find p, then $P = M - nL(1+r)$.

541. Given the annuity a, rate per cent. r, and number of years or times $= n$, to find $m =$ the amount.

Then I have $a + a(1+r) + a(1+r^2) + a(1+r^3)$, &c., to $a(1+r)^{n-1}$; by art, 3, page 180, will become due in the times 1, 2, 3, 4, and so on; the sum of the series is $\frac{a(1+r)^n - a}{r} = m$. In Logarithms: Let $n \times L.(1+r) = B$; then $A + L(b - 1) - R = M$;

Cor. 2. When m, r, and n, are given to find a. See Index $a = \frac{rm}{(1+r)^n - 1}$, or if $n \times L.((1+r) = B)$ $A = M + B - L.(b - 1)$;

Cor. 2 when a, m, and r, are given to find n,

$a(1+r)^n = mr+a \therefore (1+r)^n = \frac{mr+a}{a}$, and $n = \frac{L(mr+a)-A}{L(1+r)}$.

1. When a, n, and m are given, to find r, by arts. 86 to 92.

$\frac{a(1+r)^n - a}{(1+r)-1} = m$, or $a(1+r)^n - a = m(1+r) - m$; then $a(1+r)^n - m(1+r) + (m - a) = 0$. Let $D = \frac{M-A-N}{\frac{1}{2}(n-1)}$; $e = \frac{6}{n+1}$ also $F = \frac{L\{2(d-1)+e\}+E}{2}$; then $r = f - e$.

542. If an annuity of \$50 forborne 18 years amount to \$1342.75 cts., what rate of interest was allowed? Answer is $4\frac{1}{2}$ per cent.

Here $a = 50$, $n = 18$, and $1342.75 = m$, and $8.5 = \frac{1}{2}(n-1)$;

$0.31579 = \frac{6}{n+1} = e$, and

$M = 3.12800$	$D = 0.02044$
$A = 1.69897$	$d = \bar{1}.04819$
$N = 1.25527$	$E = \bar{1}.49940$
$8.5)0.17375(0.02044 = D$	$\bar{1},61508 = L.\{2(d-1)+e\}$
then $2\times 0.04819 = 0.09638$,	$2)\bar{1},11448$,
$e = 0.31579$	
then $2(d-1)+e = 0.41217$;	$\bar{1}.55724 = F$;

then $f = 0.36078 - 0.31579 = 0.04499 = 0.045 = r = 4\frac{1}{2}$

Given a, r, and n, to find p, In annuities at compound interest.

Then $\frac{a}{1+r} + \frac{a}{(1+r)^2} + \frac{a}{(1+r)^3}$, &c... $\frac{a}{(1+r)^n}$; therefore the present value or *sum* $= \frac{\{(1+r)^n - 1\}a}{r(1+r)^n} = p$. In Logarithms:

Let $n \times L(1+r) = B$; $\therefore A + L.(b-1) - R - B = P$.

543. When p, r, and n are given, to find a.

$a = \frac{pr(1+r)^n}{(1+r)^n - 1}$, or if $n \times L(1+r) = B$; $A = \frac{P+R+B}{L(b-1)}$.

544. When a, p, and r are given, to find n.

$(1+r)^n \times a = pr(1+r)^n + a$; $(1+r)^n = \frac{a}{a-pr}$; $\therefore n = \frac{A - L(a-pr)}{L(1+r)}$

545. Given a, n, and p, to find r.

Then $p = \frac{a(1+r)^n - a}{\{(1+r)-1\}\times(1+r)^n}$; then $p(1+r)^{n+1} - (a+p)(1+r)^n + a = 0$. Let $G = \frac{A+N-P}{\frac{1}{2}(n+1)}$; $h = \frac{6}{n-1}$;

$K = \frac{L.(h - 2\{g-1\}) + H}{2}$; then $r = h - k$. See Index.

545½. In annuities, computed by Comp. Interest, a, p, and r are given, to find m. By my Index, and also at pages 315, '6, '7, that

$(1+r)^n = \frac{a}{a-pr}$, and $(1+r)^n = \frac{m}{p}$; then $\frac{m}{p} = \frac{a}{a-pr}$; and $m =$ $\frac{pa}{a-pr}$. Cor. $p = \frac{ma}{a+mr}$. Cor. $r = \frac{(m-p)a}{mp}$, and $a = \frac{mpr}{m-p}$.

546. In computing annuities by Compound Interest, p, m, and n are given, to find a.

I have $1+r = \left(\frac{m}{p}\right)^{\frac{1}{n}}$; $r = \left(\frac{m}{p}\right)^{\frac{1}{n}} - 1$; cor. $a = \frac{rm}{(1+r)^n - 1}$; if $M - P = B$, and $\frac{M-P}{n} = C$; $A = \frac{L(c-1)+M}{L.(b-1)}$, and cor. $r = \frac{(m-p)a}{mp}$; See question 540. $n = \frac{M-P}{L(1+r)}$. See Index.

Cor. 1. When p, n, and a are given, to find m, first find r by 545, and then m by question 545. Cor. 2. Where m, n, and a are given, to find p, first find r, by 545, and then p as above.

546½. Annuities in reversion by Comp. Interest. Let $a =$ annuity, $p =$ the present worth thereof, $t =$ time of its duration, $x =$ time before it commences, and $r =$ the interest of \$1 in 1 year.

Case 1. When a, t, x, are r, given to find p. then, as before, the present worth of a to $x+t$ or x years to $\frac{\{(1+r)^{x+t}-1\}a}{r(1+r)^{x+t}}$; or $\frac{\{(1+r)^x-1\}a}{r(1+r)^x}$; then $\frac{\{(1+r)^{x+t}\}a-a}{r(1+r)^{x+t}} - \frac{\{(1+r)^x \times \}a-a}{r(1+r)^x} = p$; $\therefore p = \frac{(1+r)^t \times a - a}{r(1+r)^{x+t}}$. In Logarithms, if $t \times L(1+r) = B$; then $A + L(b-1) - R - (x+t) \times L(1+r) = P$. When p, t, x and r are given, to find a, then $A = P + R + (x+t) \times L(1+r) - L.(b-1)$. And also when p, a, t and r are given, to find x; then $\frac{A+L(b-1)-R-P-B}{L(1+r)} = x$. And when p, a, x and r are given, to find t; if $P + R + x \times L(1+r) = C$, $t = \frac{A - L(a-c)}{L(1+r)}$. Cor. 1. When p, a x and t are given to find r. S Index. Let $\frac{1}{2}(2x+n+1) = h$; $u = \frac{12h}{t^2-1}$; $\frac{A+T-P}{h} = V$; $z = u - 2(v-1)$; then $\frac{1}{2}(Z-U) = E$, and $u - e = r$.

547. The amount of an annuity in arrear, computed at simple interest, was \$215; now if the annuity had continued unpaid one year longer, it would have amounted to \$275, but if one year less, to no more than \$157 50 cts. The annuity, time, and rate.

Put $a =$ annuity; $n =$ the time; and $r =$ the interest of \$1; also \$215 $= b$, \$275 $= c$, and \$157 50 $= d$; then, by the question, I have $d = (n-1)a + \frac{(n-1)(n-2)}{1.2}ar$; $b = na + \frac{n(n-1)}{1.2}ar$;

$c=(n+1)a+\frac{(n+1)n}{1.2}ar$; or $\frac{2(b-an)}{n(n-1)}=ar$; $\frac{(c-(n+1)a\times)2}{(n+1)n}$ $=ar$; $\frac{(d-(n-1)a\times)2}{(n-1)(n-2)}=ar$; then $\frac{2(b-an)}{n(n-1}=\frac{2\{c-(n+1)a\}}{(n+1)n}$; $\therefore\ \frac{(n+1)(b-n-1)c}{n+1}=a$. Again, $\frac{2(b-an)}{n(n-1)}=\frac{(d-(n-1)a)2}{(n-1)(n-2)}$; then $a=\frac{nd-(n-2)b}{n}$; now $\frac{(n+1)(b-n-1)c}{n+1}=\frac{nd-n-2b}{n}$; $\therefore\ n^2-\frac{c-d}{d+c-2b}n=-\frac{2b}{d+c-2b}$; $\therefore\ \frac{c-d}{2d+2c-4b}-n=\sqrt{\frac{(c-d)^2-4b(2d+2c-4b)}{(2d+2c-4b)^2}}$; $\frac{(c-d)-\sqrt{(c-d)^2-4b(2d+2c-4b)}}{2d+2c-4b}$, $=n=4$.

548. Given a=annuity, r= the rate of interest, n= the number of times, and m = the sum due at the end of n times. By pp. 313, 314, I have $a, a+ar, a+2ar, a+3ar$, &c.....$a+(n-1)ar$, which will represent the several times they become due in the 1st, 2d, 3d, 4th, &c. to the nth time; then their sum $na+\frac{n(n-1)}{1.2}ar$ $=m$; and if a, n, and m be given, to find r, or (m, n, and r,) or (a, r, and m) be given, to find r, or a, or n, I have $\frac{n(n-1)}{1.2}ar=$ $m-an$, $r=\frac{(m-an)\times 2}{n.(n-1)a}$; or $2na+n(n-1)ar=2m$; therefore $\frac{2m}{2n+n(n-1)r}=a$; or $\frac{2m}{(2+(n-1)r)\times n}=a$; 2dly, I have $2an$ $+n(n-1)ar=2m$, or $2an+arn^2-anr=2m$, or arn^2+ $(2-r)an=2m$; $\therefore\ n^2+\frac{2-r}{r}n=\frac{2m}{ar}$. By art. 70, case 1, n^2+ $\frac{2-r}{r}n+\left(\frac{2-r}{2r}\right)^2=\frac{2m}{ar}+\left(\frac{2-r}{2r}\right)^2$; therefore $n+\frac{2-r}{2r}=$ $\sqrt{\left\{\frac{2m}{ar}+\left(\frac{2-r}{2r}\right)^2\right\}}$, or $n=\sqrt{\left\{\frac{2m}{ar}+\left(\frac{2-r}{2r}\right)^2\right\}}-\frac{2-r}{2r}$.

549. A's income of P $D.$ is paid in equal portions at the end of 3, 6, 9, and 12 months; and his expenses, which are as the numbers 1, 2, 3, 4, are paid at the same time: Now, if the remaining sums are lent out at rate r simple interest, during the terms 9 months, 6 months, and 3 months, respectively, (the last quarter's salary bearing no interest), and amount to $qD.$; what will be the amount of A's annual expenditure? See Index.

Ans. $\frac{8(p-q)+3pr}{2(4+r)}D.$ Ans. In $n\cdot\frac{\log. M.-\log. p}{\log. m.-\log. p}$ years.

550. If a sum (p) in (n) years amounts to m, in what time will the same amount to M?

Here $m=pR^n$; assume $M=pR^x$, where x is the time required. Now $R^n=\frac{m}{p}$; $\therefore$ n log. R = log. m — log. p, and similarly x log. R = log. M — log. p; therefore $\frac{x}{n}=\frac{\text{Log. M}-\text{Log. }p}{\text{Log. }m-\text{Log. }p}$; consequently $x=n\left(\frac{\text{Log. M}-\text{Log. }p}{\text{Log. }m-\text{Log. }p}\right)$.

51. Here $10\,P=M=PR^n=P.(1+\frac{1}{20})^n$; $\therefore$ $10=(\frac{21}{20})^n$; therefore $n(\text{log. }21-\text{log. }20)=\text{log. }10=1$. Now, log. 21 = log. $\frac{105}{5}$ = log. 105 — log. $(\frac{10}{2})$ = log. 105 — 1 + log. 2, and log. 20 = log. 10 + log. 2 = 1 + log. 2; $\therefore$ log. 21 — log. 20 = log. 105 — 2 = 2.0211893 — 2 = .0211893; consequently $n=1\div.0211893=47$ years 70.7 days. See Index.

52. Here $M=P.R^n$, (where M = the amount of P dollars for n years, R being the amount of \$1 at the end of one year.) See pp. 315—317. Here the amount must be = 2 × the principal, or $P.R^n=200$, or $100R^n=200$, or $R^n=2$. Now $R=(1+\frac{1}{20})=\$\frac{21}{20}$, allowing 5 per cent. $(\frac{21}{20})^n=2$; $\therefore$ n.log. $\frac{21}{20}$ = log. 2 = 14 years +, or $n=\frac{\text{L}.2}{\text{L}.21-\text{L}20}=\frac{.30103}{1.32222-1.30103}=\frac{30103}{2119}=$ 14 yrs. $80\frac{24}{2119}$ ds. Hence it appears that any principal will double itself, at 5 per cent. compound interest, in 14 years $80\frac{24}{2119}$ days. n may be found very nearly without logarithms, thus: since $2=R^n=(1+\frac{1}{20})^n=1+\frac{n}{20}+\frac{n(n-1)}{2.20^2}$. Neglect the terms after the third, and find n from art. 70, page 141, and art. 142.

53. Here $PR^n=M$; $\therefore$ $R=1+\frac{1}{20}=\frac{21}{20}$; and if P=1, M=100; then $(\frac{21}{20})^n=100$, therefore $n(\text{log. }21-\text{log. }20)=\text{log. }100=2$, $n=\frac{2}{\text{L}.21-\text{L}20}$, $\therefore n=\frac{2}{.0211893}=94$ years 141.4 days.

54. Here $R=\left(1+\frac{r}{m}\right)$, r being the rate per cent. due every year, and m = the number of moments in a year, = $31557600t$ moments = mt; $\therefore$ $M=PR^{n}=\left(1+\frac{r}{m}\right)^{mt}$, when logarithms may be used for expediting the calculations. See Index.

555. A person borrowed P\$ at interest. To discharge this, he invested 2\$ at the end of the first year, 4\$ at the end of the second, and 8\$ at the end of the third, and so on. How many years will elapse before this fund be large enough to discharge the debt, compound interest being allowed on both sides at a given rate?

Ans. Number of years $=\frac{\text{log. }\{P\cdot(2-R)+2\}-\text{log. R}}{\text{log. }2-\text{log. R}}$, where

$R =$ the amount of one dollar for one year at the given rate.

Let $n =$ the number of years required; then $2R^{n-1} =$ the amount of the first investment up to that time; $4R^{n-2} =$ 2d, &c.; $2^n = n$th; and $PR^n =$ the amount of the sum lent; therefore $PR^n = 2R^{n-1} + 4R^{n-1} + \ldots + 2^n = \dfrac{2^{n+1} - 2R^n}{2 - R}$, since it is in a geometrical series whose common ratio $= \dfrac{2}{R}$; and the number of terms is $= n$; hence $(P.(2 - R) + 2)R^n = 2^{n+1}$; consequently $\log.\,(P.(2 - R) + 2) + n \log.\,R = (n + 1) \log.\,2 = n.\log.\,2 + \log.\,2$; $\therefore n(\log.\,2 - \log.\,R) = \log.\,(P.(2 - R) + 2) - \log.\,2$; $\therefore$ $n = \dfrac{\log.\,(P.(2 - R) + 2) - \log.\,2}{\log.\,2 - \log.\,R}$.

556. The amount M of an annuity A, for n years, at compound interest, $= \dfrac{R^n - 1}{R - 1}A$; where $R =$ the amount of one dollar for one year, (art. 144;) hence if $S =$ the sum required, then $S = \dfrac{(\frac{21}{20})^3 - 1}{(\frac{21}{20}) - 1} \times 100 = \dfrac{21^3 - 20^3}{21 - 20} \times \dfrac{100}{20^3} = \dfrac{21^3 - 20^3}{4} = \dfrac{1261}{4} = 315.$ 25 cent.

557. First $\frac{140}{4} = \$35$, the sum paid at the end of the first quarter; also the rate per dollar for a quarter $= \frac{5}{150} \times 4 = \frac{1}{80}$; $\therefore$ $\frac{35}{80} =$ the interest for the first quarter, and $2 \times 35 =$ principal at that rate. See pages 315—320. Hence the whole interest $= \frac{35}{80} + \frac{2 \times 35}{80} + \frac{3 \times 35}{80} + \ldots \frac{11 \times 35}{80} = (1 + 2 + 3 \ldots 11)\,\frac{35}{80} = \frac{12 \times 11}{2} \times \frac{35}{80} = \frac{66 \times 35}{80} = \$28.87\frac{1}{2}$; therefore the amount required will be $(35 \times 12) + 28.87\frac{1}{2} = 448.87{,}5$, Ans.

558. Here $A = \dfrac{R^n - 1}{R - 1}.P$; (see Index;) $\therefore AR - A = PR^n - P$; $\therefore R^n = \dfrac{A(R - 1) + P}{P}$; $\therefore n \log.\,R = \log.\,(A.(R - 1) + P) - \log.\,P$; consequently the number of years required is $= n = \dfrac{\log.\,\{A.(R - 1) + P\} - \log.\,P}{\log.\,R}$. If the rate per cent. were 5, R would $= \dfrac{21}{20}$, and $n = \dfrac{\log.\,(A + 20P.) - \log.\,20 - \log.\,P}{\log.\,21 - \log.\,20}$.

559. By art. 139, page 314, the present worth of any sum, due after a certain time, is a sum such that, being put out to interest, it would amount to the given sum in that time. The discount of any sum due after a certain time is equal to the difference between that sum and its present worth, or it is equal to the interest of its present worth for that time; hence if P be the present worth of a sum A due after n years, we have $PR^n = A$; there-

fore $P=\frac{A}{R^n}$; $R=$ the amount of 1 dollar in 1 year; hence the discount $D=A-\frac{A}{R^n}=\frac{A}{R^n}(R^n-1)$; here $A=p$, $n=1\frac{1}{2}=\frac{3}{2}$; $R=1+\frac{1}{20}=\frac{21}{20}$; $D=\left(\frac{p}{(\frac{21}{20})^{\frac{3}{2}}}\times(\frac{21}{20})^{\frac{3}{2}}-1\right)=p-\frac{20^{\frac{3}{2}}}{21^{\frac{3}{2}}}p=\frac{p}{21^{\frac{3}{2}}}\left(21^{\frac{3}{2}}-20^{\frac{3}{2}}\right)$, or $D* = p-\frac{p}{(1+\frac{1}{20})^{\frac{3}{2}}}=p-\frac{p}{1+\frac{3}{40}}=\frac{3p}{43}$ nearly. * We may obtain D sufficiently accurate for practical purposes as above.

560. The present worth of an annuity $A=\frac{1-\frac{1}{R^n}}{R-1}\times A.$

Here if $A=100$; $R=1+\frac{8}{100}=1+\frac{2}{25}=\frac{27}{25}$; $n=20$; $P=\frac{(1-\frac{25}{27})^{20}}{\frac{2}{25}}\times 100=(1-(\frac{25}{27})^{20})\times 1250=983.16\frac{1}{2}$ cents.

561. The present value of a perpetuity $P=\frac{A}{R-1}$, where R is the amount of 1 dollar for 1 year, and A the annuity. See Index.

Now in this case R must $=$ the amount of one dollar in 5 years, and A the sum paid every 5 years; then $P=\frac{20}{R^5-1}=\frac{20}{\frac{21^5}{20^5}-1}=\frac{64000000}{884101}=72.39$ cents, nearly.

562. Here $D=\$\frac{15}{2}$; $A=100$; $n=1\frac{1}{2}=\frac{3}{2}$; $\therefore \frac{15}{2}=100-\frac{100}{R^{\frac{3}{2}}}$; $\therefore \frac{100}{R^{\frac{3}{2}}}=100-\frac{15}{2}=\frac{185}{2}$; $\therefore R^{\frac{3}{2}}=\frac{200}{185}=\frac{40}{37}$; therefore $\frac{3}{2}$ log. $R=$ log. 40 $-$ log. 37, $\therefore$ log. $R=\frac{2}{3}$(log. 40 $-$ log. 37); hence R, or $\left(1+\frac{r}{100}\right)=1.05335$; $r=5.335$. Otherwise, allowing simple interest, if P be the present worth of A due after n years $100r$ per cent. simple interest being allowed, we have $P+nrP=A$; $\therefore P=\frac{A}{1+rn}$; hence $D=A-\frac{A}{1+rn}$, or $\frac{15}{2}=100-\frac{100}{1+\frac{3}{2}r}=100-\frac{200}{2+3r}$; $15=200-\frac{400}{2+3r}$; therefore $185(2+3r)=400$, or $555r=400-370=30$; $\therefore r=\frac{30}{555}=\frac{2}{37}$, and $\frac{100\times 2}{37}=\frac{200}{37}=\$5.405$, the rate per cent.

563. Let $P=\frac{1-\frac{1}{R^n}}{R-1}\times A$. Question 530. $P=\frac{1-\frac{1}{R^{n+t}}}{R-1}\times A$; hence if the annuity were to continue $(n+t)$ years, I have page 316. Now it is evident the present value required is equal to the

difference between that for $(n+t)$ years and that for n; therefore it is $=\frac{1-\frac{1}{R^{n+t}}}{R-1}\times A-\frac{1-\frac{1}{R^n}}{R-1}\times A=\frac{A}{R-1}\times\left(\frac{1}{R^n}-\frac{1}{R^{n+t}}\right)$ $=\frac{A}{R^{n+t}}\times\frac{R^t-1}{R-1}$.

564. Let A be the annual rent of the estate; then its present value $=\frac{A}{R-1}$, art. 144, $=\frac{A}{1+\frac{1}{25}-1}=25A$; $\therefore$ the estate is 25 years' purchase.

565. As the time is less than a year, (see 137,) simple interest must be allowed; $\therefore$ $D=A-\frac{A}{1+nr}=400-\frac{400}{1+\frac{3}{4}(\frac{1}{100})}=\frac{22800}{1657}$ $D=\$13.76$ cents, nearly.

566. Discount (art. 138) is an allowance made on a bill, or any other debt not yet become due, in consideration of present payment. Hence it is evident that the discount on any sum must be equal to the difference between that sum and its present worth. Let A be the sum discounted, n = number of years after which it becomes due, r = the interest of \$1 for 1 year, R = the amount of 1 dollar for a year, D = discount; then (simple interest being allowed) $D=A-\frac{A}{1+nr}(a)$; for $P+nrp=A$, (P being the present value of A): P must amount in the given time to A; $\therefore$ $P=\frac{A}{1+nr}$; also (compound interest being allowed) $D=A-\frac{A}{R^n}(b)$; for $PR^n=A$, $\therefore$ $P=\frac{A}{R^n}$. If $A=100$, $n=\frac{1}{4}$; $r=\frac{1}{20}$; then $D=100-\frac{100}{1+\frac{1}{4}(\frac{1}{20})}=100-\frac{8000}{81}=\$\frac{100}{81}$. N. B. The time being less than a year, only simple interest can be allowed.

567. Let P be the present worth; then it must be such that it will amount in 5 years to \$100; $PR^5=100$, R being the amount of 1 dollar for 1 year; $P=\frac{100}{R^5}$; and if the rate per cent. were 5, R would be $=\frac{21}{20}$ and $P=\frac{100\times 20^5}{21^5}=\frac{320000000}{4084101}=78.35$ cents.

568. First, (allowing simple interest,) $D=A-\frac{A}{1+nr}=260$ $-\frac{260}{1+\frac{3}{2}(\frac{9}{200})}=260-\frac{260\times 400}{427}=\frac{7020}{427}=\$16.44$. Second, (at compound interest,) $D=A-\frac{A}{R^n}=260-\frac{260}{(1+\frac{9}{200})^{\frac{3}{2}}}=260$

$-\dfrac{260}{(1.045)^{\frac{3}{2}}}=260-\dfrac{260}{1.06825}$ (by logarithms) $=260-243.37\frac{3}{4}$
$=\$16.62\frac{1}{4}$ cents.

569. The present value is evidently = that of \$20 to continue forever, — that of \$20 a year to continue for 2 years; it is =

$$\frac{A}{R-1}-\frac{1-\frac{1}{R^n}}{R-1}\times A=\frac{A}{R^n(R-1)}=\frac{20}{(\frac{21}{20})^2(\frac{21}{20}-1)}=\frac{20^4}{21^2}=$$
$\frac{160000}{441}=362.80.$

570. The first double line by simple interest, the second by compound interest.

$$P=\frac{A}{1+nr}=\frac{75}{1+\frac{5}{4}\times\frac{1}{20}}=\frac{16\times 75}{17}=\$70.60;\text{ and second,}$$
$$P=\frac{A}{R^n}=\frac{75}{(1+\frac{1}{20})^{\frac{5}{4}}}=75\times\left(\frac{20}{21}\right)^{\frac{5}{4}}=\$70.56,\text{ either by taking}$$

3 terms of the expression of $(1+\frac{1}{20})^{\frac{5}{4}}$, or by logarithms.

570. The present worth of a sum, due after a certain time, is such a sum as would amount in that time to the given sum exactly. Hence if P be the present worth of the sum A, due after n years, we have $P+nrP=A$; $P=\dfrac{A}{1+nr}$, simple interest; $PR^n=A$; $\therefore P=\dfrac{A}{R^n}$, at compound interest; if $A=430$, $n=\dfrac{3}{4}$; $r=\dfrac{4\frac{1}{2}}{100}$ $=\dfrac{9}{200}$; $\therefore P=\dfrac{430}{1+(\frac{3}{4}\times\frac{9}{200})}=\dfrac{430\times 800}{827}=\$415.96$ nearly. The time being less than a year, compound interest cannot be allowed.

571. Here $P'=\dfrac{a}{1+\frac{b}{12}\times r}=\dfrac{12a}{12+br}$; $P''=\dfrac{c}{1+\frac{d}{12}\times r}=$ $\dfrac{12c}{12+dr}$; $\therefore P=P'+P''=\dfrac{12a}{12+br}+\dfrac{12c}{12+dr}$; therefore r^2 $+\dfrac{12(b+d)}{bd}r+\dfrac{144}{bd}=\dfrac{144(a+c)}{bdP}+\dfrac{12(ad+bc)}{bdP}r$; consequently $r^2+\dfrac{12(ad+bc-b-d)}{bdP}r=\dfrac{144(a+c-P)}{bdP}$; $\therefore r=\dfrac{6}{bdP}\Big\{(a-1)d$ $+(c-1)b\pm\sqrt{\{(a-1)d+(c-1)b\}^2+4bdP(a+c-P)}\Big\}$ by art. 70.

572. A sum, P, is due at the end of m years; find the difference between its amount of $(m+n)$ years, and the amount of its present value at the end of $(m+n)$ years, at simple interest.

573. The present value of an annuity to continue for a term of

years at a given rate of compound interest, $=m$ times the present value of the same annuity to be paid only during the latter half of the same term; required to find when the annuity will cease.

Ans. $\left(\dfrac{Pr^2 \pm mn}{1+rm}\right)$. Ans. After $2\cdot\dfrac{\log.\ (m-1)}{\log.\ (1+r)}$ years.

574. The amount of (P) at the end of $(m+n)$ years, and since it is not due until the end of (m) years, is $(P+nrP)$. Again the present value of P, due at the end of m years, is $P-\dfrac{P}{1+mr}$, whose amount for $(m+n)$ years is $\left(P-\dfrac{P}{1+rm}\right)+\left(P-\dfrac{P}{1+mr}\right)\times(m+n)r=\dfrac{mrP}{1+mr}\times(1+(m+n)r)$; $\therefore$ the difference required is $P+nrP-\dfrac{mrP}{1+mr}\times(1+mr+nr)=P\times\dfrac{1+nr-m^2r^2}{1+mr}$.

575. The present value of a freehold estate of \$100 per annum, subject to the payment of a certain sum A. at the end of every 2 yrs. is \$1000, allowing 5 per cent. comp. interest. Find the sum A.

The present value of an estate of \$100 a year, not subject to the payment of A as stated in the above.

$\dfrac{100}{\frac{21}{20}-1}=\$2000$, art. 140, $\dfrac{A}{R-1}$; hence it appears that the present value of the sum (A), paid every 2 years, $=\$1000$. Now the amount of one dollar in one year, R^2 is its amount in 2 years, and the form $\dfrac{A}{R-1}$ for the present value of a perpetuity, becomes $\dfrac{A}{R^2-1}$. When the payments become due every 2 years, instead of every 1 year, I have $\dfrac{A}{R^2-1}=1000$; $\therefore A=1000(R^2-1)=(\frac{21}{20})^2\times1000-1000=\frac{441000}{400}-1000=1102\frac{1}{2}-1000=102\frac{1}{2}$.

576. Here $D=A-\dfrac{A}{1+nr}=\$125\frac{1}{2}-\dfrac{\$125\frac{1}{2}}{1+3\times\frac{9}{200}}$; now $\dfrac{125\frac{1}{2}}{1+\frac{27}{200}}=\dfrac{50200}{454}=\$110.58$ nearly.

577. Let n be the whole number of years, after which the annuity will cease, (R) the amount of one dollar in 1 year at the given rate, and A the annuity; then the present value of A for the whole term $=$ the first expression, and also the present value of the first half of n years will be the second expression, and therefore the present value of the latter half will then follow:

$$\frac{1-\frac{1}{R^n}}{R-1}\times A;\quad \frac{1-\frac{1}{R^{\frac{n}{2}}}}{R-1}\times A;\quad \frac{1-\frac{1}{R^n}}{R-1}\times A-\frac{1-\frac{1}{R^{\frac{n}{2}}}}{R-1}A=\frac{A}{R-1}\left(\frac{1}{R^{\frac{n}{2}}}-\frac{1}{R^n}\right);$$

$\frac{1-\frac{1}{R^n}}{R-1}A = m \times \frac{A}{R-1} \times \left(\frac{1}{R^{\frac{n}{2}}} - \frac{1}{R^{\frac{n}{2}}}\right)$; $\therefore 1 - \frac{1}{R^n} = m\left(\frac{1}{R^{\frac{n}{2}}} - \frac{1}{R^n}\right)$,

$\therefore R^n = mR^{\frac{n}{2}} - m + 1$; $\therefore R^n - mR^{\frac{n}{2}} = 1 - m$, and by art. 70,
$\therefore R^{\frac{n}{2}} = \frac{m}{2} \pm \frac{m-2}{2} = m - 1$, or 1, $\therefore \frac{n}{2}$ log. R = log. $(m-1)$, or log. $1(=0)$; $\therefore n = 2\frac{\log.(m-1)}{\log. R}$, or 0, the number of years.

579. Here $D = A - \frac{A}{1+nr} = 100 - \frac{100}{1+\frac{1}{20}} = 100 - \frac{2000}{21} = \frac{100}{21} = \$4.76$. Again, the interest of $\$\frac{100}{21}$, at 5 per cent., for one year, $= \frac{100}{21} \times \frac{1}{20} = \$\frac{5}{21} = 0.23,8$ mills *Ans.*

578. A gives p\$. to B on condition that he receives an annuity for (n) years: what must be the annuity, reckoning compound interest at a given rate? See Index.

Ans. $\frac{pr^n}{r^n - 1}\left\{r - 1\right\}$, where r = amount of \$1 for a year.

580. Required the present worth of an annuity of \$$p$ for n years, payable every instant in equal portions, interest also being converted into principal as fast as it becomes due. See Index.

Ans. Present worth $= \frac{p}{r-1}\left\{1 - \frac{1}{r^{mn}}\right\}$, where m = the number of instants in one year, and r = amount of \$1 for one instant.

581. If the annuity be a dollars, the sum due every instant will be $= \frac{a}{(365\frac{1}{4}) \times 24 \times 60 \times 60} = \frac{a}{m}$, where $m = 3600 \times 6 \times 1461 = 31557600$. Let R be the amount of \$1 for one instant; then $1 : R :: A : RA$ = the amount at the end of the 2d instant; $\therefore A + RA = (1+R)A$ = sum due at the end of the 2d instant. In the same manner I have $R(1+R).A$ = amount at the end of the 3d instant, and $(1+R+R^2)A$ = sum due at the end of the 3d instant. Similarly, $(1+R+R^2+R^3+\ldots R^{p-1})A$ = the sum due at the end of (p) instant $= \frac{R^p - 1}{R-1} \times A$. Now the number of instants in n years $= nm$; let therefore P be the present value required; then $\therefore R^m$ = the amount of \$1 in a year; the amount of P in n years $= P.R^{mn}$; $\therefore P.R^{mn} = \frac{R^{mn}-1}{R-1}A$; $\therefore P = \frac{1-\frac{1}{R^{mn}}}{R-1}A$.

582. Let r be the interest of one dollar for one year; then $R = 1 + \frac{r}{m}$, (m being the number of instants in a year;) then, by ap-

proximate, consequently $R^m = \left(1+\frac{r}{m}\right)^m = 1 + r + \frac{m-1}{2m}r^2 + \frac{(m-1)(m-2)}{2.3.m^2}r^3$ &c. $= 1 + r + \frac{r^2}{2} + \frac{r^3}{2.3} + \ldots$ nearly, when m is very large $= e^r$, (e being the base of hyperbolic logarithms;) consequently $R = e^{\frac{r}{m}}$; hence $P = \frac{1 - \frac{1}{e^{rn}}}{e^{\frac{r}{m}} - 1}A$ nearly.

583. If \$52500 can be sunk, by equal annual instalments, in 14 years, when interest is allowed on the yearly balances, what is the yearly payment?

1. This question does not appear to be clearly stated. It is evidently to find an annuity for 14 years, which shall be worth 52500 dollars; but whether this is to be considered as the ultimate value, or the present worth, does not clearly appear. In this latter case, the solution will be thus: Let R denote 1.05, the amount of \$1 for one payment, $n = 14$, $A = 52500$ the amount, and a the annuity required; then, by 4, $a = \frac{R-1}{R^n-1} \times AR^n = \$5303.76$, the annual payment in that case. But on the former supposition it will be $\frac{R-1}{R^n-1} \times A = \$2657.41\frac{1}{2}$ for the annual payment.

2. Put $52500 = a$, 14 years $= t$, the amount of \$1 for 1 year $= r$, and x the annual payment. Then $a - x$ is the first year's balance, the interest of which is $ar - xr$, and the second year's balance $= ar - xr - x$, its interest is $ar^2 - xr^2 - xr$, and so on to the time t. $\therefore ar^t - xr^{t-1} \ldots - xr - x = 0$, and $x + xr + xr^2 + xr^3 \ldots xr^t = ar^t$; hence the sum of this series $= \frac{r-1}{r^t-1} \times art$ $= 5303.7584$, taking $r = 1.05$.

3. Put x for the part taken from the principal of the first year, and a for the amount of \$1 annuity forborne 14 years at 5 per cent. compound interest; then as $1 : a :: x : 52500$, which gives \$2678.76, and $52500 \times .05 = 2625$; this added to the former gives \$5303.76, the yearly payment.

4. Let $p = 52500$, $n = 14$, $a = 1.05$, $x =$ the yearly payment. Then the principal remaining after 1 year $= ap - x$; after 2 years $= a^2p - ax - x$; after 3 years $= a^3p - a^2x - ax - x$; after n years $= a^np - a^{n-1}x - a^{n-2}x - a^{n-3}x \ldots - ax - x$. This principal consists of two parts, the one being a^np, and the other inverted the geometrical series $x + ax + a^2x + a^3x \ldots + a^{n-1}x$, the sum of which geometrical progression is $\frac{a^n-1}{a-1} \times a$, which must, by

the question, be $=a^n p$; hence then $x=\frac{a-1}{a^n-1}\times pa^n=\$5303.76$, the annual payment.

584. If 100 dollars be due 1 year hence, 200 dollars 2 years hence, 300 dollars 3 years hence, and 400 dollars 4 years hence, at what time must all be paid together, without loss to debtor or creditor, allowing five per cent. compound interest?

The present worth of the 4 sums is $\frac{100}{1.05}+\frac{200}{1.05^2}+\frac{300}{1.05^3}+\frac{400}{1.05^4}$ $=864.875$. And by art. 144, cor. 1, the $\frac{(\log.1000-\log.864.875)}{\log.1.05}$ $=2.9754$ years, the time in which $864{,}87\frac{1}{2}$ will amount to 1000, which therefore is the true equated time.

585. To find the amount when the principal is increased not only by the interest, but also by some other sum at the same time. The amount of the original principal p in n years is pR^n, and if A be the sum that is continually added, the first A will be at interest $n-1$ years, the second will be at interest $n-2$ years, &c., and therefore the sum of their amounts is $AR^{n-1}+AR^{n-2}+\ldots.AR^{n-n}$, or $A(R^{n-1}+R^{n-2}+\ldots.1)$. Now the terms within the parenthesis form a geometrical progression, whose first term is R^{n-1}, and ratio R, consequently the sum will be $A\times\frac{R^n-1}{R-1}$; $\therefore$ the whole amount is $pR^n+A\frac{R^n-1}{r}$, or $a=\frac{p(R^{n+1}-1)}{r}$, when $A=p$. If, however, A is not added to the nth year, then we have $a=pR^n+\frac{AR(R^{n-1}-1)}{r}$, or when $a=p$; $a=\frac{pR(R^n-1)}{r}$.

Cor. 1. If instead of $p=A$, we have $p=0$, then $a=\frac{A(R^n-1)}{r}$; which expresses the amount of an annuity A, at compound interest, left unpaid for n years. Cor. 2. If P be the present value of the annuity A for n years, P must be such that if it were put out at compound interest for n years, it would amount to the same sum as the annuity; that is, we must have

$PR^n=\frac{A(R^n-1)}{r}$; consequently $P=\frac{A(1-\frac{1}{R^n})}{r}$. Cor. 3. If n be infinite, then $\frac{1}{R^n}$ will vanish, and therefore $P=\frac{A}{r}$.

586. An old gentleman bequeathed 100,000 dollars to his four sons, in such proportion, that every son, at the time of their successively coming of age, will possess the same sum as his fortune;

what legacy did the father give to each, supposing their ages were 12, 13, 14, 15 years, and allowing compound interest at 5 per cent. per annum?

1. Put $r = 1.05$ the amount of 1 dollar for one year, $m = 100{,}000$ dollars; and let the four shares be denoted by x, rx, r^2x, r^3x; then $x + rx + r^2x + r^3x = m$, hence $x = m \div (1 + r - r^2 + r^3) =$ 23201.18 dolls., the first share; then the other three shares are 24361.242, and 25579.304, and 26858.27. For, by table 3, the amount of each of these sums will be 35992.65 dollars.

2. Put $p = 100000$, and $r = 1.05$; then will the ratios of the shares be denoted by $1, r, r^2, r^3$, the sum of which is $4.310125 = a$. Then as $a : r^3 :: p : 26858.27$ dollars, the eldest son's share.

$a : r^2 :: p : 25579.30$	"	the second	"	"
$a : r :: p : 24361.24$	"	the third	"	"
$a : 1 :: p : 23201.18$	"	the fourth	"	"

3. The present worths of 1 dollar, forborne during their respective minorities, are .74622, and .71068, and .67684, and .64462 dolls., the sum of which is 2.77836; consequently $2.77836 : 100000 :: .74622 : 26858.27$ dolls., the sum left to the eldest son. In like manner the sums bequeathed to the others are found to be 25579.30, and 24361.24, and 23201.18 dollars, Ans. as before.

587. A sum of money, p, is to be divided among A, B, C, in such a manner, that at the end of a, b, c years, when they respectively come of age, they are to possess equal sums: required the share of each, allowing compound interest at a given rate.

Ans. A's share $= p \cdot \dfrac{r^{b+c-a}}{r^{b+c-a} + r^b + r^c}$; B's share $= p \cdot \dfrac{r^{c+a-b}}{r^{b+c-a} + r^b + r^c}$; and C's share $= p \cdot \dfrac{r^{b+a-c}}{r^{b+c-a} + r^b + r^c}$.

588. A lends B 1000 dollars, for which B repays him as follows, viz. at the end of three months 180 dollars, of 5 months 150 dollars, of six months 140 dollars, of eight months 100 dollars, of nine months 90 dollars, of ten months 120 dollars, and at the year's end 250 dollars. The rate of interest is required.

Let x^{12} = rate per annum; then x (according to art. 137) will be the rate for one month, x^2 for two months, &c., whence the present value of 180 dolls. to be received at the end of 3 months will be $\frac{180}{x^3}$, &c.; $\therefore \frac{180}{x^3} + \frac{150}{x^5} + \frac{140}{x^6} + \frac{100}{x^8} + \frac{90}{x^9} + \frac{120}{x^{10}} + \frac{250}{x^{12}} =$ 1000 (per quest.); and therefore $x^{12} - .18x^9 - .15x^7 - .14x^6 - .1x^4 - .09x^3 - .12x^2 = .25$. Solved, $x = 1.003852$; and $x^{12} = 1.047216$; and consequently 4.72 the rate per cent. required.

589. A gentleman bought an estate in houses for 1500 dollars, which, being let, brought him in 120 dollars per annum, clear of all expenses and deductions. At the end of 10 years, (most of the

houses being out of repair, and he not choosing to be at the expense of fitting them up,) he sold the whole estate again for 800 dollars. The question is, to find what interest he made of his money.

Let $a = 1500$ dollars, $b = 800$ dollars, $c = 120$ dollars, and x the required rate of interest; Then will $ax =$ principal and interest, and $ax - c =$ amount after the first payment is deducted. And in the same manner we have $ax^{10} - cx^9 - cx^8 - cx^7 - cx^6 - cx^5 - cx^4 - cx^3 - cx^2 - cx - c = b$ (per question), or $ax^{10} - c(x^{10} - 1) \div (x - 1) = b$; therefore $ax^{11} - (a - c)x^{10} - bx + b + c = 0$. Solved, $x = 1.04142$, &c., and the rate of interest required \$4.14 per cent. See Index.

590. How long must a capital a, \$3600, remain at the interest p, 5, to become as much as a capital a', \$5000, at the interest $p' = 4$, for n, 12 years? Ans. $\dfrac{\log. a' + n \log. p' - \log. a}{\log. p}$ years = 16 nearly.

591. A capital a is put out at the interest p, and at the expiration of each year the interest is added to the principal; but at the same time it increases or diminishes yearly by the same sum b. What will this capital amount to n years hence? Ans. It is $= ap^n \pm \dfrac{b(p^n - 1)}{p - 1}$, where the upper sign obtains for the capital increased by b, and the lower the capital diminished by b.

592. A capital a is lent out at the rate p. In what time will it become a', if the capital, with the interest and compound interest added to it, be augmented or diminished yearly by the sum b? Ans. Log. $n = \dfrac{\log. \{(p-1)a' \pm b\} - \log. \{(p-1) \pm b\}}{\log. p}$; ($n$ the number of years.) The upper sign of $\pm$ denotes the addition of b, the lower the deduction b.

593. If, in the preceding problem, b be yearly taken away, and b be greater than the interest of the capital a; in this case, in how many years will the capital be spent?

Ans. Log. $n = \dfrac{\log. b - \log. \{b - (p-1)a\}}{\log. p}$ gives the number of yrs.

594. What is the present value w of an annuity r at p per cent. which a person has to enjoy n years?

If an annuity r, having n years to run, be worth in cash w, what is its amount? Ans. $w = \dfrac{(p^n - 1)r}{(p-1)p^n}$. Ans. $r = \dfrac{(p-1)p^n w}{p^n - 1}$.

596. How long has an annuity to run, if its present value be considered as equal to w dollars, the interest being 5 per cent.?

Ans. Log. $n = \dfrac{\log. r - \log. \{r - (p-1)w\}}{\log. p}$.

R = amount of 1 dollar for 1 year at the given rate.

597. An annuity r is required for the given space of n' years, whose present value is equal to another annuity r for n years; if both be calculated at p per cent., what is the annuity?*

598. But how great is the required annuity, if it be payable for m years; consequently, if computing after a lapse of m years, it be payable n' years after this period?

$$* \text{ Ans. } r' = \frac{(p^n-1)p^{n'-n}}{p^n-1}r. \quad \text{Ans. } r' = \frac{(p^n-1)p^{n'-n+m}}{p^{n'}-1}.$$

599. The amount of an annuity of \$1 to continue n years = \$P, and the amount of the same annuity to continue $2n$ years = \$Q; what is the rate per cent. allowing compound interest?

$$\text{Ans. } 100 \cdot \left\{ \left(\frac{Q-P}{P}\right)^{\frac{1}{n}} - 1 \right\}.$$

600. Given (s) the sum of an arithmetic progression, (n), the number of terms, and (p) the product of the first and last terms, to determine the progression.

$$\text{First term} = \frac{s}{n} \pm \left\{ \frac{s^2}{n^2} - p \right\}^{\frac{1}{2}}, \text{ and com. dif.} = \mp \frac{2}{n-1} \cdot \left\{ \frac{s^2}{n^2} - p \right\}^{\frac{1}{2}}.$$

where the upper or lower sign must be used, according as the progression is a decreasing or an increasing one.

601. Seven years ago, I was just three times as old as my eldest son; but seven years hence, if we should both live, I shall be only twice as old. What are our present ages? Ans. 49 and 21.

602. Find the ratio of the length and breadth of a rectangular field, consisting of 2 acres of ground, that shall have the same perimeter as a square field consisting of 4 acres. Ans. $\frac{2+\sqrt{2}}{2-\sqrt{2}}$

603. A person being asked how many horses he kept, said, "For want of room in my stable, I am obliged to put 8 of them out to hire; but I am now building a new one, twice as large as the former, which, when finished, will enable me to accommodate 8 horses more than I now have. How many did he keep? Ans. 24.

604. A company of 18 persons, consisting of men and women, having dined together at an inn, their reckoning came to £9 18s., in the settling of which each of them paid as many shillings as there were men in company. How many men and women were there? Ans. 11 men and 7 women.

605. Find two geometrical mean proportionals between 3 and 24, and four geometrical means between 3 and 96.

Ans. 6 and 12; and 6, 12, 24, and 48.

606. Supposing that 19 pounds of gold weigh 18 pounds in water, and 10 pounds of silver 9 pounds in water, and that a mass of gold and silver of 106 pounds weighs 99 pounds in water; it is required to find the quantity of gold and silver contained in the mass. Ans. 76 pounds of gold, and 30 pounds of silver.

Let x and y, denote the weight of gold and silver in each mass then $x+y=100$, and As $19 : 18 :: x : \frac{18}{19}x$; $10 : 9 :: y : \frac{9}{10}y$; and $\frac{18}{19}x+\frac{9}{40}y=99$, $\frac{2}{19}x+\frac{1}{10}y=11$, Ans $x=76$ and $y=30$.

607. The dimensions of a rectangular floor are such, that if it had been 2 feet broader and 3 feet longer, it would have been 64 square feet larger, and if it had been 3 feet broader and 2 feet longer, it would have been 68 square feet larger. What is the length and breadth of the floor? Ans. 14 feet long and 10 broad.

Let $(x+2)(y+3)=64-xy$, or $2x+3y=58$ and $(x+3y)(y+2)=68+xy$ or $3x+2y=62$, and $x=14$, $y=10$ Ans.

608. Find three numbers in arithmetical progression, such that the square of each, added to the product of the other two, shall make 576, 612, and 792 respectively. Ans. 12, 18, and 24.

609. Find two such whole numbers, x and y, that the sum of the aliquot parts of x shall be two thirds of y, and the sum of the aliquot parts of y three fourths of x. Ans. $x=20$, $y=33$.

610. Find three numbers in harmonical proportion, such that their three differences shall be in geometrical proportion. *

611. The sum of two numbers is 2, and the sum of their ninth powers 32; required the numbers by a quadratic equation.

Ans. * $\sqrt{5}-1$, $6+2\sqrt{5}$, and 2. Ans. $1\pm\frac{1}{3}\sqrt{(6\sqrt{34}-33)}$.

612. The sum of the two extremes of six numbers, in geometrical progression, is 165, and the sum of the four means 150; required the terms of the progression, by a quadratic equation.

Ans. 5, 10, 20, 40, 80, and 160.

613. Given $xy(x^2+y^2)=3$, and $x^2y^2(x^4+y^4)=7$, to find the values of x and y by a quadratic equation.

Ans. $x=\frac{1}{2}(\sqrt{5}+1)$, $y=\frac{1}{2}(\sqrt{5}-1)$.

614. Given $x+y+z=23$, $xy+xz+yz=167$, and $xyz=385$, to find the values of x, y, and z. Ans. $x=5$, $y=7$, $z=11$.

615. Given $x(y+z)=207$, and $y(x+z)=87$, $z.(x+y)$ 240, to find the values of x, y, and z. Ans. $x=9$, $y=3$, $z=20$.

616. Given $x^2+xy+y^2=5$, and $x^4+x^2y^2+y^4=11$, to find the values of x and y by a quadratic equation.

Ans. $x=\frac{2}{5}\sqrt{10}+\frac{1}{5}\sqrt{5}$, $y=\frac{2}{5}\sqrt{10}-\frac{1}{5}\sqrt{5}$.

617. Given $x^2+yz=920$, $y^2+xz=980$, and $z^2+xy=1000$, to find the values of x, y, and z by a biquadratic equation, this being one of the least dimensions to which the question can be reduced. Ans. $x=19.5991$, $y=22.7788$, and $z=23.5276$.

518. Required the sum of n terms of the series $1^2+2^2x+3^2x^2+4^2x^3+5^2x^4+$ &c.

Ans. $\frac{1+x}{(1-x)^3}$.

519. Find the sum of the series $\frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}+\frac{1}{8.10.12}$ &c. continued ad infinitum. Ans. $\frac{1}{32}$

620. Required the sum of the series $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\frac{4}{7.9.11}+$ &c. continued ad infinitum. Ans. $\frac{1}{8}$.

621. Find the sum of the series $\frac{1}{2.3.4.5}+\frac{4}{5.6.7.8}+\frac{7}{8.9.10.11}$ $+$ &c. continued to infinity. Ans. $\frac{3}{280}$.

622. Given $1+3x+6x^2+10x^3+15x^4+$ &c. ad infinitum $=10$, to find the value of x. Ans. $x=1-\frac{1}{10}\sqrt[3]{100}$.

623. Given $\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\frac{1}{3840}x^5+$ &c. ad infinitum $=\frac{1}{4}$, to find x. Ans. $x=\frac{1}{2}-\frac{1}{16}+\frac{1}{96}-\frac{1}{512}+\frac{1}{2560}-$ &c.

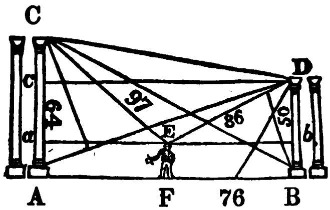
624. Find the sum of 100 terms of the series $(1\times2)+(2\times3)+(3\times4)+(4\times5)+(5\times6)+$ &c. Ans. 343400.

625. Required the sum of 25 terms of the series $1\times35+2\times36+3\times37+4\times38+5\times39+$ &c. which gives the number of shot in a complete oblong pile, consisting of 25 tiers, the number of shot in the uppermost row being 35. Ans. 16575.

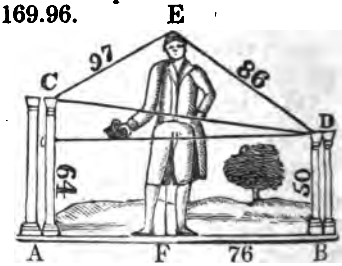
626. Required the approximate value of the series $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-$ &c. continued ad infinitum. Ans. .822467.

627. Find the approximate value of the hypergeometrical series $1.2-2.3+2.3.4-2.3.4.5+2.3.4.5.6-$ &c. continued ad infinitum. See Index. Ans. .59634736.

628. There are two columns, in the ruins of Persepolis, left standing upright; one is 64 feet above the plane, the other 50: Between these, in a right line, stands an ancient statue, the head whereof is 97 feet from the summit of the higher, and 86 feet from the top of the lower column; the base whereof measures just 76 feet to the centre of the figure's base. Required the distance of the tops of the columns. CD = 169.96.



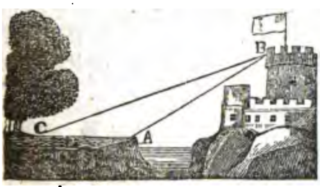
CD = 157.03 ft.

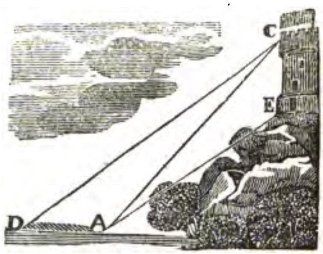


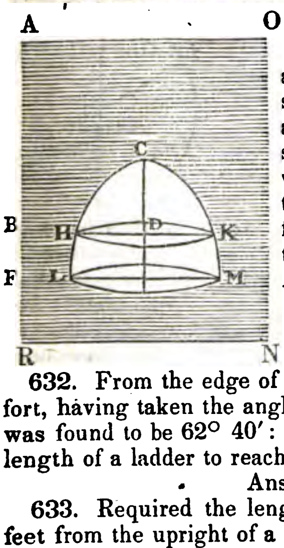
629. Wanting to know the height of an inaccessible tower; at the least distance from it, on the same horizontal plane, I took its angle of elevation equal to 58°; then going 300 ft = AC directly from it, found the angle there to be only 32°: Find its height, and my distance from it at the first station. Ans. $\begin{cases}\text{height } 307.53\\ \text{distance } 192.15\end{cases}$

630. Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inaccessible hill; I took the angle of elevation of the top of the hill 40°, C the top of the tower 51°; then measuring in a line directly from it to the distance of 200 ft AD farther, I found the angle to the top of the tower to be 33° 45′. What is the height of the tower? Ans. 93.33148 ft.

Figure to 629.







629. If a diving bell, of the form of a parabolic conoid, be let down into the sea to the several depths of 5, 10, 15, and 20 fathoms; it is required to assign the respective heights to which the water will rise within it: its axis and the diameter of its base being each 8 feet, and the quicksilver in the barometer standing at 30. 9 inches.

Ans. at 5 fath's water rises			2.03546 ft.
at 10	"	"	3.06393 "
at 15	"	"	3.70267 "
at 20	"	"	4.14658 "

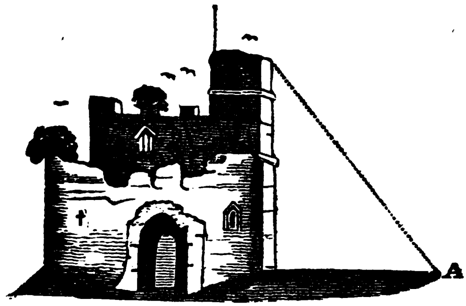
632. From the edge of a ditch, of 36 feet wide, surrounding a fort, having taken the angle of elevation of the top of the wall, it was found to be 62° 40′: required the height of the wall, and the length of a ladder to reach from my station to the top of it.

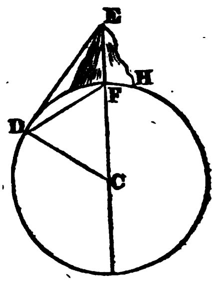
Ans. height of wall 69.64, ladder 78.4 feet.

633. Required the length of a shore, which being to strut 11 feet from the upright of a building, will support a jamb 23 feet 10 inches from the ground. Ans. 26 feet 3 inches.

634. At 170 feet distance from the bottom of a tower, the angle of its elevation was found to be $52^\circ\ 30'$. Required the altitude of the tower. Ans. 221.55 feet.

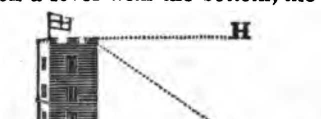
635. If the Peak of Teneriffe be $2\frac{1}{2}$ miles high, and the angle taken at the top of it, as formed between a plumb-line and a line conceived to touch the earth in the horizon, or farthest visible point, be $88^\circ\ 2'$; required to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly globular. Ans. dist. 135.943, diam. 7918 miles.



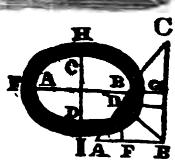


636. From the top of a tower, by the sea-side, of 143 feet high, it was observed that the angle of depression of a ship's bottom, then at anchor, measured 35°; what was the ship's distance from the bottom of the wall? Ans. 204.22 feet.

37. What is the perpendicnlar height of a hill, its angle of elevation, taken at the bottom of it, being 46°, and 200 yards further off, on a level with the bottom, the angle was 31°? Ans. 286.28 yds.



38. If, from a right-angled triangle, whose base AB is 12 and perpendicular BC = 16 ft., a line be drawn parallel to the perpendicular, cutting off a triangle whose area is 24 sq. feet; required the sides of this triangle, AFD. Ans. 6, 8, and 10.

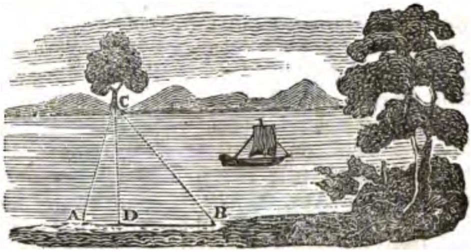


39. Wanting to know the distance between a house and a mill, which were seen at a distance on the other side of a river, I meas-

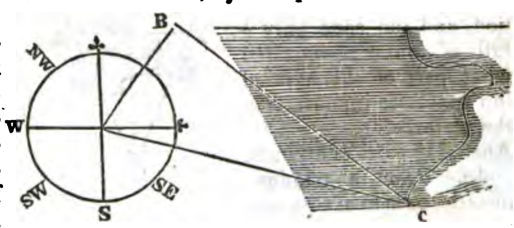
ured a base line along the side where I was, of 600 yards, and at each end of it took the angles subtended by the other end and the house and mill, which were as follow, viz. at one end the angles were 58° 20′ and 95° 20′, and at the other end the like angles were 53° 30′ and 98° 45′. What then was the distance between the house and mill?

Ans. 959.5866 yards.

40. Wanting to know the breadth of a river, I measured a base of 500 yards in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree, close to the bank on the other side of the river, to be 53° and 79° 12′. What was the perpendicular breadth of the river? Ans. 529.48 yards.

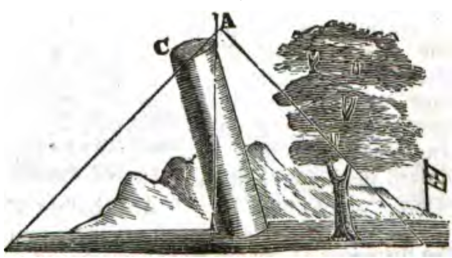


41. A point of land was observed, by a ship at sea, to bear east-by-south; and after sailing north-east 12 miles, it was found to bear south-east-by-east. It is required to determine the place of that headland, and the ship's distance from it at the last observation.



Ans. 26.0728 miles.

42. If a round pillar of 30 feet diameter be raised on a plane, inclined to the horizon in an angle of 75°, or the shaft inclining 15° out of the perpendicular: what length will it bear before it overset. Ans. 111.96 ft.

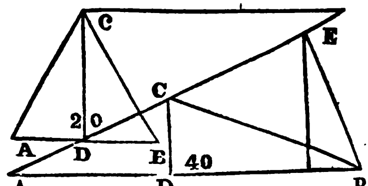


43. If BD, in figure 46, represent a portion of the earth's surface, and D the point where the levelling instrnment is placed, then LB will be the difference between the true and the apparent level; and you are required to demonstrate that, for distances not exceeding 5 or 6 miles measured on the earth's surface, BL, estimated in feet, is equal to $\frac{2}{3}$ of the square of BD, taken in miles.

Ans. $17\frac{2}{3}$ feet.

44. Given two sides of an obtuse-angled triangle, which are 20 and 40 poles. Required the third side, that the triangle may contain just an acre of land.

Ans. 58.884, and 23.09 poles.

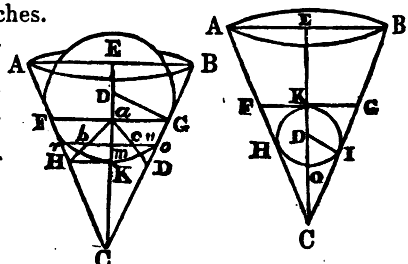


45. If a heavy sphere, whose diameter is 4 inches, be let fall into a conical glass, full of water, whose diameter is 5, and altitude 6 inches; it is required to determine how much water will run over. Ans. 26.272 cubic inches, or near $\frac{35}{47}$ parts of a pint.

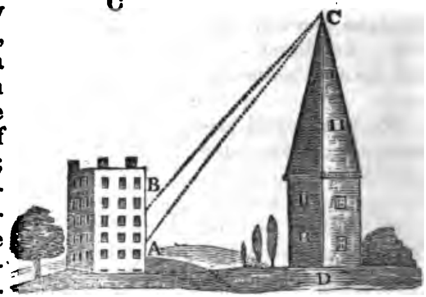
47. The cone being still the same, and $\frac{1}{5}$ full of water; required the diameter of a sphere which shall be just all covered by the water. Ans. 2.445996 inches.

46. The dimensions of a sphere and cone being the same as in the last question, and the cone only $\frac{1}{5}$ full of water; required what part of the axis of the sphere is immersed in the water.

Ans. .546 parts of an inch.



48. From a window near the bottom of a house, which seemed to be on a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple equal 40°; then from another window, 18 feet directly above the former, the like angle was 37° 30′; required the height and distance of the steeple.



Ans. height 210.44; distance 250.79.

49. In a garrison besieged are three remarkable objects, A, B, C, the distances of which from each other are discovered by means

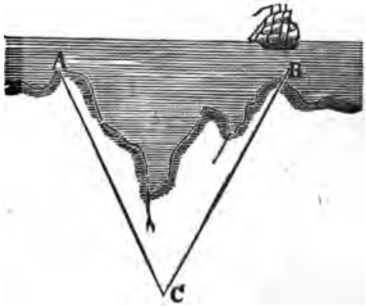
of a map of the place, and are as follow, viz. AB 266¼, AC 530, BC327½ yards. Now, having to erect a battery against it, at a certain spot without the place, and being desirous to know whether my distances from the three objects be such, as that they may from thence be battered with effect, I took, with an instrument, the horizontal angles subtended by these objects from the stations, S and found them to be as follow, viz. the angle ASB 13¼°, and the angle BSC 29° 50′. Required the three distances, SA, SB, SC; the object B being situated nearest me, and between the two others A and C. Ans. SA 757.14; SB 537.10; SC 655.30.

50. Being on the side of a river, and wanting to know the distance to a house which was seen at a distance on the other side, I measured out for a base 400 yards in a right line by the side of the river, and found that the two angles, one at each end of this line, subtended by the other end and the house, were 68° 2′ and 73° 15′. What then was the distance between each station and the house?

Ans. 593.09 and 612.385 yards.

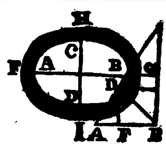
51. Required the same as in example 49, when the object B is the farthest from my station, but still seen between the two others as to angular position, and those angles being thus: the angle ASB 33° 45′, and BSC 22° 30′; also the 3 distances, AB 600, AC 800, BC 400 yds. Ans. SA 710.3; SB 1041.85; SC 934.14.

52. Wanting to know the extent of a piece of water, or distance between two headlands, I measured from each of them to a certain point inland, and found the two distances to be 735 yards and 840 yards; also the horizontal angle subtended between these two lines was 55° 40′. What was the distance required?



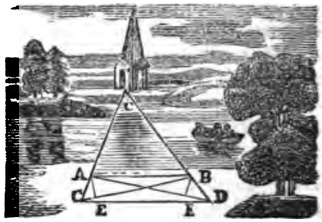
Ans. 741.2 yards.

53. The ellipse in Grosvenor-square measures 840 links across the longest way, and 612 the shortest, within the rails; now the walls being 14 inches thick, what ground do they enclose, and what do they stand upon?

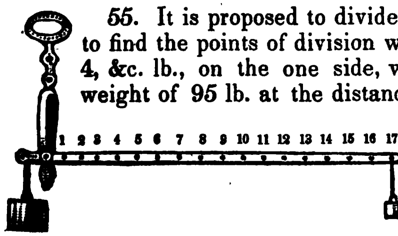


Ans. 4 acres, 0 roods, 6 poles.

54. Wanting to know my distance from an inaccessible object O, on the other side of a river, and having no instrument for taking angles, but only a chain or cord for measuring distances; from each of two stations, A and B, which were taken at 500 yards asunder, I measured in a direct line from the object O 100 yards, viz. AC and BD each equal to 100 yards; also the diagonal AD measured 550 yards, and the diagonal BC 560. What then was the distance of the object O from each station A and B?



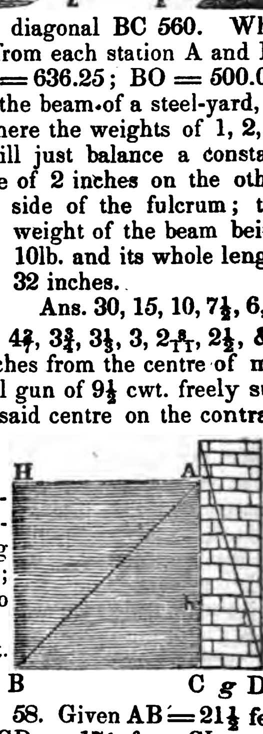
Ans. AO = 636.25; BO = 500.09.



55. It is proposed to divide the beam of a steel-yard, or to find the points of division where the weights of 1, 2, 3, 4, &c. lb., on the one side, will just balance a constant weight of 95 lb. at the distance of 2 inches on the other side of the fulcrum; the weight of the beam being 10lb. and its whole length 32 inches.

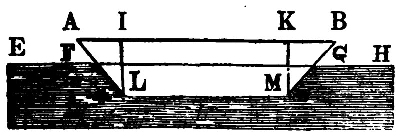
Ans. 30, 15, 10, $7\frac{1}{2}$, 6, 5, $4\frac{2}{7}$, $3\frac{3}{4}$, $3\frac{1}{3}$, 3, $2\frac{8}{11}$, $2\frac{1}{2}$, &c.

56. What weight hung on at 70 inches from the centre of motion of a steel-yard, will balance a small gun of $9\frac{1}{2}$ cwt. freely suspended at 2 inches distance from the said centre on the contrary side? Ans. $30\frac{2}{5}$lbs.



57. To find the thickness of an upright rectangular wall, necessary to support a body of water; the water being 10 feet deep, and the wall 12 feet high; also the specific gravity of the wall to that of the water as 11 to 7.

Ans. 4.204374 feet.



58. Given AB = $21\frac{1}{2}$ feet, CD = $17\frac{1}{6}$ feet, CI = $2\frac{1}{4}$ feet, $b = 4\frac{3}{4}$ feet. Required the weight of the pontoon and its load, when it is immersed to the depth CL, of $1\frac{1}{2}$ feet.

59. Let the weight of such a pontoon to be 900lbs., what is the greatest weight it will carry?-Ans. $8287\frac{3}{4}$ lbs.nely Ans. $12014\frac{1}{16}$ lbs.

60. Suppose the weight of the above pontoon and its load to be 6000lbs. how deep will it sink in water? Ans. 13.064 inches.

61. Find the quantity of pressure against a dam or sluice, across a canal, which is 20 feet wide at top, 14 at bottom, and 8 feet depth of water. Ans. 14.28292 tons.

62. If R and r be the radii of two spheres inscribed in a cone, so that the greater may touch the less, and also the base of the cone; demonstrate that the capacity of the cone is $= \dfrac{8\pi R^5}{3r(R-r)}$.

63. Let ABC, fig. 50, be the profile, or perpendicular section of a breast-work, and EP that of the ditch. Now, suppose the area of the section ABC is 88 feet, the depth of the ditch RD 6 feet, ER = SO = 3 feet; what is the breadth of the ditch at top when the sections of the ditch and the breast-work are equal; that is, when the earth thrown out of the ditch is sufficient to make the breast-work? See appendix for Ans. to the six following.

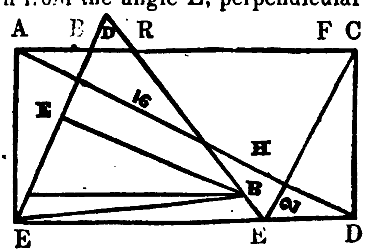
64. And what must be the breadth of the ditch at top, the depth and width at bottom remaining the same, when the profile of the breast-work remains the same, and the earth, in consequence of removal, occupies $\frac{1}{12}$th more space than it did before it was taken out of the ditch? See figure 54, Index.

65. The four sides of a trapezium are $6\frac{1}{4}$, $15\frac{3}{5}$, 12, and 9 respectively. The first two of these sides make a right angle. Required the area of the quadrilateral. Also, when the same four sides form a quadrilateral inscriptible in a circle, find its area, angles, and diagonals. See figure 55.

66. Find the ratio of the surfaces of the torrid zone, the two temperate, and the two frigid zones, of the earth; supposing the two tropics to be 23° 28′ from the equator, and the two polar circles to be 23° 28′ from their respective poles.

67. A cone, whose altitude is 63, and diameter of base 32, is to be cut, by sections parallel to the base, into four portions of equal curve surface: required the respective distances from the vertex, measured on the slant side, at which the sections are to be made.

68. If the line EFB be drawn from the angle E, perpendicular to the diagonal AD of a right-angled triangle parallelogram, and BF, FD, are given, find the sides of the parallelogram.



Let AF = x, EF = y, BF = b, DF = c. The triangles AFB, AFE, and DFE are similar, $\therefore b : x :: x : \dfrac{x^2}{b} =$ FE $= y$, and $b : x :: y$ or $\dfrac{x^2}{b} : c$; whence $\dfrac{x^3}{b} = bc$, and $x^3 = b^2c$, and

$x = \sqrt[3]{(b^2c)}$; then AE $= \sqrt{(x^2 + y^2)}$, and ED $= \sqrt{(c^2 + y^2)}$.

69. There are 2 circles, BDA and BFC, touching in B, and if DE be perpendicular to BA at the centre E, then there is given AC and DF, to find the diameters. See Index, Fig. 63.

Let radius BE $= x$, DF $= b$, CA $= a$, then FE $= x - b$, EC $= x - a$, then $(FF)^2 =$ BF $\times$ EC, (case 4, 17;) $\therefore x^2 - 2bx + b^2 = x^2 - ax$, and $2bx - ax = b^2$, and $x = \frac{b^2}{2b - a}$; whence BC $= 2x - a$.

68. Seven men bought a grinding-stone of 60 inches diameter, each paying $\frac{1}{7}$ part of the expense; what part of the diameter must each grind down for his share ?Ans. the 1st, 4.4508, 2d, 4.8400, 3d, 5.3535. 4th, 6.0765,5 th,7.2079, 6th, 9.3935, 7th, 22.6778 inches.

69 What dollars principal, being put out at its equal value per cent, at simple interest, for an equal number of years, will raise an interest equal to half the principal?

Let \$100 principal $= a$, and for the required principal, &c. put x; then $a : x :: x : x^2 \div a$, the interest for one year, and as 1 (year) is to x (years) so is $x^2 \div a$ to $x^3 \div a$, the interest for x years, $\therefore$ per question, $x^3 \div a = \frac{1}{2}x$; reduced, $x = \sqrt{\frac{1}{2}a} =$ \$7.07106. Ans.

Showing the Sum to which \$1 *Principal will increase at Compound Interest in any number of Years not exceeding* 21.

Yrs.	3	3½	4	4½	5	6	7
1	1.0300	1.0350	1.0400	1.0450	1.0500	1.06 = R	1.07000
2	1.0609	1.0712	1.0816	1.0920	1.1025	1.1236 = R^2	1.14490
3	1.0927	1.1087	1.1249	1.1412	1.1576	1.191016 = R^3	1.22504
4	1.1255	1.1475	1.1699	1.1925	1.2155	1.26247696	1.31079
5	1.1593	1.1877	1.2167	1.2462	1.2763	1.3382255776	1.40255
6	1.1948	1.2293	1.2653	1.3023	1.3401	1.4185191122	1.50073
7	1.2299	1.2723	1.3159	1.3609	1.4071	1.5036302590	1.60578
8	1.2668	1.3168	1.3686	1.4221	1.4775	1.5938480745	1.71818
9	1.3048	1.3629	1.4233	1.4861	1.5513	1.6894789590	1.83845
10	1.3439	1.4106	1.4802	1.5530	1.6289	1.7908476965	1.96715
11	1.3842	1.4600	1.5895	1.6229	1.7103	1.8982985583	2.10485
12	1.4258	1.5111	1.6010	1.6959	1.7959	2.0121964718	2.25219
13	1.4685	1.5640	1.6651	1.7722	1.8856	2.1329282601	2.40984
14	1.5126	1.6187	1.7317	1.8519	1.9799	2.2609039557	2.57853
15	1.5580	1.6753	1.8009	1.9353	2.0789	2.3965581931	2.75903
16	1.6047	1.7340	1.8730	2.0224	2.1829	2.5403516847	2.95216
17	1.6528	1.7947	1.9479	2.1134	2.2920	2.6927727857	3.15881
18	1.7024	1.8575	2.0258	2.2085	2.4066	2.8543391529	3.37993
19	1.7535	1.9225	2.1068	2.3079	2.5270	3.0255995021	3.61652
20	1.8061	1.9828	2.1911	2.4117	2.6533	3.2071354722	3.86968
21	1.8603	2.0594	2.2788	2.5202	2.7860	3.3995636005	4.14056

A TABLE exhibiting the period in which the Population of a Country has a tendency to DOUBLE *itself, from an estimate of its increase* PER CENT. *taken at the end of every ten years.*

Per centage Increase in ten years.	Nume. value of $\frac{1}{10}\{\text{Log.}\ (100+\pi) - 2\}$	Per. of doubl'g Lg. 2, or 3010300 $\div \frac{1}{10}\{\text{Log.}\ (100+\pi) - 2\}$	Per centage Increase in ten years.	Nume. value of $\frac{1}{10}\{\text{Log.}\ (100+\pi) - 2\}$	Pe. of doubl. l.2, or 3010300 $\div \frac{1}{10}\{\text{l.}\ (100+\pi) - 2\}$
$\pi = 1.0$	,00043214	696.60	$\pi = 20.0$	,00791812	38.01
1.5	,00064660	465.55	20.5	,00809870	37.17
2.0	,00086002	350.02	21.0	,00827854	36.36
2.5	,00107239	280.70	21.5	,00845763	35.59
3.0	,00128372	234.49	22.0	,00863598	34.85
3.5	,00149403	201.48	22.5	,00881361	34.15
4.0	,00170333	176.73	23.0	,00899051	33.48
4.5	,00191163	157.47	23.5	,00916670	32.83
5.0	,00211893	142.06	24.0	,00934217	32.22
5.5	,00232525	129.46	24.5	,00951694	31.63
6.0	,00253059	118.95	25.0	,00919100	31.06
6.5	,00273496	110.06	25.5	,00986437	30.51
7.0	,00293838	102.44	26.0	,01003705	29.99
7.5	,00314085	95.84	26.5	,01020905	29.48
8.0	,00334238	90.06	27.0	,01038037	28.99
8.5	,00354297	84.96	27.5	,01055102	28.53
9.0	,00374265	80.43	28.0	,01072100	28.07
9.5	,00394141	76.37	28.5	,01089031	27.64
10.0	,00413927	72.72	29.0	,01105897	27.22
10.5	,00433623	69.42	29.5	,01122698	26.81
11.0	,00453230	66.41	30.0	,01139434	26.41
11.5	,00472749	63.67	30.5	,01156105	26.03
12.0	,00492180	61.16	31.0	,01172713	25.67
12.5	,00511525	58.84	31.5	,01189258	25.31
13.0	,00530784	56.71	32.0	,01205739	24.96
13.5	,00549959	54.73	32.5	,01222159	24.63
14.0	,00569049	52.90	33.0	,01238516	24.30
14.5	,00588055	51.19	33.5	,01254813	23.99
15.0	,00606978	49.59	34.0	,01271048	23.68
15.5	,00625820	48.10	34.5	,01287223	23.38
16.0	,00644580	46.70	35.0	,01303338	23.09
16.5	,00663259	45.38	35.5	,01319393	22.81
17.0	,00681859	44.14	36.0	,01335389	22.54
17.5	,00700379	42.98	36.5	,01351327	22.27
18.0	,00718820	41.87	37.0	,01367206	22.01
18.5	,00737184	40.83	37.5	,01383027	21.76
19.0	,00755470	39.84	38.0	,01398791	21.52
19.5	,00773679	38.91	38.5	,01414498	21.28

Per centage Increase in ten Years.

Numerical value of $\frac{1}{10}\{\text{Log.}\ 100+\pi) - 2\}$

Period in yrs. of doubling. L.2 or .3016300, $\div \frac{1}{10}\{\text{L.}\ (100+\pi) - 2\}$

$n = 39.0$	,01430148	21.04	$n = 45$	,01613680	18.65
39.5	,01445742	20.82	46	,01643529	18.31
40.0	,01461820	20.59	47	,01673173	17.99
41.	,01492191	20.17	48	,01702617	17.68
42	,01522883	19.76	49	,01731863	17.38
43	,01553360	19.37	50	,01760913	17.09
44	,01583625	19.00			

A TABLE of Reciprocals, Squares, Cubes, and Roots.

Num.	Reciprocal.	Square.	Cube.	Square Root.	Cube Root.
1	1	1	1	1.0000000000	1.000000
2	5	4	8	1.4142135624	1.259921
3	3333333	9	27	1.7320508076	1.442250
4	25	16	64	2.0000000000	1.587401
5	2	25	125	2.2360679775	1.709976
6	1666666	36	216	2.4494897428	1.817121
7	1428571	49	343	2.6457513111	1.912933
8	125	64	512	2.8284271247	2.000000
9	1111111	81	729	3.0000000000	2.080084
10	1	100	1000	3.1622776602	2.154435
11	0909090	121	1331	3.3166247904	2.223980
12	0833333	144	1728	3.4641016151	2.289428
13	0769230	169	2197	3.6055512755	2,351335
14	0714285	196	2744	3.7416573868	2.410142
15	0666666	225	3375	3.8729833462	2.466212
16	0625	256	4096	4.0000000000	2.519842
17	0588235	289	4913	4.1231056256	2.571282
18	0555555	324	5832	4.2426406871	2.620741
19	0526316	361	6859	4.3588989435	2.668402
20	05	400	8000	4.4721359550	2.714418
21	0476190	441	9261	4.5825756950	2.758923
22	0454545	484	10648	4.6904157598	2.802039
23	0434783	529	12167	4.7958315233	2.843867
24	0416666	576	13824	4.8989794856	2 884499
25	04	625	15625	5.0000000000	2.924018
26	0384615	676	17576	5.0990195136	2.962496
27	0370370	729	19683	5.1961524227	3.000000
28	0357143	784	21952	5.2915026221	3.036589
29	0344828	841	24389	5.3851648071	3.072317
30	0333333	900	27000	5.4772255751	3.107232
31	0322581	961	29791	5.5677643628	3.141381
32	03125	1024	32768	5.6568542495	3.174802
33	0303030	1089	35937	5.7445626465	3.207534
34	0294118	1156	39304	5.8309518948	3.239612

Present Value of $1 for 21 years, discounting at Comp. Interest.

Yrs.	at 3 per ct.	3½ per ct.	4 per ct.	4½ per ct.	5 per ct.	6 per ct.
1	.9708738	.9061836	.9615385	.9569378	.9523809	.943396
2	.9425959	.9335107	.9245562	.9157299	.9070295	.889996
3	.9151417	.9019427	.8889964	.8762966	.8638376	.839619
4	.8884870	.8714422	.8548042	.8385613	.8227025	.792094
5	.8626088	.8419732	.8219271	.8024511	.7835262	.747258
6	.8374843	.8135006	.7903145	.7678957	.7462154	.704961
7	.8130915	.7859910	.7599178	.7348285	.7106813	.665057
8	.7894092	.7594116	.7306902	.7031851	.6768394	.627412
9	.7664167	.7337310	.7025867	.6729044	.6446089	.591898
10	.7440939	.7089188	.6755642	.6439277	.6139133	.558395
11	.7224213	.6849457	.6495809	.6161988	.5846793	.526788
12	.7013799	.6617833	.6245971	.5896639	.5568374	.496969
13	.6809513	.6391041	.6005741	.5642716	.5303214	.468839
14	.6611178	.6177818	.5774751	.5399729	.5050679	.442301
15	.6418619	.5968906	.5552645	.5167204	.4810171	.417265
16	.6231669	.5767059	.5339082	.4944693	.4581115	.393646
17	.6050164	.5572038	.5133733	.4731764	.4362967	.371364
18	.5873946	.5383611	.4936281	.4528004	.4155207	.350344
19	.5702860	.5201557	.4746424	.4433018	.3957340	.330513
20	.5536758	.5025659	.4566870	.4146429	.3768895	.311805
21	.5375493	.5855709	.4388336	.3967874	.3589424	.294155

Amount of $1 Annuity for 21 years, calculated at Comp. Interest.

1	1.0000000	1.00000	1.00000	1.	1.	1.000000
2	2.0300000	2.03500	2·04000	2.04500	2.05000	2.060000
3	3.0909000	3.10623	3.12160	3.13703	3.15250	3.183600
4	4.1836270	4.21494	4.24646	4.27819	4.31013	4.374616
5	5.3091358	5.36247	5.41632	5.47071	5.52563	5.637092
6	6.4684099	6.55015	6.63298	6.71689	6.80191	6.975316
7	7.6624622	7.77941	7.89829	8.01915	8.14204	8.393838
8	8.8923361	9.05169	9.21423	9.38001	9.54911	9.897468
9	10.1591061	10.36880	10.58280	10.80211	11.02656	11.491319
10	11.4638793	11.73139	12.00611	12.28821	12.57789	13.180793
11	12.8077957	13.14199	13.48635	13.84118	14.20679	14.791643
12	14.1920296	14.60196	15.02581	15.46403	15.91713	16.869941
13	15.6177905	16.11303	16.62684	17.15991	17.71298	18.882138
14	17.0863242	17.67699	18.29191	18.93211	19.59863	21.015066
15	18.5989139	19.29568	20.02359	20.78405	21.57856	23.275970
16	20.1568813	20.97103	21.82453	22.71934	23.65749	25.672528
17	21.7615877	22.70502	23.69751	24.74171	25.84037	28.212880
18	23.4144354	24.49969	25.64541	26.85508	28.13238	30.905653
19	25.1168684	26.35718	27.67123	29.06356	30.53900	33.759892
20	26.8703745	28.27968	29.77808	31.37142	33.06595	36.785591
21	28.6764857	30.26947	31.96920	33.78314	35.71925	39.992727

A TABLE of the present worth of $1 annuity for 21 years.

Yrs	3 per ct.	3½ per ct.	4 per ct.	4½ per ct.	5 per ct.	6 per ct.
1	0.970874	0.966184	0.961539	0.956938	0.952381	0.9524
2	1.913470	1.899694	1.886095	1.872668	1.859410	1.8334
3	2.828611	2.801637	2.775091	2.748964	2.723248	2.6730
4	3.717098	3.673079	3.629895	3.587526	3.545951	3.4651
5	4.579707	4.515052	4.451822	4.389977	4.329477	4.2124
6	5.417191	5.328553	5.242137	5.157873	5.075692	4.9173
7	6.230283	6.114544	6.002055	5.892701	5.786373	5.5824
8	7.019692	6.873956	6.732745	6.595886	6.463213	6.2098
9	7.786109	7.607687	7.435332	7.268791	7.107822	6.8017
10	8.530203	8.316605	8.110896	7.912718	7.721735	7.3601
11	9.252624	9.001551	8.760477	8.528917	8.306414	7.8869
12	9.954004	9.663334	9.385074	9.118581	8.863252	8.3838
13	10.634955	10.302739	9.985647	9.682852	9.393573	8.8527
14	11.296073	10.920520	10.563122	10.222825	9.898641	9.2950
15	11.937935	11.517411	11.118387	10.739546	10.379658	9.7123
16	12.561102	12.094117	11.652296	11.234015	10.837770	10.1059
17	13.166119	12.651321	12.165669	11.707191	11.274066	10.4773
18	13.753513	13.189682	12.659297	12.159992	11.689587	10.8276
19	14.323799	13.709837	13.133939	12.593294	12.085321	11.1581
20	14.877475	14.212403	13.590326	13.007937	12.462210	11.4699
21	15.415024	14.697974	14.029160	13.404724	12.821153	11.7641

A TABLE of the amount of which $1 will purchase for 21 years.

1	1.0300000	1.0350000	1.0400000	1.045000	1.0500000	1.06000
2	.5226108	1.5264005	.5301961	.5339976	.5378049	.54543
3	.3535304	.3569342	.3603485	.3637734	.3672086	.37411
4	.2690271	.2722511	.2754901	.2787437	.2820118	.28859
5	.2183546	.2214814	.2246271	.2277916	.2309748	.23739
6	.1845975	.1876682	.1907619	.1938784	.1970175	.20336
7	.1605064	.1635445	.1666096	.1697015	.1728198	.17913
8	.1424564	.1454767	.1485278	.1516097	.1547218	.16103
9	.1284339	.1314460	.1344930	.1375745	.1406901	.14702
10	.1172305	.1202414	.1232909	.1263788	.1295046	.13586
11	.1080775	.1110920	.1141490	.1172482	.1203889	.12079
12	.1004621	.1034840	.1065522	.1096662	.1128254	.11927
13	.0940295	.0970616	.1001437	.1032754	.1064558	.11296
14	.0885263	.0915707	.0946690	.0978203	.1010240	.10758
15	.0837666	.0868251	.0899411	.0931138	.0963423	.10299
16	.0796109	.0826848	.0858200	.0890154	.0922699	.09895
17	.0759535	.0790431	.0821985	.0854176	.0886991	.09544
18	.0727087	.0758168	.0789933	.0822369	.0855462	.09235
19	.0698139	.0729403	.0761386	.0794073	.0827450	.08962
20	.0672157	.0703611	.0735818	.0768761	.0802426	.08718
21	.0648718	.0680366	.0712801	.0746006	.0779961	.08500

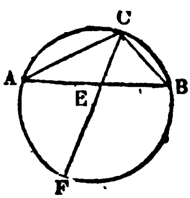
The Analysis and Construction of Geometrical Problems.

Geometrical Analysis is the way by which we proceed from the *quæsitum*, or thing demanded, granted for the moment to be known, till we have connected it by a series of consequences with something anteriorly known, or placed it among the number of principles known to be true.

Analysis may be distinguished into two kinds. In the one, which was named by Pappus contemplative, it is proposed to ascertain the truth or the falsehood of a proposition advanced as a theorem; the other is referred to the solution of problems, or to the investigation of unknown relations. In the analysis of a *theorem* we assume as true, or as previously existing, the subject of the proposition advanced, and proceed by the consequences of the hypothesis to something known to be either true or false; and if the result be thus found true, the proposition advanced is likewise true—if false, false. The direct demonstration is afterwards formed, by taking up again, in an inverted order, the several parts of the analysis. When a *problem* is under consideration, we first suppose it resolved, and then pursue the consequences thence derived till we come to something known. If the ultimate result thus obtained be comprised in what geometers call data, (that is, such data as are consistent,) the question proposed may be resolved; the construction is also constituted by taking the parts of the analysis in an inverted order. If the last result of the analysis involve any impossibility or contradiction to pre-established conclusions, it will prove evidently, in this case as well as in the former, that the thing proposed is impossible.

In illustration of these remarks, the following examples are given.

1. It is required to draw, in a given segment of a circle, from the extremes of the base A and B, two lines, AC, BC, meeting at a point C in the circumference, such that they shall have to each other a given ratio, viz. that of M to N.



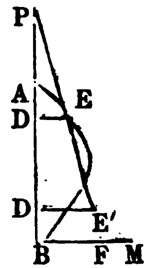
Analysis. Suppose that the thing is effected, that is to say, that AC : CB :: M : N, and let the base AB of the segment be cut in the same ratio in the point E. Then EC, being drawn, will bisect the angle ACB (by th. 83, Geom.); consequently, if the circle be completed, and CE be produced to meet it in F, the remaining circumference will also be bisected in F, or have FA = FB, because those arcs are the double measures of equal angles: therefore the point F, as well as E, being given, the point C is also given.

Construction. Let the given base of the segment AB be cut in

the point E in the assigned ratio of M to N, and complete the circle: bisect the remaining circumference in F; join FE, and produce it till it meet the circumference in C; then drawing CA, CB, the thing is done.

Demonstration. Since the arc FA = the arc FB, the angle ACF = angle BCF by theor. 49, Geom.; therefore AC : CB :: AE : EB, by theor. 83. But AE : EB :: M : N, by construction; therefore AC : CB :: M : N.

2. From a given circle to cut off an arc such that the sum of m times the sine, and n times the versed sine, may be equal to a given line.

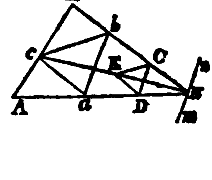


Anal. Suppose it done, and that AEE′B is the given circle, BE′E the required arc, ED its sine, BD its versed sine; in DA (produced if necessary) take BP an nth part of the given sum; join PE, and produce it to meet BF $\perp$ to AB, or $\parallel$ to ED, in the point F. Then, since $m.ED + n.BD = n.BP = n.PD + n.BD$; consequently $m.ED = n.PD$; hence PD : ED :: $m : n$. But PD : ED :: (by sim. tria.) PB : BF; therefore PB : BF :: $m : n$. Now PB is given, therefore BF is given in magnitude, and being at right angles to PB, is also given in position; therefore the point F is given, and consequently PF given in position; and therefore the point E, its intersection with the circumference of the circle AEE′B, or the arc BE is given. Hence the following—

Constr. From B, the extremity of any diameter AB of the given circle, draw BM at right angles to AB; in AB (produced if necessary) take BP an nth part of the given sum; and on BM take BF so that BF : BP :: $n : m$. Join PF, meeting the circumference of the circle in E and E′, and BE or BE′ is the arc required.

Demon. From the points E and E′ draw ED and E′D′ at right angles to AB. Then, since BF : BP :: $n : m$, and (by sim. tria.) BF : BP :: DE : BP; therefore DE : DP :: $n : m$. Hence $m.DE = n.DP$; add to each $n.BD$, then will $m.DE + n.BD = n.BD + n.DP = n.PB$, or the given sum.

3. In a given triangle ABH, to inscribe another triangle abc, similar to a given one, having one of its sides parallel to a line mBn given by position, and the angular points a, b, c, situate in the sides AB, BH, AH, of the triangle ABH respectively.



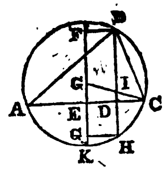
Anal. Suppose the thing done, and that abc is inscribed as required. Through any point C in BH draw CD parallel to mBn, or to ab, and cutting AB in D; draw CE parallel to bc, and DE to ac, intersecting each other in E. The triangles DEC, acb, are similar,

and DC : ab :: CE : bc; also BDC, Bab, are similar, and DC : ab :: BC : Bb. Therefore, BC : CE :: Bb : bc; and they are about equal angles, consequently B, E, c, are in a right line.

Constr. From any point C in BH, draw CD parallel to nm; on CD constitute a triangle CDE similar to the given one; and through its angle E draw BE, which produce till it cuts AH in c; through c draw ca parallel to ED, and cb parallel to EC; join ab, then abc is the triangle required, having its side ab parallel to mn, and being similar to the given triangle.

Demon. For, because of the parallel lines ac, DE, and cb, EC, the quadrilaterals BDEC and Bacb, are similar; and therefore the proportional lines DC, ab, cutting off equal angles BDC, Bab; BCD, Bba; must make the angles EDC, ECD, respectively, equal to the angles cab, cba; while ab is parallel to DC, which is parallel to mBn, by construction.

4. Given in a plane triangle, the vertical angle, the perpendicular, and the rectangle of the segments of the base, made by that perpendicular; to construct the triangle.



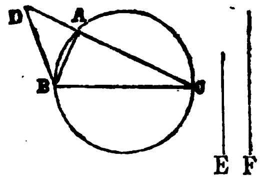
Anal. Suppose ABC the triangle required, BD the given perpendicular to the base AC, produce it to meet the periphery of the circumscribing circle ABCH, whose centre is O, in H; then, by th. 61, Geom. the rectangle BD.DH = AD.DC, the given rectangle; hence, since BD is given, DH and BH are given; therefore BI = HI is given; as also ID = OE; and the angle EOC is = ABC the given one, because EOC is measured by the arc KC, and ABC by half the arc AKC or by KC. Consequently EC and AC = 2EC are given. Whence this

Constr. Find DH such that DB.DH = the given rectangle, or find DH $= \frac{\text{AD.DC}}{\text{BD}}$; then on any right line GF take FE = the given perpendicular, and EG = DH; bisect FG in O, and make EOC = the given vertical angle; then will OC cut EC, drawn perpendicular to OE, in C. With centre O and radius OC, describe a circle, cutting CE produced in A; through F parallel to AC draw FB, to cut the circle in B; join AB, CB, and ABC is the triangle required.

Remark. In a similar manner we may proceed, when it is required to divide a given angle into two parts, the rectangle of whose tangents may be of a given magnitude. See prob. 40, Simpson's Select Exercises.

5. From the extremities B and C of the base of a given segment of a circle BAC, it is required to draw two lines BA, CA, meeting

at a point A in the circumference, so that they shall have a given ratio to each other; as, for instance, that of E to F.



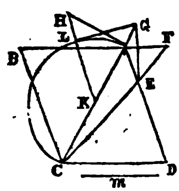
Anal. Suppose the thing done, or that the point A is found: (that is, such as to render AC : AB :: F : E) and let BD be drawn, making the angle CBD equal to CAB, and meeting CA produced in D.

Then the angle C being common to the two triangles CBA, CBD, and the angles CAB, CBD, being equal, they are equiangular; and consequently AC : AB :: CB : CD. Whence CB being given, BD is also given in position and magnitude.

Constr. Draw BB, making the angle CBD equal to that which is contained in the given segment CAD, and CB to BD in the given ratio of F to E. Join CD, cutting the segment in A; and draw BA, AC. These are the lines required.

Demon. The triangles CBA, CBD, being equiangular, AC : AB :: CB : BD :: F : E, which is the given ratio by construction.

6. From one angle C of a given rhombus ABCD, to draw a line, such that the part intercepted by the sides BA, AD, which contain the opposite angle A, shall be of a given length, as *m*.



Anal. Suppose the thing done; and make the $\angle$ CEG $=$ $\angle$ CAF.

Then by sim. tria. BA : AF :: CE : EF, and CA : AF :: CE : EG, whence CA . EG $=$ AF . CE $=$ BA . EF, or CA : AB :: EF ($=m$) : EG. And since $\angle$ CAF $=$ $\angle$ BAC $=$ $\angle$ CAD, the whole $\angle$ GAE $=$ CAF or CEG. Hence the triangles AGE, CGE, being similar, CG : EG :: EG : GA, or CG . GA $=$ EG^2. But EG is already known, being a fourth proportional to CA, BA, and *m* (as shown above): wherefore also the rectangle CG . GA is known. Whence arises the following—

Const. Upon the diameter of the rhombus CA, describe a semicircle, and make AH perpendicular to AC, and $=$ to the fourth proportional to AC, AB, and *m*. From the centre of the circle K, draw KH, cutting the circumference in L, and make KG $=$ KH; and join GL. Then, from the point G, apply GE $=$ GL; and through C and E draw CEF, meeting BA produced in F. The line EF will be that required.

Demon. Since the sides KG, KL, of the triangle KGL, are equal to the sides KH, KA, of the triangle KHA, and the $\angle$ K is common, we have $\angle$ KGL $=$ $\angle$ KAH, and the side GL (or GE) $=$ AH. But the $\angle$ HAK being right (constr.), the $\angle$ KLG is also

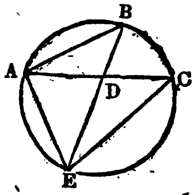
right, and GL is a tangent to the circle at L. Whence CG.GA $=$ GL^2 $=$ AH^2 $=$ GE^2, or CG : GE :: GE : GA; and therefore the triangles CGE, AGE, being similar, the $\angle$ CEG $= \angle$ EAG. Again, the $\angle$ GAF being equal to BAC or CAD, the whole $\angle$ EAG $=$ CAF; and consequently CEG $=$ CAF. Hence the triangles CGE, CFA, being similar, CA : AF :: CE : EG, or CA.EG $=$ AF.CE. Also, since AE, BC, are parallel, BA : AF :: CE : EF; or BA.EF $=$ AF.CE; whence, also, BA.EF $=$ CA.EG, or CA : BA :: EF : EG. But CA : BA :: *m* : AH (by constr.) whence EF : EG :: *m* : AH; and therefore, EG, as before shown, being equal to AH, EF is also equal to *m*.

Analysis and Demonstration of Theorems.

THEOREM 1. The square of a line bisecting the vertical angle of any triangle, together with the rectangle of the segments of the base made by that line, is equal to the rectangle contained under the sides of the triangle.

Let ABC be any triangle, and BD drawn to bisect the vertical angle, and cut the base in D; then will BD^2 + AD.DC $=$ AB.BC.

Anal. Suppose the theorem to be true; that is, suppose BD^2 + AD.DC $=$ AB.BC. Then, by way of preparation, or construction, or to obtain something on which to found our analysis, let a circle be described about the triangle ABC, and let BD be produced to meet it in E; and join EC.



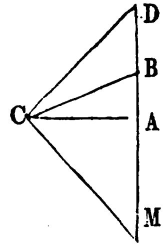
Then AD.DC $=$ BD.DE. Add BD^2 to each of these, and BD^2 + AD.DC $=$ BD^2 + BD.DE. But BD^2 + BD.DE $=$ EB.ED; and BD^2 + AD.DC $=$ AB.BC, by hypothesis. Hence, also, AB.BC $=$ EB.BD.

Now this we shall find to be true by an elementary theorem;* and hence the theorem which forms its foundation is also true.

Demon. Describe a circle ABCE about the triangle, and produce BD to E, and join EC. Because the triangles ABD, EBC, are similar, having the angles at A and E equal, and likewise the angles at B, by hypothesis, equal, we shall have AB : BD :: BE : BC. Therefore, AB.BC $=$ EB.BD. But EB.BD $=$ BD^2 + BD.DE, and BD.DE $=$ AD.DC. Therefore, finally, AB.BC $=$ BD^2 + AD.DC.

* For the triangles ABD, EBC, are similar, having the angles at A and E equal, and the angles at B in each of them equal, by the hypothesis. Hence AB : BD :: BE : BC; that is, AB.BC $=$ EB.BD.

2. If one of the angles C, of a right-angled triangle CAD, be bisected by a line CB meeting AD in B, then $2CA^2 : CA^2 - AB^2 :: AD : AB$.



Anal. Suppose it true, and through C draw CM perpendicular to CD, meeting DA produced in M; then, by hypothesis, $2CA^2 : CA^2 - AB^2 :: AD : AB$. But $CA^2 = MA.AD$, by similar triangles, DCA and ACM; therefore $2MA.AD : 2MA.AD - AB^2 :: AD : AB :: 2MA.AD : 2MA.AB$; whence $2MA.AD - AB^2 = 2MA.AB$. And adding $AM^2 + AB^2$ to each of these equals, we have $DM.MA = AM^2 + AB^2 + 2AM.AB$. Now $DM.MA = MC^2$, and $AM^2 + AB^2 + 2AM.AB = MB^2$. Hence $MC = MB$, and the angle $MBC = MCB$. Therefore the angle BCA is equal to the angle BCD.

Demon. The triangle MBC is isosceles, having the defects from a right angle of the angles at the base BC equal. Hence MB^2 $(= MC^2) = CA^2 + AM^2$. And taking $MA^2 + AB^2$ from each of their equals, there remains $2MA.AB = CA^2 - AB^2$; and as $MA.AD = CA^2$, we have $2CA^2 : CA^2 - AB^2 :: 2MA.AD : 2MA.AB :: AD : AB$.

It will be perceived, that the steps of the analysis and demonstration are, all through, the reverse of each other.

Application of Algebra to Geometry.

When it is proposed to resolve a geometrical problem algebraically, or by algebra, it is proper, in the first place, to draw a figure that shall represent the several parts or conditions of the problem, and to suppose that figure to be the true one. Then, having considered attentively the nature of the problem, the figure is next to be prepared for a solution, if necessary, by producing or drawing such lines in it as appear most conducive to that end. This done, the usual symbols or letters, for known and unknown quantities, are employed to denote the several parts of the figure, both the known and unknown parts, or as many of them as necessary, as also such unknown line or lines as may be easiest found, whether required or not. Then proceed to the operation, by observing the relations that the several parts of the figure have to each other; from which, and the proper theorems in Nulty's Elements of Geometry, make out as many equations independent of each other, as there are unknown quantities employed in them: the resolution of which equations, in the same manner as in arithmetical problems, will determine the unknown quantities, and resolve the problem proposed.

As no general rule can be given for drawing the lines and selecting the fittest quantities to substitute for, so as always to bring out the most simple conclusions, because different problems require different modes of solution; the best way to gain experience, is to try the solution of the same problem in different ways, and then apply that which succeeds best, to other cases of the same kind, when they afterwards occur. The following particular directions, however, may be of some use.

1st. In preparing the figure, by drawing lines, let them be either parallel or perpendicular to other lines in the figure, or so as to form similar triangles. And if an angle be given, it will be proper to let the perpendicular be opposite to that angle, and to fall from one end of a given line, if possible.

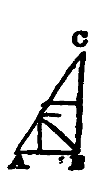
2d. In selecting the quantities proper to substitute for, those are to be chosen, whether required or not, which lie nearest the known or given parts of the figure, and by means of which the next adjacent parts may be expressed by addition and subtraction only, without using surds.

3d. When two lines or quantities are alike related to other parts of the figure or problem, the best way is, not to make use of either of them separately, but to substitute for their sum, or difference, or rectangle, or the sum of their alternate quotients, or for some line or lines, in the figure, to which they have both the same relation.

4th. When the area, or perimeter, of a figure, is given, or such parts of it as have only a remote relation to the parts required, it is sometimes of use to assume another figure similar to the proposed one, having one side equal to unity, or some other unknown quantity. For, hence the other parts of the figure may be found, by the known proportions of the like sides, or parts, and so an equation be obtained. For examples, take the following problems; and for the principles of their geometrical construction, pp. 577—582.

Problem 1. In a right-angled triangle, having given the base (3), and the sum of the hypotenuse and perpendicular (9), to find both these two sides.

Let ABC represent the proposed triangle right-angled at B. Put the base AB $=3=b$, and the sum AC + BC of the hypotenuse and perpendicular $=9=s$; also, let x denote the hypotenuse AC, and y the perpendicular BC.



Then by the question, - - - $x+y=s$,
And by theorem 34 - - - $x^2=y^2+b^2$,
By transposing y in the 1st. equation gives $x=s-y$,
This value of x substituted in the 2d, gives $s^2-2sy+y^2=y^2+b^2$,
Taking away y^2 on both sides leaves - $s^2-2sy=b^2$,

By transpos. $2sy$ and b^2, gives - - $s^2 - b^2 = 2sy$,

And dividing by $2s$, gives - - - $\frac{s^2 - b^2}{2s} = y = 4 = BC$.

Hence $x = s - y = 5 = AC$.

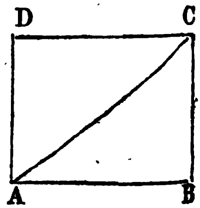
N. B.—In this solution and some of the following ones, the notation is made by using as many unknown letters, x and y, as there are unknown sides of the triangle, a separate letter for each: in preference to using only one unknown letter for one side, and expressing the other unknown side in terms of that letter and the given sum or difference of the sides; though this latter way would render the solution shorter and sooner; because the former way gives occasion for more diversified practice in reducing equations; which is the very end and reason for which these problems are given at all.

2. In a right-angled triangle, having given the hypotenuse (5), and the sum of the base and perpendicular (7), to find both these two sides.

Let ABC represent the proposed triangle right-angled at B. Put the given hypotenuse $AC = 5 = a$, and the sum $AB + BC$ of the base and perpendicular $= 7 = s$; also let x denote the base AB, and y the perpendicular BC.

Then by the question $x + y = s$,
And by theorem 34 $x^2 + y^2 = a^2$,
By transposing y in the 1st, gives $x = s - y$,
By substitu. this value for x, gives $s^2 - 2sy + 2y^2 = a^2$,
By transposing s^2, gives $2y^2 - 2sy = a^2 - s^2$,
By dividing by 2, gives $y^2 - sy = \frac{1}{2}a^2 - \frac{1}{2}s^2$,
By completing the square, gives $y^2 - sy + \frac{1}{4}s^2 = \frac{1}{2}a^2 - \frac{1}{4}s^2$,
By extracting the root, gives $y - \frac{1}{2}s = \sqrt{\{\frac{1}{2}a^2 - \frac{1}{4}s^2\}}$,
By transposing the $\frac{1}{2}s$, gives $y = \frac{1}{2}s \pm \sqrt{\{\frac{1}{2}a^2 - \frac{1}{4}s^2\}}$
$= 4$ and 3, the values of x and y.

3. In a rectangle, having given the diagonal (10), and the perimeter, or sum of all the four sides (28), to find each of the sides severally.

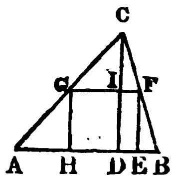


Let ABCD be the proposed rectangle; and put the diagonal $AC = 10 = d$, and half the perimeter $AB + BC$ or $AD + DC = 14 = a$; also put one side $AB = x$, and the other side $BC = y$.

Hence, by right-angled triangles $x^2 + y^2 = d^2$,
And by the question $x + y = a$,

Then by transpos. y in the 2d, gives $x = a - y$,
This value substitu. in the 1st, gives $a^2 - 2ay + 2y^2 = d^2$,
Transposing a^2, gives $2y^2 - 2ay = d^2 - a^2$
And dividing by 2, gives $y^2 - ay = \frac{1}{2}d^2 - \frac{1}{2}a^2$,
By completing the square, it is $y^2 - ay + \frac{1}{4}a^2 = \frac{1}{2}d^2 - \frac{1}{4}a^2$,
And extracting the root, gives $y - \frac{1}{2}a = \sqrt{\{\frac{1}{2}d^2 - \frac{1}{4}a^2\}}$,
And transposing $\frac{1}{2}a$, gives $y = \frac{1}{2}a \pm \sqrt{\{\frac{1}{2}d^2 - \frac{1}{4}a^2\}}$
$= 8$, or 6, the values of x and y.

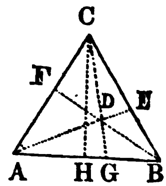
4. Having given the base and perpendicular of any triangle, to find the side of a square inscribed in the same.



Let ABC represent the given triangle, and EFGH its inscribed square. Put the base AB $= b$, the perpendicular CD $= a$, and the side of the square GF or GH $=$ DI $= x$; then will CI $=$ CD $-$ DI $= a - x$.

Then, because the like lines in the similar triangles ABC, GFC, are proportional (by theor. 84, Geom.), AB : CD :: GE : CI, that is, $b : a :: x : a - x$. Hence, by multiplying extremes and means, $ab - bx = ax$, and by transposing bx, gives $ab = ax + bx$; then dividing by $a + b$, gives $x = \frac{ab}{a+b} =$ GF or GH, the side of the inscribed square; which therefore is of the same magnitude, whatever the species or the angles of the triangle may be.

5. In an equilateral triangle, having given the lengths of the three perpendiculars, drawn from a certain point within, on the three sides; to determine the sides.



Let ABC represent the equilateral triangle, and DE, DF, DG, the given perpendiculars from the point D. Draw the lines DA, DB, DC, to the three angular points; and let fall the perpendicular CH on the base AB. Put the three given perpendiculars, DE $= a$, DF $= b$, and DG $= c$, and put $x =$ AH or BH, half the side of the equilateral triangle. Then is AC or BC $= 2x$, and by right-angled triangles the perpendicular CH $= \sqrt{\{AC^2 - AH^2\}}$ $= \sqrt{\{4x^2 - x^2\}} = \sqrt{3x^2} = x\sqrt{3}$.

Now, since the area or space of a rectangle is expressed by the product of the base and height (cor. 2, th. 81, Geom.), and since a triangle is equal to half a rectangle of equal base and height (cor. 1, th. 26), it follows, that the whole triangle ABC is $= \frac{1}{2}AB \times CH$ $= x \times x\sqrt{3} = x^2\sqrt{3}$

the triangle $ABD = \frac{1}{2}AB \times DG = x \times c = cx$,
the triangle $BCD = \frac{1}{2}BC \times DE = x \times a = ax$,
the triangle $ACD = \frac{1}{2}AC \times DF = x \times b = bx$.

But the three last triangles make up, or are equal to, the whole former or great triangle; that is, $x^2\sqrt{3} = ax + bx + cx$; hence, dividing by x, gives $x\sqrt{3} = a + b + c$, and dividing by $\sqrt{3}$, gives $x = \frac{a+b+c}{\sqrt{3}}$, half the side of the triangle sought.

Also, since the whole perpendicular CH is $= x\sqrt{3}$, it is therefore $= a + b + c$. That is, the whole perpendicular CH is just equal to the sum of all the smaller perpendiculars DE + DF + DG taken together, wherever the point D is situated; a property common to all regular polygons.

6. In a right-angled triangle, having given the base (3), and the difference between the hypotenuse and perpendicular (1); to find both these two sides.

Put x for the hypotenuse, b for the base (3) of the triangle, and d for the difference (1) of the hypotenuse and perpendicular. Then, because (Geom. theorem 9) the hypotenuse is greater than the perpendicular, $x - d =$ the perpendicular. But (Geom. theorem 34) $x^2 = (x-d)^2 + b^2 = x^2 - 2dx + d^2 + b^2$, that is, $2dx = d^2 + b^2$ and $x = \frac{d^2+b^2}{2d} = \frac{1^2+3^2}{2 \times 1} = \frac{10}{2} = 5$.

7. In a right-angled triangle, having given the hypotenuse (5), and the difference between the base and perpendicular (1), to determine both these two sides.

Put x for the less side about the right angle, d for the difference (1) of the base and perpendicular, also a for the hypotenuse (5). Then (Geom. theor. 34) $a^2 = x^2 + (x+d)^2 = x^2 + x^2 + 2dx + d^2$, $= 2x^2 + 2dx + d^2$, that is, $x^2 + dx = \frac{a^2-d^2}{2}$. Completing the square, extracting the square root, and transposing, $x = -\frac{1}{2}d \pm \sqrt{(\frac{a^2-d^2}{2} + \frac{1}{4}d^2)} =$ (subst. known values) $-\frac{1}{2} \pm \sqrt{(\frac{25-1}{2} + \frac{1}{4})}$ $= -\frac{1}{2} \pm 3\frac{1}{2} = 3$ or -4, of which values the last (being irrelevant) is to be rejected.

8. Having given the area, or measure of the space, of a rectangle, inscribed in a given triangle, to determine the sides of the rectangle.

If a be put for the perpendicular, and b for the base of the triangle; c for the area of the rectangle, and x for the length of the rectangle parallel to the base of the triangle, it will be as $b : a :: x : \frac{ax}{b}$ the difference of the altitudes of the rectangle and triangle. Therefore the altitude of the rectangle is $a - \frac{a}{b}x$. But c

$= x\left(a - \frac{ax}{b}\right) = ax - \frac{ax^2}{b}$, that is, $x = \frac{1}{2}b \pm \sqrt{\left(\frac{b^2}{4} - \frac{bc}{a}\right)}$ the length of the rectangle; consequently $\left\{\frac{c}{\frac{1}{2}b \pm \sqrt{\left(\frac{b}{4} - \frac{bc}{a}\right)}}\right\} =$ the breadth; which were to be determined.

9. In a triangle, having given the ratio of the two sides, together with the segments of the base, made by a perpendicular from the vertical angle; to determine the sides of the triangle.

Put a and b for the two segments of the base, x for the side adjacent to a, y for the other side of the triangle, and $m : n$, the ratio x has to y. Then, (since $m : n :: x : y$) it follows that $y = \frac{nx}{m}$. But $x^2 - a^2 = y^2 - b^2 = \frac{n^2x^2}{m^2} - b^2$. That is, $m^2x^2 - n^2x^2 = m^2a^2 - m^2b^2$.

$$\left.\begin{array}{l} \text{Hence } x = \sqrt{\left(\dfrac{m^2a^2 - m^2b^2}{m^2 - n^2}\right)} \\ \text{And } y = \dfrac{n}{m}\sqrt{\left(\dfrac{m^2a^2 - m^2b^2}{m^2 - n^2}\right)} \end{array}\right\} \text{which were to be determined.}$$

10. In a triangle, having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base; to find the sides of the triangle.

Put b for half the base, c for the line from the vertex, a for the sum of the two sides, and x for one of the sides. Then $a - x$ is the other side of the triangle. But (theor. 38) $2b^2 + 2c^2 = a^2 - 2ax + 2x^2$. Therefore $x = \frac{1}{2}a \pm \sqrt{(b^2 + c^2 - \frac{1}{4}a^2)}$, and $a - x = \frac{1}{2}a \pm \sqrt{(b^2 + c^2 - \frac{1}{4}a^2)}$. Ans.

11. In a triangle, having given the two sides about the vertical angle, with the line bisecting that angle, and terminating in the base; to find the base.

Let a represent one of the sides of the triangle, b the other side, c the line bisecting the vertical angle, and x the base; it will be, as $(a + b) : x :: a : \frac{ax}{a + b}$ the segment of the base, adjacent to a. And, as $(a + b) : x :: b : \frac{bx}{a + b}$ the segment of the base, adjacent to b. Therefore (because in any triangle having the vertical angle bisected, the rectangle of the two sides less the rectangle of the segments of the base is equal to the square of the line bisecting the vertical angle,) (theor. 44) $ab - \left(\frac{ax \times bx}{a + b}\right) = c^2$. Hence $x^2 =$

$\frac{a^3b + 2a^2b^2 + ab^3 - a^2c^2 - 2a^2cb - b^2c^2}{ab}$; which equation resolved, $x = (a + b)\sqrt{(\frac{ab - c^2}{ab})}$ = the base of the triangle.

12. To determine a right-angled triangle; having given the lengths of two lines drawn from the acute angles to the middle of the opposite sides.

If a be put for the line joining the acute angle at the base and the middle of the perpendicular, b for the line from the middle of the base to the vertical angle, x for half the base, and y for half the perpendicular; then $(2y)^2 = b^2 - x^2$; or $y^2 = \frac{b^2 - x^2}{4}$. Also $a^2 = \frac{b^2 - x^2}{4} + 4x^2$, or $4a^2 - b^2 = 15x^2$, that is, $x = \sqrt{(\frac{4a^2 - b^2}{15})}$, hence the base of the triangle $= 2\sqrt{(\frac{4a^2 - b^2}{15})}$ and $y^2 = \frac{b^2 - x^2}{4} = \frac{16b^2 - 4a^2}{4 \times 15}$, or $y = \sqrt{(\frac{4b^2 - a^2}{15})}$. Therefore $2y = 2\sqrt{(\frac{4b^2 - a^2}{15})}$ = the perpendicular; $\sqrt{(\frac{12a^2 + 12b^2}{15})}$ = the hypotenuse.

13. To determine a right-angled triangle; having given the perimeter, and the radius of its inscribed circle.

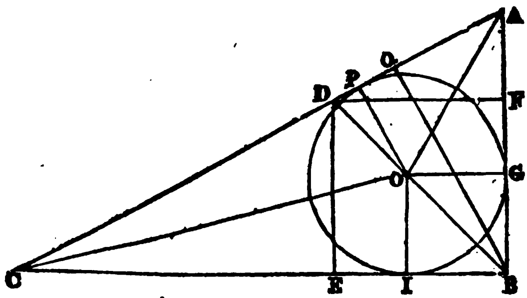
Put a for the radius of the inscribed circle, p for the perimeter of the triangle, x for half the sum of the base and perpendicular, and y for half the difference; then $p - 2x$ = the hypotenuse, and $\frac{ap}{2}$ = the area of the triangle. But $x^2 - y^2 = ap$, and $(x + y)^2 + (x - y)^2$ being equal to $(p - 2x)^2$ or $p^2 - 4px + 4x^2$, it follows that $p^2 - 4px + 4x^2 = 2x^2 + 2y^2$. And $2 \times (x^2 - y^2) = 2 \times ap$.

$$\text{Therefore by addition,} \left\{\begin{array}{l} 2x^2 - 2y^2 = \quad 2ap \\ 2x^2 + 2y^2 = p^2 \quad * \quad - 4px + 4x^2 \end{array}\right\}$$
$$4x^2 + \; * \; = p^2 + 2ap - 4px + 4x^2, \text{ or}$$

$p^2 + 2ap - 4px = 0$. Hence $x = \frac{p^2 + 2ap}{4p} = \frac{1}{2}(\frac{1}{2}p + a)$

And $y = \sqrt{(x^2 - ap)} = \sqrt{\{\frac{1}{4}(\frac{1}{2}p + a)^2 - ap.\}}$ Therefore
$\frac{1}{2}(\frac{1}{2}p + a) + \sqrt{\{\frac{1}{4}(\frac{1}{2}p + a)^2 - ap\}}$ = the base or perp.
$\frac{1}{2}(\frac{1}{2}p + a) - \sqrt{\{\frac{1}{4}(\frac{1}{2}p + a)^2 - ap\}}$ = the perp. or base.

Otherwise, let ABC be the given triangle, PIG the inscribed circle, whereof OP, OG, OI, are three radii at right angles to the sides of the triangle, CP is equal to CI, and AP to AG; also GB and BI are equal. Put p for the perimeter, r for the given radius, x for GA, and y for CI. Then BC $= r + y$; AB $= r + x$; and AC $= x + y$.



But $x+y+(r+y)+(r+x)=p$, or $x+y=\frac{p-2r}{2}=\frac{1}{2}p$ $-r$, and $(x+y)^2=(r+y)^2$; or $r(x+y)=xy-r^2$.

In the 1st equation $y=\frac{1}{2}p-r-x$; substitute, therefore, this value for y in the 2d equation, and reduce.

$x^2-(\frac{1}{2}p-r)x=-\frac{1}{2}rp$. Therefore

$x=\frac{1}{2}(\frac{1}{2}p-r)\pm\sqrt{\{\frac{1}{4}(\frac{1}{2}p-r)^2-\frac{1}{2}rp\}}$ And

$y=\frac{1}{2}(\frac{1}{2}p-r)\mp\sqrt{\{\frac{1}{4}(\frac{1}{2}p-r)^2-\frac{1}{2}rp\}}$ Hence, by adding r to each, $AB=\frac{1}{2}(\frac{1}{2}+pr)\pm\sqrt{\{\frac{1}{4}(\frac{1}{2}p-r)^2-\frac{1}{2}rp\}}$

$BC=\frac{1}{2}(\frac{1}{2}+pr)\mp\sqrt{\{\frac{1}{4}(\frac{1}{2}p-r)^2-\frac{1}{2}rp\}}$.

14. To determine a triangle; having given the base, the perpendicular, and the ratio of the two sides.

Let a represent the base, x one of the segments of the base by the perpendicular, r the side of the triangle adjacent to x, s the other side of the triangle, b the perpendicular, and $m:n$ the ratio of $r:s$.

Then $a-x$ is the other segment of the base. Also $x^2+b^2=r^2$, and $x^2+b^2+a^2-2ax=s^2$. But $m^2:n^2::(x^2+b^2):(x^2+b^2+a^2-2ax.)$ Therefore $(m^2-n^2)x^2-2am^2x=(n^2-m^2)b^2-a^2m^2$. That is, $x^2-\frac{2am^2}{m^2-n^2}x=-b^2-\frac{a^2m^2}{m^2-n^2}$, which equation resolved, $x=\frac{am^2}{m^2-n^2}\pm\sqrt{\left\{\frac{a^2m^4}{(m^2-n^2)^2}-\frac{a^2m^2}{m^2-n^2}-b^2.\right\}}$

$$r=\sqrt{\left\{b^2+\left(\frac{am^2}{m^2-n^2}\pm\sqrt{\left\{\frac{a^2m^4}{(m^2-n^2)^2}\frac{a^2m^2}{m^2-n^2}-b^2\right)^2.\right\}}\right\}}$$

And

$$s=\frac{n}{m}\left\{b^2+\left(\frac{am^2}{m^2-n^2}\pm\sqrt{\left\{\frac{a^2m^4}{(m^2-n^2)^2}-\frac{a^2m^2}{m^2-n^2}-b^2\right)^2.\right\}}\right\}^{\frac{1}{2}}$$

15. To determine a right-angled triangle; having given the hypotenuse, and the side of the inscribed square.

Put a for the hypotenuse, c for the side of the given square, x for

the perpendicular, and y for the base of the triangle. Then as $x : y :: (x - c) : c$; or $y(x - c) = cx$, that is, $xy = c(x + y)$. But $a^2 = x^2 + y^2$. Add $2xy$ to both sides of the equation, $a^2 + 2xy = x^2 + 2xy + y^2$. Substitute $2c(x + y)$ for $2xy$ on the first side, $a^2 + 2c(x + y) = x^2 + 2xy + y^2$ or $a^2 = (x + y)^2 - 2c(x + y)$. $\therefore x + y = c + \surd(a^2 + c^2)$, and assume $s = c + \surd(a^2 + c^2)$. Then having the sides $= s$, and the hypotenuse $= a$, x or $y = \frac{1}{2}s \pm \surd(\frac{1}{2}a^2 - \frac{1}{4}s^2)$.

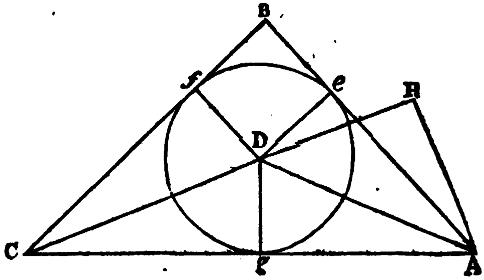
16. To determine the radii of three equal circles, described in a given circle, to touch each other and also the circumference of the given circle.

See Question 75, then $x = 2a\surd 3 - 3a = a(2\surd 3 - 3)$.

17. In a right-angled triangle, having given the perimeter, or sum of all the sides, and the perpendicular let fall from the right angle on the hypotenuse; to determine the triangle, that is, its sides.

Put $p =$ the perimeter of the triangle, a for the perpendicular falling on the hypotenuse, (x and y denote the greater and less sides) if unequal, $z =$ the hypotenuse, then $z = p - x - y$, and $z^2 = x^2 + y^2 = p^2 - 2p(x + y) + x^2 + 2xy + y^2$, $\therefore p(x + y) - \frac{1}{2}p^2 = xy$, and by comparison $(a + p) \times (x + y) = ap + \frac{1}{2}p^2$, $\therefore$ $x + y = p\left(\frac{a + \frac{1}{2}p}{a + p}\right)$, and $y = p\left(\frac{a + \frac{1}{2}p}{a + p}\right) - x$, put $s = x + y = p\left(\frac{a + \frac{1}{2}p}{a + p}\right)$. $z = p - y - x = p - s = b$; the sum of the two sides $= s$, and the hypotenuse $= b$ in the right-angled triangle. Hence (by prob. 11, page), x or $y = \frac{1}{2}s \pm \surd(\frac{1}{2}b^2 - \frac{1}{4}s^2)$. Therefore the triangle is determined.

18. To determine a right-angled triangle; having given the hypotenuse, and the difference of two lines drawn from the two acute angles to the centre of the inscribed circle.



In the annexed figure let CD be the greater, and AD the less of the two lines of which the difference is given, and let DH be a production of CD, and AH perpendicular to CH, AH is equal to HA, because the angle ADH is equal to the sum of the angles ACD, CAD, together equal to half a right angle, and the angle at H a right angle. If,

therefore, a be put for AC the hypotenuse, x for CD, y for AD, b for the difference of x and y, r for DH, and s for AH, then (Geom. theor. 4.) $r = s = \frac{y}{\sqrt{2}}$. But $x^2 + y^2 + 2rx = a^2 = x^2 + y^2 + xy\sqrt{2}$. Now substituting $x - b$ for y, and c for the $\sqrt{2}$, it will be $x^2 - 2bx + b^2 + x^2 + cx^2 - cbx = a^2$, that is, $(2 + c)x^2 - (2 + c)bx + b^2 = a^2$. Whence $x = \frac{b}{2} \pm \sqrt{(\frac{a^2 - b^2}{2 + c} + \frac{1}{4}b^2)}$, and $y = \sqrt{(\frac{a^2 - b^2}{2 + c} + \frac{1}{4}b^2)} - \frac{1}{2}b$. Consequently the radius of the inscribed circle is known, and the triangle determined.

For, put $m = \frac{1}{2}b \pm \sqrt{(\frac{a^2 - b^2}{2 + c} + \frac{1}{4}b^2)} = \text{CD}$.

$n = \sqrt{(\frac{a^2 - b^2}{2 + c} + \frac{1}{4}b^2)} - \frac{1}{2}b = \text{AD}$.

And let De, Df, Dg, be three radii at right angles to the sides of the triangle; likewise put w for Ag, and z for Cg.

$z^2 - w^2 = m^2 - n^2$.

Also $z + w = a$; and by division

$z - w = \frac{m^2 - n^2}{a}$, that is, $z = \frac{a^2 + m^2 - n^2}{2a}$; $w = \frac{a^2 - m^2 + n^2}{2a}$

and $\text{D}g = \sqrt{\left\{ m^2 - \frac{(a^2 + m^2 - n^2)^2}{4a^2} \right\}}$ the radius of the inscribed circle.

19. Given the base, the perpendicular, and the difference of the two other sides, to determine the triangle.

Put a for half the base, d for the difference of the two sides, b for the perpendicular, x for the excess of the greater segment of the base above a, y for the greater side of the triangle, and z for the less.

Then $y^2 = b^2 + (a + x)^2$, or $y = \sqrt{\{b^2 + (a + x)^2.\}}$ Also $z^2 = b^2 + (a - x)^2$, or $z = \sqrt{\{b^2 + (a - x)^2.\}}$

But $\sqrt{\{b^2 + (a + x)^2\}} - d = \sqrt{\{b^2 + (a - x)^2.\}}$ Therefore $b^2 + (a + x)^2 - 2d\sqrt{\{b^2 + (a + x)^2\}} + d^2 = b^2 + (a - x)^2$: and by red. $4ax + d^2 = 2d\sqrt{\{b^2 + (a + x)^2,\}}$ or $16a^2x^2 + 8ad^2x + d^4 = 4d^2(b^2 + a^2 + 2ax + x^2)$.

Consequently, $x = \sqrt{\left\{ \frac{4d^2(a^2 + b^2) - d^4}{16a^2 - 4d^2} \right\}}$, which value put equal c.

$$\left. \begin{array}{l} y = \sqrt{\{(a + c)^2 + b^2.\}} \\ z = \sqrt{\{(a - c)^2 + b^2.\}} \end{array} \right\} \text{ which were to be determined.}$$

20. Determine a triangle, having given the base, the perpendicular, and the rectangle, or product of the two sides.

Let a represent the perpendicular, b half the base, c the product of the two sides, x the excess of the greater segment of the base by the perpendicular above b; and put y for the greater side of the triangle, z for the less.

Then $y=\sqrt{\{a^2+(b+x)^2\}}=\dfrac{c}{z}$

$$z=\sqrt{\{a^2+(b-x)^2\}}=\frac{c}{y}=\frac{c}{\sqrt{\{a^2+(b+x)^2\}}}$$

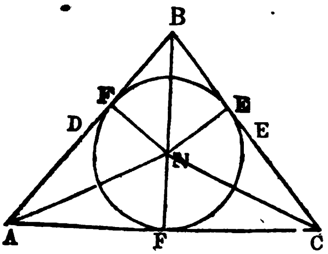
(or squared,) $a^2+(b-x)^2=\dfrac{c^2}{a^2+(b+x)^2}$. Hence $c^2=(a^2+b^2)^2+x^4+2(a^2-b^2)x^2$, that is, $x^4+2(a^2-b^2)x^2=(a^2+b^2)^2-c^2$. And, this equation resolved, by page 81,

$$x^2=-(a^2-b^2)\pm\sqrt{\{(a^2+b^2)^2-c^2+(a^2-b^2)^2\}}$$
$$\text{or } x=\sqrt{\{(b^2-a^2)\pm\sqrt{\{(a^2+b^2)^2-c^2+(a^2-b^2)^2\}},\}}$$

which value put $=d$.

$$\left.\begin{array}{l} y=\sqrt{\{a^2+(b+d)^2\}} \\ z=\sqrt{\{a^2+(b-d)^2\}} \end{array}\right\}\text{ which were to be determined.}$$

21. In a triangle having given all the three sides, find the radius of the inscribed circle.



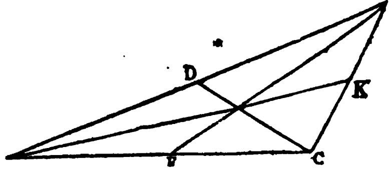
$$\left.\begin{array}{l}\text{Let AC}=a\\ \text{AB}=b\\ \text{BC}=c\end{array}\right\}\begin{array}{l}\text{radius AO}=x.\\ \text{Area of }\triangle\text{ ABC}\\ \qquad =s.\end{array}$$

$$\text{Then }\triangle\text{ AOC}=\tfrac{1}{2}a.x$$
$$\triangle\text{ BOC}=\tfrac{1}{2}b.x$$
$$\triangle\text{ AOB}=\tfrac{1}{2}c.x$$

$$\therefore \tfrac{1}{2}a.x+\tfrac{1}{2}b.x+\tfrac{1}{2}c.x=s.$$

That is, $x=\dfrac{2s}{a+b+c}$, which was required.

22. Determine a triangle, having given the lengths of three lines drawn from the three angles to the middle of the opposite sides.



Let ABC be the triangle, and AF, BE, and CD the three given lines. If a represent AF, b CD, and c BE; also if x be put for BC, y for AB, and z for AC; then (the. 38) $y^2+z^2=2a^2+\frac{1}{2}x^2$, that is, $y^2+z^2-\frac{1}{2}x^2=2a^2$. For the same reason

$x^2+z^2-\frac{1}{2}y^2=2b^2$. And

$x^2+y^2-\frac{1}{2}z^2=2c^2$. Comparing the 1st equation with double the sum of the 2d and 3d,

$4\frac{1}{2}x^2=2(2b^2+2c^2-a^2)$, or divided by $4\frac{1}{2}$, and the square root extracted,

$$\left.\begin{aligned} x&=\tfrac{2}{3}\sqrt{(2b^2+2c^2-a^2)}\\ y&=\tfrac{2}{3}\sqrt{(2a^2+2c^2-b^2)}\\ z&=\tfrac{2}{3}\sqrt{(2a^2+2b^2-c^2)}\end{aligned}\right\}$$

23. To determine a right-angled triangle; having given the side of the inscribed square, and the radius of the inscribed circle.

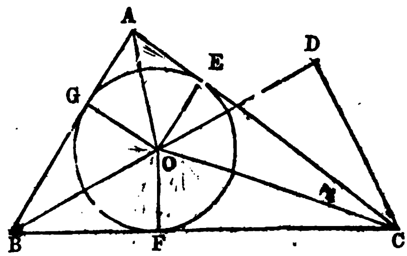
Let ABC be the proposed triangle, [See fig. on p. 589,] BFDE the inscribed square, OG and OP radii of the inscribed circle at right angles to AB and AC, and BD a diagonal of the inscribed square; also let BQ be perpendicular to AC from the right angle. Put a for the side of the given square, b for the radius of the given circle, and x for the segment AQ of the base AC by the perpendicular BQ; then FG $= a - b$, since GB = OG; and $a-b : a :: b : \frac{ab}{a-b}=$ BQ (because GF : BF :: OD : BD :: OP = BQ).

Wherefore (since BD $=\sqrt{2a^2}$), DQ $=\sqrt{\left\{2a^2-\frac{a^2b^2}{(a-b)^2}\right\}}$.

Let this value of DQ be recognized in c, and put d for $\frac{ab}{a-b}$.

It is as $x : d :: d : \frac{d^2}{x}=$ CQ [theor. 87]. And $x+c : \frac{d^2}{x}-c :: x : d$.* Hence $dx+dc=d^2-cx$. That is, $x=\frac{d^2-dc}{d+c}$ = AQ, wherefore the triangle is determined.

24. Determine a triangle and the radius of the inscribed circle, having given the lengths of three lines drawn from the three angles to the centre of that circle.



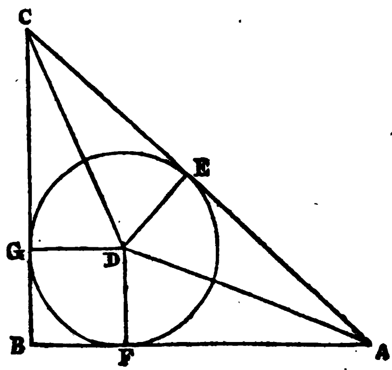
Having constructed the figure, draw lines from the centre of the circle to the points of contact, as OG, OF, OE. Also produce one of the given lines (as BO) indefinitely beyond the centre, and on it produced, from either of the other angles let fall a per-

* For the triangles AFD, DE, are similar; also the triangle AQB is similar to AFD, and consequently to DEB. Therefore AD : DC :: AF : DE (= DF) :: AQ : QB.

pendicular, (as CD). Then because the angles ABC, BCA, and CAB are together equal to two right angles, the angles OBC, OCB, and OAC are together equal to half of two right angles. But the angle OBC together with the angle OCB, is equal to the angle COD, therefore the angles COD, OAE are equal to the angles COD and OCD, each to each, and either pair to a right angle. Put $a = $ BO, $b = $ AO, $c = $ CO, and put x for the radius of the inscribed circle. Then because AO : OE :: CO : OD, $b : x :: c : \frac{c}{b}x = $ OD. Therefore CD $= \sqrt{(c^2 - \frac{c^2x^2}{b^2})} = \frac{c}{b}\sqrt{(b^2 - x^2)}$. But (theor. 36) $BC^2 = BO^2 + OO^2 + 2(BO \times OD)$, or BC $= \sqrt{\{\frac{b(a^2 + c^2) + 2acx}{b}.\}}$ Now BC : CD :: BO : OF. That is, $\sqrt{\{\frac{b(a^2 + c^2) + 2acx}{b}\}} : \frac{c}{b}\sqrt{(b^2 - x^2)} :: a : x$. Hence $bx^2\{2 \times (a^2 + c^2) + 2acx\} = a^2c^2(b^2 - x^2)$, and by reduction, $x^3 + (\frac{a^2b^2 + a^2c^2 + b^2c^2}{2abc})x^2 = \frac{1}{2}abc$, an equation in which x is determinable, and, x known, the triangle is determined.

The equation, however, arising from this problem, as it contains three dimensions of the unknown quantity, admits of no geometrical construction by means of a circle and right lines.

25. To determine a right-angled triangle; having given the hypotenuse, and the radius of the inscribed circle.

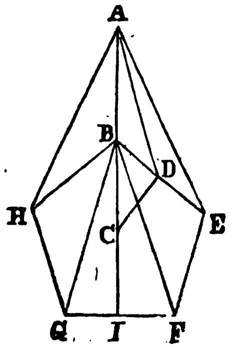


Construct a right-angled triangle, and inscribe in it a circle; also draw right lines from the centre to the points of contact, and to the two acute angles, as in the figure ABCDEFG. Put x for the base (AB), y for the perpendicular (BC), and let a represent the given radius (DG, DE or DF), b the hypotenuse (AC).

Then, because GB and DF are equal, $y - a$ is the expression for CG; and $x - a = $ AF. Also CE and CG are equal [Geom. 2 cor. the. lxi]; and AF is equal to AE. AE + CE = AC; that is, $(y - a) + (x - a) = b$, or $x + y = 2a + b$. Now $x^2 + y^2 = b^2$; comparing, therefore, the double of this equation, with the square of the preceding,

$x^2 - 2xy + y^2 = b^2 - 4ab - 4a^2$. Hence
$x - y = \sqrt{(b^2 - 4ab - 4a^2)}$, consequently
$x = a + \frac{1}{2}b \pm \frac{1}{2}\sqrt{(b^2 - 4ab - 4a^2)}$, and
$y = a + \frac{1}{2}b \pm \frac{1}{2}\sqrt{(b^2 - 4ab - 4a^2)}$. Therefore the triangle is determined.

26. A dodecædron is a solid composed of twelve regular pentagonal pyramids, whose verticles meet in the centre of the circumscribing sphere, and the bases of the pyramids form the superficies of the dodecædron. Now suppose a dodecædron having the side of each pentagon composing the superficies thereof eight inches, and supposing every two of its composing pyramids to be hollowed out in the form of the greatest hemisphere, cylinder, cube, cone, triangular pyramid, and square pyramid: What will the remainder of the dodecædron weigh, after having been hollowed or scooped out as above described, supposing each cubic foot of the matter of which it is composed to weigh 60 lbs.?



Let ABEFGH be one of the twelve pyramids constituting the dodecædron, and C the centre of its base; draw CD perpendicular to BE, and the rest of the lines as in the figure.

Putting A = 8 = BE one side of the pentagon or base, by Mensuration, I have $AC = A\sqrt{\frac{25 + 11\sqrt{5}}{40}} = 1.113516A$; and $CD = \frac{1}{2}A \times \tan. 54° = .68819095A$ = radius of the base of the cone. And, since the greatest inscribed cylinder is known to be $\frac{4}{9}$ of the cone, we shall have $\frac{13}{9}CD^2 \times 3.14159, \&c. \times \frac{1}{3}CA = 2.14916A^2 \times \frac{1}{3}CA$ = the cone and cylinder together. But the greatest triangle in the pentagon is BFG, and the greatest quadrangle is BFGH; the sum of these two is the $\triangle$ GHB + 2$\triangle$ BFG. Now the $\triangle$ GHB = GH $\times$ HB $\times$ $\frac{1}{2}$S. $\angle$ H = $\frac{1}{2}A^2 \times$ S. 108° = $.47552825A^2$, and 2 $\triangle$ BFG = BI $\times$ GF = $A^2 \times \frac{1}{2}$ tangent $\angle$ BFI = $A^2 \times \frac{1}{2}$ tang. 72° = $1.53884175A^2$; the sum of these two is $2.01437A^2$ = the bases of the triangular and quadrangular pyramids together; and consequently their contents together will be $2.01437A^2 \times \frac{1}{3}AC$. Adding this to the sum of the cone and cylinder above found, we have $4.16353A^2 \times \frac{1}{3}AC = 1.545386A^3$ = the sum of these four solids.

Again, $DA = \sqrt{(AC^2 + CD^2)} = \frac{3 + \sqrt{5}}{4}A = 1.309071A$,

and by sim. triangles $AD : DC :: CA : \frac{AC \times CD}{DA} = \frac{5 + 3\sqrt{5}}{20}A$

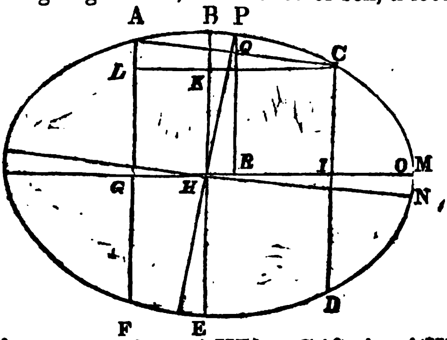
$= .5854102A =$ the radius of the inscribed hemisphere; and consequently its solidity is $(.5854102A)^3 \times \frac{2}{3} \times 3.14159$, &c. $= .4201837A^3$.

Also, a square may be inscribed in a pentagon by placing one of its sides parallel to a side of the pentagon; and, by calculation, it appears that if z be the side of the square, $z \div 1.0604974$ will be the side of the pentagon; hence, if this square be one face of the cube, and the pentagon the section of the pentagonal pyramid parallel to the base and distant from it the height of the cube, we shall have, by similar figures, $AC : BE :: AC - z : z \div 1.0604974$;

hence $z = \dfrac{1.0604974AC}{1.0604974A + AC} \times A = .5431802A =$ the side of the cube; and consequently the cube of this, or $.1602626A^3 =$ the content of it.

Collecting now the three sums together, and multiplying by 2, we have $4.2516646A^3$ for the contents of the twelve cavities. But, by Mensuration, $5A^3\sqrt{\dfrac{47 + 21\sqrt{5}}{40}} = 7.6631189A^3 =$ the content of the dodecædron. Consequently the difference of these two $= 3.4114543A^3 = 1746.6646016$ inches $= 1.0108013$ feet $=$ solidity remaining; which, at 60 lbs. a foot, weighs 60.648077 lbs. Ans.

27. In gauging a spheroidical ale cask, I found the diameter of one head to measure 18.1 inches, that of the other 16, the bung diameter 20, and the distance between the two heads 20.6 inches; also, by the cask lying a little obliquely, I observed that the liquor just rose to, or touched the upper extremities of the two heads. Having noted these dimensions, I was informed that there were in the cask a ball of iron weighing 60 lbs., another ball of lead weighing 90 lbs., and a cube of box, a foot square. What quantity of liquor was in the cask?



Let ACDF be the cask, BE the conjugate axe, and HM the semi-transverse CL ∥ to HM, HN ∥ to AC, HQP the semiconjugate of HN, and NO and QR ∥ BE.

By the nature of the ellipse, we have $\sqrt{(HB^2 - GA^2)} : \sqrt{(HB^2 - IC^2)} :: LK : KC$; and by compos. &c. $\sqrt{(HB^2 - GA^2)} + \sqrt{(HB^2 - IC^2)} : LC\ (= LK + KC)$

$$\because \begin{cases} (HB^2 - GA^2)^{\frac{1}{2}} : LK = \dfrac{LC \times \sqrt{(HB^2 - GA^2)}}{\sqrt{(HB^2 - GA^2)} + \sqrt{(HB^2 - IC^2)}} \\ = \dfrac{20.6 \times \sqrt{(10^2 - 9.05^2)}}{\sqrt{(10^2 - 9.05^2)} + \sqrt{(10^2 - 8^2)}} = 8.5463. \\ (HB^2 - IC^2)^{\frac{1}{2}} : KC = \dfrac{LC \times \sqrt{(HB^2 - IC^2)}}{\sqrt{(HB^2 - GA^2)} + \sqrt{(HB^2 - IC^2)}} \\ = \dfrac{20.6 \times \sqrt{(10^2 - 8^2)}}{\sqrt{(10^2 - 9.05^2)} + \sqrt{(10^2 - 8^2)}} = 12.0537. \end{cases}$$

And, $\sqrt{(EK \times KB)} : KC :: BH : HM = \dfrac{BH \times KC}{\sqrt{(EK \times KB)}} = \dfrac{120.537}{6} = 20.0895.$

Also, by right-angled $\triangle$ s, $AC = \sqrt{(CL^2 + LA^2)} = \sqrt{(20.6^2 + 1.05^2)} = 20.62674$; and sine of $\angle ACL = \dfrac{LA}{AC} = .0509048$ $=$ sine $\angle NHM$, which put $= s$, and its cosine $= c$.

Then, by the nature of the ellipse,

$MH^2 : HB^2 :: MH^2 - HO^2 : ON^2$;

but $HO = HN \times c$, and $ON = NH \times s$, wherefore,

$MH^2 : HB^2 :: MH^2 - NH^2 \times c^2 : HN^2 \times s^2$, or by subtraction

$MH^2 - HB^2 : HB^2 :: MH^2 - HN^2 \times (s^2 + c^2) : HN^2 \times s^2$,

or since $s^2 + c^2 = 1$

$MH^2 - HB^2 : HB^2 :: MH^2 - HN^2 : HN^2 \times s^2$, or

$(MH^2 - HB^2) \times s^2 : HB^2 :: MH^2 - HN^2 : HN^2$,

and by addition

$(MH^2 - HB^2) \times s^2 + HB^2 : HB^2 :: MH^2 : HN^2$, and hence

$$NH = \frac{MH \times HB}{\sqrt{(HB^2 + s^2 \times (MH^2 - HB^2))}} = \frac{MH \times HB}{\sqrt{(HB^2 \times c^2 + MH^2 \times s^2)}} = 20.01094.$$

Also, its semi-conjugate $PH = \sqrt{(MH^2 + HB^2 - HN^2)} = 10.15629$. But $HR = \dfrac{IH - GH}{2} = 1.7537$; and $RQ = \dfrac{GA + IC}{2} = 6.525$; and, by right-angled triangles, $HQ = \sqrt{(QR^2 + RH^2)} = 8.703511$; consequently $QP = PH - HQ = 1.45278$.

Then, by cor. 2, page 275 Mensuration, the content of the segment APC will be 3.14159 &c. $\times\ MH \times HB^2 \times QP^2 \times \dfrac{HP - \frac{1}{3}PQ}{PH^2} = 122.9787$ inches $=$ the vacuity.

But the content of the box cube is $12^3 = 1728$; all of which will be immersed, because it is heavier than ale. Also, using Ward's table of specific gravities, the solidity of the iron ball will be

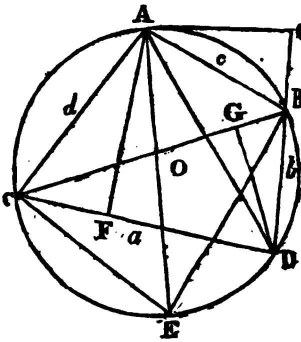
$\frac{60 \times 16}{4.422979} = 217.0483$; and that of the leaden one $\frac{90 \times 16}{6.553885} =$ 219.717.

The sum of these three added to the above segment, gives 2287.744 for the sum of the deductions.

Now the content of the whole cask is $(2EB^2 + FA^2) \times .2618HG + (2EB^2 + CD^2) \times .2618\,HI = (2EB^2 \times GI + FA^2 \times HG + CD^2 \times HI) \times .2618 = 5855.3142$.

From this whole content then taking the above sum of the deductions, we obtain 3567.5702 inches; which divided by 282, we have 12.65096 for the number of gallons required.

27. Having given the sides, $a = 6$, $b = 4$, $c = 5$, and $d = 3$, of a trapezium, inscribed in a circle, to determine the diameter of the circle.



Let ACE denote the circle, ABDC the inscribed trapezium; draw the diameter AE, join EC, EB; the angles ACE, ABE, being each in a semicircle, are right angles (Geom. p. 52). $\therefore$ $CE = \sqrt{\{AE^2 - AC^2\}}$, $EB = \sqrt{\{AE^2 - AB^2\}}$, (Geom. p. 35). Join CB, then $AB.CE + AC.EB = AE.CB$ (Geom. p. 233); put $AB = a$, $AC = b$, $CD = c$, $DB = d$, $AE = 2r$, then the equation becomes $a\sqrt{\{4r^2 - b^2\}} + b\sqrt{\{4r^2 - a^2\}} = 2r.CB$; in like manner we have, $c\sqrt{\{4r^2 - d^2\}} + d\sqrt{\{4r^2 - c^2\}} = 2r.CB$, $\therefore$ $a\sqrt{\{4r^2 - b^2\}} + b\sqrt{\{4r^2 - a^2\}} = c\sqrt{\{4r^2 - d^2\}} + d\sqrt{\{4r^2 - c^2\}}$ (1), in the same manner I find $b\sqrt{\{4r^2 - c^2\}} + c\sqrt{\{4r^2 - b^2\}} = d\sqrt{\{4r^2 - a^2\}} + a\sqrt{\{4r^2 - d^2\}}$, or $b\sqrt{\{4r^2 - c^2\}} - a\sqrt{\{4r^2 - d^2\}} = d\sqrt{\{4r^2 - a^2\}} - c\sqrt{\{4r^2 - b^2\}}$, (2).

By squaring (1) and reducing I have $2(a^2 + b^2 - c^2 - d^2)r^2 - (a^2b^2 - c^2d^2) = cd\sqrt{\{4r^2 - d^2\}} \times \sqrt{\{4r^2 - c^2\}} - ab\sqrt{\{4r^2 - b^2\}} \times \sqrt{\{4r^2 - a^2\}}$ (3), and by squaring we have (2), $4b^2r^2 - b^2c^2 - 2ab\sqrt{\{4r^2 - c^2\}} \times \sqrt{\{4r^2 - d^2\}} + 4a^2r^2 - a^2d^2 = 4r^2d^2 - a^2d^2 - 2dc\sqrt{\{4r^2 - a^2\}} \times \sqrt{\{4r^2 - b^2\}} + 4c^2r^2 - c^2b^2$, and reducing $2(a^2 + b^2 - c^2 - d^2)r^2 = ab\sqrt{\{4r^2 - c^2\}} \times \sqrt{\{4r^2 - d^2\}} - dc\sqrt{\{4r^2 - a^2\}} \times \sqrt{\{4r^2 - b^2\}}$ (4), eliminating $\sqrt{\{4r^2 - d^2\}} \times \sqrt{\{4r^2 - c^2\}}$ by (3) and (4), we have $2(ab - cd) \times (a^2 + b^2 - c^2 - d^2)r^2 - (a^2b^2 - c^2d^2)ab = -(a^2b^2 - c^2d^2) \times \sqrt{\{4r^2 - a^2\}} \times \sqrt{\{4r^2 - b^2\}}$, or by rejecting the factor $ab - cd$, $2(a^2 + b^2 - c^2 - d^2)r^2 - (ab + cd)ab = -(ab + cd) \times \sqrt{\{4r^2 - a^2\}} \times \sqrt{\{4r^2 - b^2\}}$ (5) by squaring (5), $4(a^2 + b^2 - c^2 -$

$d^2)^2r^4 - 4ab(ab + cd).(a^2 + b^2 - c^2 - d^2)r^2 + a^2b^2(ab + cd)^2 = (ab + cd)^2 \times (4r^2 - a^2) \times (4r^2 - b^2) = (ab + cd)^2 \times (16r^4 - 4r^2(a^2 + b^2) + a^2b^2)$, or $4(a^2 + b^2 - c^2 - d^2)^2r^2 - 4ab(ab + cd)(a^2 + b^2 - c^2 - d^2) = (ab + cd)^2 \times (16r^2 - 4(a^2 + b^2))$.

Or $\{(a^2 + b^2 - c^2 - d^2)^2 - 4(ab + cd)^2\}r^2 = \{ab(a^2 + b^2 - c^2 - d^2) - (ab + cd) \times (a^2 + b^2 \times (ab + cd) = - \{ab(c^2 + d^2) + cd(a^2 + b^2)\} \times \{ab + cd\}$ or there results $\{4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2\}r^2 = \{a^2b(c + d^2) + cd(a^2 + b^2)\} \times (ab + cd)$, consequently we have

$$r = \sqrt{\left\{\frac{\{ab(c^2 + d^2) + cd(a^2 + b^2)\} \times (ab + cd)}{\{4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2\}}\right\}}$$ Or I have

$$r = \sqrt{\left\{\frac{\{ab + cd\} \times \{ac + bd\} \times \{ad + bc\}}{4 \times \{ab + cd\}^2 - \{a^2 + b^2 - c^2 - d^2\}^2}\right\}}, \quad (6).$$

Again we have $4 \times \{ab + cd\}^2 - \{a^2 + b^2 - c^2 - d^2\}^2 = \{a^2 + b^2 + 2ab - c^2 - d^2 + 2cd\} \times \{c^2 + d^2 + 2cd - a^2 - b^2 + 2ab\} = \{a + b)^2 - (c - d)^2\} \times \{(c + d)^2 - (a - b)^2\} = \{a + b + c - d\} \times (a + b + d - c) \times (a + c + d - b) \times (b + c + d - a)$; put $a + b + c + d = 2s$, then $a + b + c - d = 2(s - d)$, $a + b + d - c = 2(s - c)$, $a + c + d - b = 2(s - b)$, and $b + c + d - a = 2(s - a)$, consequently we have

$$r = \frac{1}{4}\sqrt{\left\{\frac{(ab + cd) \times (ac + bd) \times (ad + bc)}{(s - a) \times (s - b) \times (s - c) \times (s - d)}\right\}} = 7.3722$$

as required (7).

Cor. If $d = 0$ the trapezium becomes a triangle and (7) reduces to $r = \frac{1}{4} \times \frac{abc}{\sqrt{((s - a) \times (s - b) \times (s - c) \times s)}}$ = the radius of the circle which will circumscribe the triangle.

Otherwise, put the angle CDB $= x$ (by trig. p.   ) $a^2 + b^2 - 2ab.\cos. x = BC^2 = c^2 + d^2 - 2cd.\cos. (180^\circ - x) = c^2 + d^2 + 2cd.\cos. x$. Hence $\cos. x = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$. The area of DCB $= ab.\sin. x =$ CB.DG. But BC $= \sqrt{(a^2 + b^2 - 2ab.\cos. x)} = m$. Hence DG $= \frac{ab.\sin. x}{m}$ (by Euc. VI. C) $ab =$ D $\times$ DG (the diameter of the circumscribing circle,)

$$ab = \frac{ab.\sin. x}{m} \times D, \therefore D = \frac{m}{\sin. x} = m, \cos. c.x.$$

28. In a given circle inscribe an equilateral triangle; and within this triangle describe a circle, &c.; then if r = radius of the first circle, find the sum of the areas of all the circles and all the triangles ad infinitum.

It is easy to see that the radius of the first circle is twice that of the second, the second twice that of the third, and so on indefinitely,

$\therefore$ the radii are r, $\frac{1}{2}r$, $\frac{1}{4}r$, $\frac{1}{8}r$, and so on. Put $p = 3.14159265$ &c. then the areas of the circles are pr^2, $\frac{1}{4}pr^2$, $\frac{1}{16}pr^2$, $\frac{1}{64}pr^2$, &c., (Geom. p. 291) hence $pr^2 + \frac{1}{4}pr^2 +$ &c. $= pr^2(1 + \frac{1}{4} + \frac{1}{16} +$ &c. ad infin.) $=$ the sum of the circles required. Put this sum $= s$, now $1 + \frac{1}{4} + \frac{1}{16} +$ &c. is a decreasing geometrical progression whose ratio of decrease $= \frac{1}{4}$, hence by the common rule for finding the sum of such a series, we have $1 + \frac{1}{4} + \frac{1}{16} +$ &c. $= 1 \div 1 - \frac{1}{4} = 1 \div \frac{3}{4} = \frac{4}{3}$, $\therefore s = \frac{4}{3}pr^2 =$ the sum of all the circles, as required.

It is also evident that the sides of the first triangle are twice those of the second, and so on as before. By putting $a = b = c$ and $s = \frac{3}{2}a$ in the cor. to the solution of page 598, there results $r = a \div \sqrt{3}$ and $a = r\sqrt{3} =$ the expression for the side of the equilateral triangle inscribed in the circle whose radius $= r$; by the common rule for the area of the equilateral triangle we have $(a^2\sqrt{3}) \div 4 =$ the area of the first triangle, and $\{(\frac{1}{2}a)^2\} \times \frac{1}{4}\sqrt{3}$ $=$ that of the second, and $\{(\frac{1}{4}a)^2\} \times \frac{1}{4}\sqrt{3} =$ that of the third, and so on. Let s' denote the sum of these areas, and we have

$$s' = \frac{a^2\sqrt{3}}{4}\left\{1 + \frac{1}{4} + \frac{1}{16} + \text{\&c.}\right\} = \frac{a^2}{\sqrt{3}} = r^2\sqrt{3}, \quad \text{Ans.}$$

If $a = 10$ feet, then $s = 57.735$ square feet very nearly.

Remark. If within any triangle we inscribe another by joining the middle of its sides, and within this second triangle we inscribe another by similar means, and so on, the sum of the triangles so formed together with the first is easily found in an analogous manner to the methods used in the above solution. For let s denote the area of the first triangle, then it is evident that $\frac{1}{4}s$ is the area of the second, $\frac{1}{16}s$ that of the third, and so on; let s denote the sum of these areas continued ad infinitum; then $s' = s(1 + \frac{1}{4} + \frac{1}{16}$ $+$ &c.) ad infinitum, $= 4s \div 3$ as required.

29. If from any point within an equilateral triangle, perpendiculars be drawn to the three sides, their sum is equal to a perpendicular drawn from one of the angles on the opposite side. Required proof.

From the point within the triangle draw straight lines to all the angles of the triangle, and they will evidently divide it into three triangles, whose bases are all equal to each other, being each one of the sides of the equilateral triangle. Let $a =$ one of the sides of the equilateral triangle, $p =$ the altitude of the triangle, then (Geom. p. 176) if $A =$ its area, we have $A = \frac{1}{2}ap$; also let x, y, z denote the perpendiculars from the point within to the sides of the triangle, then $\frac{1}{2}(ax + ay + az) =$ the sum of the three triangles into which the triangle was divided $= A$, $\therefore \frac{1}{2}a(x + y + z)$ $= \frac{1}{2}ap$, and $x + y + z = p$, as was to be proved.

30. Given the four sides of a quadrilateral inscribed in a circle, to find the diagonals.

Let ABDC (see fig. on page 598) be the inscribed quadrilateral, having AD, CB for its diagonals, put AB $= a$, AC $= b$, CD $= c$, BD $= d$, CB $= x$, AD $= y$. Now (Geom. p.) $xy = ac + bd$, (1), also (Geom. p.) $ab + cd : ad + bc :: y : x$, (2), or $ab + cd : ab + bc :: xy : x^2$, $\therefore$ by (1)

$$x = \sqrt{\left\{\frac{(ad + cb) \times (ac + bd)}{ad + bc}\right\}} \text{ hence}$$

$$y = \sqrt{\left\{\frac{(ab + cd) \times (ac + bd)}{ad + bc}\right\}} \text{ which are the diagonals.}$$

31. If a, b, c, d, be the four sides of a quadrilateral, inscribed in a circle, and $s = a + b + c + d$, it is required to prove that the area $= \sqrt{\{(\frac{1}{2}s - a)(\frac{1}{2}s - b)(\frac{1}{2}s - c)(\frac{1}{2}s - d)\}}$.

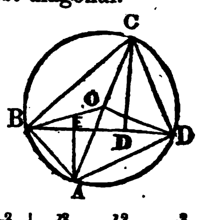
Let ABDC (see fig. to page 598) be the quadrilateral, inscribed in the circle ACE, from the angle A draw the perpendiculars AF, AG to the sides CD, BD, respectively, let A denote the area of the trapezium, then $\frac{1}{2}$(AF.CD) $=$ the area of the triangle ACD, and $\frac{1}{2}$(AG.BD) that of the triangle ABD, (Geom. p. 176); but these triangles make up the trapezium, $\therefore$ A $= \frac{1}{2}$(AF.CD $+$ AG.BD). Now the two angles ACF, ABD, when added make two right angles (Geom. p. 130) also ABD $+$ ABG $=$ two right angles (Geom. p. 28.) $\therefore$ the angle ACF $=$ ABG, and ACE being acute, ABD is obtuse, and the perpendicular AG falls without the triangle ABD; hence (Geom. pp. 191, 192,) $AD^2 = AC^2 + CD^2 - 2CD.CF$, and $AD^2 = AB^2 + BD^2 + 2BD.BG$, whence $AB^2 + BD^2 + 2BD.BG = AC^2 + CD^2 - 2CD.CF$, or $BD.BG + CD.CF = \frac{1}{2}(AC^2 + CD^2 - AB^2 - BD^2)$, (1), put AB $= a$, AC $= b$, CD $= c$, BD $= d$, and (1) becomes $BG \times d + CF \times c = \frac{1}{2}(b^2 + c^2 - a^2 - d^2)$, (2). Now since the angles at F and G are right, they are equal (Geom. p. 27), and since ACF $=$ ABG, the triangles ACF, AGB are equiangular, (Geom. p. 74), $\therefore$ similar, and AC : CF :: AB : BG, AC : AB :: AF : AG (Geom. p. 202), or $BG = \frac{CF.AB}{AC} = \frac{CF \times a}{b}$ and $AG = \frac{AF \times a}{b}$, hence $A = \frac{AF}{2b}(cb + ad)$ (3), also (2) gives $\frac{CF}{b} = \frac{b^2 + c^2 - a^2 - d^2}{2(cb + ad)}$, but CF $= \sqrt{(AC^2 - AF^2)} = \sqrt{(b^2 - AF^2)}$, hence $\sqrt{(1 - \frac{AF^2}{b^2})} = \frac{b^2 + c^2 - a^2 - d^2}{2(bc + ad)}$, or $AF^2 \div b^2 = 1 - \left\{\frac{b^2 + c^2 - a^2 - d^2}{2(cb + ad)}\right\}^2 = \frac{4(bc + ad)^2 - (b^2 + c^2 - a^2 - d^2)^2}{4(bc + ad)^2}$.

Or $\frac{AF}{b} = \frac{\sqrt{\{4(bc+ad)^2 - (b^2+c^2-a^2-d^2)^2\}}}{2(bc+ad)}$, this value when substituted in (3) gives $A = \frac{\sqrt{\{4(bc+ad)^2 - (b^2+c^2-a^2-d^2)^2\}}}{4}$, (4). Now $4(bc+ad)^2 - (b^2+c^2-a^2-d^2)^2 = (b^2+c^2+2bc-a^2-d^2+2ad).(a^2+d^2+2ad-b^2-c^2+2bc) = (\{b+c\}^2 - \{a-d\}^2)(\{a+d\}^2 - \{b-c\}^2) = \{a+b+c-d\}.\{b+c+d-a\}.\{a+d+b-c\}.\{a+d+c-b\} =$ (since $s = a+b+c+d$) $= \{s-2a\}\{s-2b\}\{s-2c\}.\{s-2d\}$, hence (4) becomes by substitution $A = \sqrt{(\frac{1}{2}s-a).(\frac{1}{2}s-b).(\frac{1}{2}s-c).(\frac{1}{2}s-d)}$ as required.

Otherwise. It has been proved (Geom. p.) that the product of the three sides of any plane triangle $=$ its surface multiplied by twice the diameter of its circumscribed circle; hence (supposing the same notation as in problems 27, 28, and the present problem,) $AC.CD.AD =$ the area of the triangle $ACD \times 4r$, and $AB.BD.AD =$ the area of the triangle $ABD \times 4r$, or (since the area of the two triangles $=$ the area of the trapezium $= a$), by addition $(AC.CD + AB.BD) \times AD = (bc+ad) \times AD = 4Ar$, or substituting the value of $AD = y = \sqrt{\left\{\frac{(ab+cd)\times(ac+bd)}{ad+bc}\right\}}$ as found in prob. 28, I have $A = \sqrt{((ab+cd)\times(ac+bd)(ad+bc))} \div 4r =$ (by (7) found in the solution of problem 27,) $(\sqrt{(S-a).(S-b).(S-c).(S-d)}$, which agrees with the result found above, for $S = \frac{1}{2}s$.

Cor. If one of the sides (as (d) for example $= 0$,) the trapezium becomes a triangle and the area $= \sqrt{\{S(S-a).(S-b).(S-c)\}}$ which agrees with the common rule for the area of a triangle when the three sides are given.

32. Having given the area and its four sides respectively of a trapezium, to determine the length of the greatest diagonal.



Put $AB = a$, $BC = b$, $CD = c$, $AD = d$, angle $BAD = x$, angle $BCD = y$, and the double area $= e$, then by Young's Trigonometry, page 47, Amer. edit. $a^2 + d^2 - 2ad \cos. x = BD^2 = b^2 + c^2 - 2bc \cos. y$, therefore $2ad \cos. x - 2bc \cos. y = a^2 + d^2 - b^2 - c^2$.

Again, double the area of the triangle $BAD = ad \sin. x$, and double the area of the triangle $BCD = bc \sin. y$; therefore, putting $2r = a^2 + d^2 - b^2 - c^2$, we have $ad \sin. x + bc \sin. y = e$ and $ad \cos. x - bc \cos. y = r$. Hence $a^2d^2 \sin.^2 x = (e - bc \sin. y)^2$, and $a^2d^2 \cos.^2 x = (r + bc \cos. y)^2$, therefore by addition, and remarking that $\sin.^2 x + \cos.^2 x = 1$, and $\sin.^2 y + \cos.^2 y = 1$, $a^2d^2 = (e - bc \sin. y)^2 + (r + bc \cos. y)^2$

$= e^2 - 2ecb \sin. y + r^2 + 2rbc \cos. y + b^2c^2$, or $r \cos. y - e \sin. y = \frac{a^2d^2 - b^2c^2 - e^2 - r^2}{2bc} = m$; whence $r \cos. y - m = e \sin. y$; or, $r^2 \cos.^2 y - 2rm \cos. y + m^2 = e^2 \sin.^2 y = e^2 - e^2 \cos.^2 y$, therefore $(r^2 + e^2) \cos.^2 y - 2rm \cos. y = e^2 - m^2$, and, dividing by $r^2 + e^2$ and completing the $\square$, $\cos. y = \frac{1}{r^2 + e^2} \times \{e\sqrt{(r^2 + e^2 - m^2)} + rm\}$, and the diagonal BD $= \sqrt{(b^2 + c^2 - 2bc \cos. y)}$.

Or since $(bc \sin. y)^2 = (e - ad \sin. x)^2$ and $(bc \cos. y)^2 = (r - ad \cos. x)^2$, we have as before

$$e \sin. x + r \cos. x = \frac{a^2d^2 - b^2c^2 + e^2 + r^2}{2ad} = n, \text{ therefore}$$

$\cos. x = \frac{1}{r^2 + e^2} \times \{r\sqrt{(r^2 + e^2 - n^2)} + rn\}$, and the diagonal BD is then $= \sqrt{(a^2 + d^2 - 2ad \cos. x)}$.

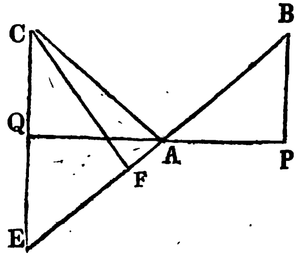
Here it may be remarked, that when the value of e is such as to make either m^2 or n^2 greater than $r^2 + e^2$, the part under the radical becomes negative, and consequently the problem does not then admit of a solution. Therefore the limit of possibility, or the case in which the area is the greatest possible, will be when m^2 and n^2 are each equal to $r^2 + e^2$. The values of m and n are then equal, but have a different sign, and the above expressions for $\cos. y$ and $\cos. x$ give $\cos. y = \frac{rm}{r^2 + e^2}$ or $\frac{r}{\sqrt{(r^2 + e^2)}}$; $\cos. x = \frac{rn}{r^2 + e^2}$ or $\frac{-r}{\sqrt{(r^2 + e^2)}}$; therefore $\cos. y$ is $= -\cos. x$, and consequently the one angle is the supplement of the other, and the trapezium is inscribed in a circle. Again, because $m = -n$,

$\frac{a^2d^2 - b^2c^2 - e^2 - r^2}{2bc} = -\frac{a^2d^2 - b^2c^2 - e^2 + r^2}{2ad}$; therefore by reduction $e^2 = (ad + bc)^2 - r^2$, or $e = \sqrt{\{(ad + bc)^2 - r^2\}}$. Hence, when the four sides are given, if the double area be greater than $\sqrt{\{(ad + bc)^2 - r^2\}}$ the problem is impossible.

The common rule for finding the area of a trapezium capable of being inscribed in a circle, when the four sides are known, may be deduced immediately from the above expression for e, by considering that $(ad + bc)^2 - r^2$ is $= (ad + bc + r)\,(ad + bc - r) = \frac{1}{4}(2ad + 2bc + a^2 + d^2 - b^2 - c^2)\,(2ad + 2bc - a^2 - d^2 + b^2 + c^2) = \frac{1}{4}(a + d + c - b)\,(a + d - c + b)\,(b + c + a - d)\,(b + c + d - a)$.

33. Two lines AB, AC, drawn from the same point A, being given both in position and length, to draw another PQ through that point, so that two perpendiculars BP, CQ, falling thereon from the

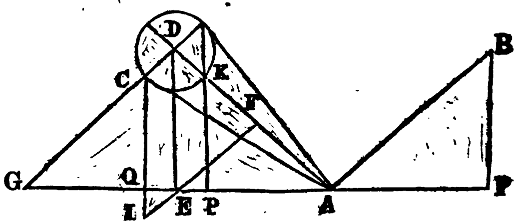
extremes of the two given lines, may form two triangles ABP ACQ, equal to each other.



Let BA and CQ be produced to meet in E; and upon BE let fall the perpendicular CF; and put CF $= a$, AF $= b$, AB $= c$, and EF $= x$. The areas of similar triangles being in proportion as the squares of their homologous sides, and both the triangles AEQ, ABP, being similar to ECF, we have $CE^2(a^2 + x^2) : \frac{1}{2}CF \times EF(\frac{1}{2}ax) :: AB^2\ (c^2) :$ area $ABP = \frac{\frac{1}{2}ac^2x}{a^2 + x^2}$. And, $a^2 + x^2 : \frac{1}{2}ax :: (x + b)^2\ (= AB^2)$: area $AEQ = \frac{\frac{1}{2}ax \times (x + b)^2}{a^2 + x^2}$. This, taken from the area AEC $(= \frac{1}{2}a \times (x + b))$ leaves the area $ACQ = \frac{\frac{1}{2}a \times (x + b) \times (a^2 - bx)}{a^2 + x^2}$. Which being equal to ABP by hypothesis, we therefore have $(x + b) \times (aa - bx) = c^2x$; from whence, making $d = \frac{c^2 - a^2}{b} + b$, x is found $= \sqrt{(a^2 + \frac{1}{4}d^2)} - \frac{1}{2}d$.

GEOMETRICALLY.—If Ab be supposed perpendicular, and equal to AB, and bp perpendicular to PQ, the triangle Abp being similar to ABP, will also be equal to it; and consequently equal to ACQ. And, if these equal triangles be, successively, taken from the trapezium AQCb, the remainders pbCQ and ACb will likewise be equal.

Hence it appears that the parallelogram under Cb and DE, whose angle is CDE (supposing DE parallel to, and an arithmetical mean between CQ and bp) is equal to the given triangle ACb; and consequently the altitude thereof equal to half that of the triangle.



Whence it is evident that the point E must fall somewhere in a line FI drawn through the middle of AD (or AC) parallel to bC

(Nulty's Geom. p.). But the point E, since the angle ADE is a right one, will likewise fall in the circumference of a semicircle described upon the diameter AD (Nulty's Geom. p.). And therefore FE, being a radius, must be equal to AF; and consequently DG = AD, supposing DC produced to meet AQ in G.

Therefore, in order to the geometrical construction, having made A*b* perpendicular and equal to AB, and drawn AD to the middle of C*b* (as above intimated), let DG, in DC produced, be taken equal to AD; and from G, through A, draw GP, and the thing is done.

It often happens that the demonstration of a geometrical construction, to be the most neat and elegant, proceeds upon principles very different from those whereby we first arrived at such construction. The case above is an instance of it; where, from the similar triangles, it is manifest that GQ (Ap) : QC :: Gp (AQ) : pb; and therefore $\frac{1}{2}AQ \times QC = \frac{1}{2}Ap \times pb = \frac{1}{2}BP \times AP$.